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ON MIXED INTERPOLATING-SMOOTHING SPLINES AND THE ν-SPLINE

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Abstract. In [BV93], Bezhaev and Vasilenko characterize the "mixed interpolating-smoothing spline" in the abstract setting of a Hilbert space. In this paper, we derive a characterization under slightly more general conditions. This is specialized to the finite-dimensional case, where, in particular, we show that the ν-spline (a piecewise polynomial alternative to splines in "tension", see [N74]) is a special case of the mixed interpolating-smoothing spline, a limiting case of smoothing splines as certain weights increase to infinity, and a limiting case of near-interpolants ([K99a]) as certain tolerances decrease to zero.

Key words. splines, interpolation, smoothing, approximation

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1. Introduction

Let X, Y, Z_1 and Z_2 be Hilbert spaces with inner products $\langle\, , \rangle_X$, $\langle\, , \rangle_Y$, $\langle\, , \rangle_{Z_1}$ and $\langle\, , \rangle_{Z_2}$, respectively, and let $T : X \longrightarrow Y$, $\Lambda_1 : X \longrightarrow Z_1$, $\Lambda_2 : X \longrightarrow Z_2$ and $\pi : Z_2 \longrightarrow Z_2$ be bounded linear maps. For $z_1 \in Z_1$, $z_2 \in Z_2$ and $\rho > 0$, the **mixed interpolating-smoothing spline** is a solution to the problem

$$\text{(1.1)} \qquad \underset{f\in\Lambda_1^{-1}\{z_1\}}{\text{minimize}} \ \|Tf\|_Y^2 + \rho\,\|\pi(\Lambda_2 f - z_2)\|_{Z_2}^2,$$

as defined in [BV93] but when π is the identity map on Z_2 and ρ is replaced by $1/\rho$. In particular, (1.1) reduces to spline interpolation when $Z_2 = \emptyset$, and to smoothing when $Z_1 = \emptyset$.

The solutions to (1.1) (when π is the identity map) are characterized in [BV93] by the method of Lagrange multipliers. For this, it is observed that the solutions to (1.1) correspond to stationary points of the "Lagrangian" functional

$$\mathcal{L}(f, v) := \frac{1}{2}\|Tf\|_Y^2 + \frac{1}{2}\rho\,\|\pi(\Lambda_2 f - z_2)\|_{Z_2}^2 + \langle v, \Lambda_1 f - z_1\rangle_{Z_1}$$

for some "Lagrange multiplier" $v \in Z_2$, and are identified by setting the partial derivatives of $\mathcal{L}$ with respect to f and v to zero. In the present paper, the solutions to (1.1) are characterized by the theory of best approximation in a Hilbert space, an application of the projection theorem, as done for the separate problems of pure smoothing and pure interpolation in [A92] (following his earlier work in the 1960's), [AL68], [BV93] and [dB98]. This abstract characterization is refined by specializing (1.1) to the case that Z_1 and Z_2 are finite-dimensional, and further refined when the splines are represented in a certain "dual" basis.

We then apply the abstract results to problems of the form

$$\text{(1.2)} \qquad \underset{\substack{f^{(j-1)}(t_i)=z_{ij}\\ (i,j)\in\iota}}{\text{minimize}} \int_0^1 |f^{(m)}(s)|^2 \,\mathrm{d}\,s \ + \sum_{(i,j)\notin\iota} w_{ij}\,|f^{(j-1)}(t_i) - z_{ij}|^2$$

for some subset ι of $n \times m$ and for non-negative **weights** w_{ij}. The solutions here are polynomial splines. Problem (1.2) reduces to the problem of best interpolation when $\iota = n \times m$, and to the problem of smoothing when $\iota = \emptyset$. This generalizes the standard problem of smoothing studied in [So64], [R67], [R71], [dB78], [W90], [D93] and [dB98], for example, to include the "Hermite-type" functionals $f \longmapsto f^{(j-1)}(t_i)$ and possibly zero weights (the reciprocal of the weights appear in the characterizations given in those papers that include the weights).

A third instance of (1.2) is when $\iota = \{(i, 1)\}$ and $z_{ij} = 0$ for $j > 1$, in which case the solutions to (1.2) are ν-splines, as defined in [N74] (when $m = 2$). In this case, the weights w_{ij} for $j > 1$ are the so-called "tension" parameters. Moreover, by letting some of the weights go to ∞ in the problem of pure smoothing, we show that ν-splines are a special limiting case of smoothing splines. As shown in [K99b], the problem

$$\text{(1.3)} \qquad \text{minimize } \left\{ \int_0^1 |f^{(m)}(s)|^2 \,\mathrm{d}\,s : |f^{(j-1)}(t_i) - z_{ij}| \le \varepsilon_{ij},\ i{=}1{:}n,\ j{=}1{:}m \right\}$$

of "near-interpolation" ([K99a]) yields the same solutions as the problem of smoothing for certain weights w_{ij} that depend on the **tolerances** ε_{ij}, and vice-versa (here, zero weights correspond to inactive constraints). As a consequence, we also verify here that the ν-spline is a limiting case of near-interpolation when $\varepsilon_{ij} \longrightarrow 0$ for some ij.

As a final example, we will apply the theory to the problem

$$\underset{\substack{f^{(j-1)}(t_i)=z_{ij} \\ (i,j)\in\iota}}{\text{minimize}} \int_0^1 |f''(s)+\alpha f'(s)|^2 \,\mathrm{d}\, s \;+ \sum_{(i,j)\notin\iota} w_{ij}\,|f^{(j-1)}(t_i) - z_{ij}|^2,$$

the solutions of which are "splines in tension" (with smoothing), generalized here to include the Hermite-type functionals. The solutions here are piecewise exponentials, in contrast to the previous examples where the solutions are piecewise polynomial. Although not discussed in this paper, thin plate splines and thin plate splines in tension are also special cases of (1.1).

2. Hilbert space structure and additional notation

To characterize the mixed interpolating-smoothing spline, (1.1) is reformulated as a problem of best approximation in the Hilbert space

$$H := Y \times Z_2$$

with inner product

$$\langle\, ,\rangle_H := \langle\, ,\rangle_Y + \rho\,\langle\, ,\rangle_{Z_2}.$$

With

$$e : X \longrightarrow H : f \longmapsto (Tf, \pi\Lambda_2 f),$$

problem (1.1) can be restated

(2.1) $$\underset{f\in\Lambda_1^{-1}\{z_1\}}{\text{minimize}} \;\|e(f) - (0,\pi z_2)\|_H^2.$$

The setup is illustrated in Figure 2.2.

$$\begin{tikzcd}
\Lambda_1^{-1}\{z_1\} \subset X \arrow[r, "T"] \arrow[d, "\Lambda_2"'] \arrow[dr, "e"] & Y \arrow[d] \\
Z_2 \arrow[r] & H = Y \times Z_2
\end{tikzcd}$$

Figure 2.2. Hilbert space structure.

In addition to the notation above, let $Z := Z_1 \times Z_2$ and $\Lambda : X \longrightarrow Z : f \longmapsto (\Lambda_1 f; \Lambda_2 f)$ (with ";" indicating a column vector, i.e., a row map). Let $T^* : Y \longrightarrow X$ and $\Lambda^* : Z \longrightarrow X$ denote the usual adjoint maps (with identifications $Z \approx Z^*$ and $X \approx X^*$). To keep the notation below precise, let I_{ii} be the identity map on Z_i, 0_i the zero element of Z_i, and 0_{ij} the zero map from Z_i to Z_j, $i = 1, 2$ and $j = 1, 2$, and let 0_X be the zero element of X, 0_Y the zero element of Y, and $0 \in \mathbb{R}$.

3. Characterizations

The characterizations (and existence) of solutions to (1.1), as stated below, are based on the observation that σ is a solution iff $e(\sigma)$ is a best approximation to $(0_Y, \pi z_2)$ from $e(\Lambda_1^{-1}\{z_1\})$. In the Hilbert space H, and assuming that $e(\Lambda_1^{-1}\{z_1\})$ is closed, this is the case iff $e(\sigma) - (0_Y, \pi z_2)$ is orthogonal to $e(\Lambda_1^{-1}\{z_1\})$ (a consequence of the "projection theorem" for closed convex sets in Hilbert spaces).

Theorem 3.1. *Solutions to (1.1) exist when* $z_1 \in \operatorname{ran} \Lambda_1$, *and they are unique if* $\ker T \cap \ker \Lambda_1 \cap \ker(\pi \Lambda_2) = \{0_X\}$.

Proof: Since $z_1 \in \operatorname{ran} \Lambda_1$ and Λ_1 is bounded, $\Lambda_1^{-1}\{z_1\}$ is a nonempty closed and affine subspace of X, which, under the bounded linear map e, $e(\Lambda_1^{-1}\{z_1\})$ becomes a closed and affine subset of H. Indeed, $\Lambda_1^{-1}\{z_1\}$ is closed since Λ_1 is bounded, and e is bounded since T and $\pi\Lambda_2$ are bounded. Therefore, there exists a unique element $(y, z) \in e(\Lambda_1^{-1}\{z_1\})$ of minimal distance to $(0_Y, \pi z_2) \in H$, and any $\sigma \in \Lambda_1^{-1}\{z_1\}$ such that $e(\sigma) = (y, z)$ solves (1.1). This establishes existence.

Assume that σ_1 and σ_2 are solutions to (1.1). Then $e(\sigma_1)=e(\sigma_2)=(y, z)$ and $\Lambda_1\sigma_1=\Lambda_1\sigma_2$, implying that $(\sigma_1 - \sigma_2) \in \ker T \cap \ker \Lambda_1 \cap \ker(\pi\Lambda_2)$, and so $\sigma_1 = \sigma_2$ when $\ker T \cap \ker \Lambda_1 \cap \ker(\pi\Lambda_2) = \{0_X\}$. □

Proposition 3.2. *Let* $\sigma \in X$. *Then* σ *is a solution to (1.1) iff*

$$\begin{bmatrix} \Lambda_1 & 0_{11} \\ (T^*T + \rho\,(\pi\Lambda_2)^*\pi\Lambda_2) & \Lambda_1^* \end{bmatrix} \begin{bmatrix} \sigma \\ v \end{bmatrix} = \begin{bmatrix} z_1 \\ \rho\,(\pi\Lambda_2)^*\pi z_2 \end{bmatrix} \tag{3.3}$$

for some $v \in Z_1$.

Proof: Note first that $\Lambda_1\sigma = z_1$ when (3.3) holds, implying that $z_1 \in \operatorname{ran} \Lambda_1$, which, by Theorem 3.1, implies existence. As stated above, σ solves (1.1) iff $e(\sigma)$ is a best approximation to $(0_Y, \pi z_2)$ from $e(\Lambda_1^{-1}\{z_1\})$, which holds iff

$$e(\sigma) - (0_Y, \pi z_2) \;\perp\; e(\Lambda_1^{-1}\{z_1\}),$$

iff

$$\langle e(\sigma) - (0_Y, \pi z_2), e(\sigma) - e(f)\rangle_H = 0$$

for all $f \in \Lambda_1^{-1}\{z_1\}$, and, since $\Lambda_1^{-1}\{z_1\}$ is parallel to $\ker \Lambda_1$, iff

$$\langle e(\sigma) - (0_Y, \pi z_2), e(f)\rangle_H = 0$$

for all $f \in \ker \Lambda_1$, i.e.,

$$\langle T\sigma, Tf\rangle_Y + \rho\,\langle \pi(\Lambda_2\sigma - z_2), \pi\Lambda_2 f\rangle_{Z_2} = 0.$$

On passing to the adjoint maps,

$$\langle T^*T\sigma, f\rangle_X + \rho\,\langle (\pi\Lambda_2)^*\pi(\Lambda_2\sigma - z_2), f\rangle_X = 0$$

for all $f \in \ker \Lambda_1$. Therefore,

$$T^*T\sigma + \rho\,(\pi\Lambda_2)^*\pi(\Lambda_2\sigma - z_2) \in (\ker \Lambda_1)^\perp = \operatorname{ran} \Lambda_1^*$$

(the last equality since X and Z are Banach spaces and Λ_1 is bounded, by [R91: Theorem 4.12]), and so

$$T^*T\sigma + \rho\,(\pi\Lambda_2)^*\pi(\Lambda_2\sigma - z_2) + \Lambda_1^* v = 0_X$$

for some $v \in Z_1$. This, along with the interpolatory condition $\Lambda_1\sigma = z_1$, establishes (3.3). □

Equation (3.3) is identical to Equation (1.10) in [BV93] when $\pi = I_2$.

Corollary 3.4. *If σ solves (1.1), then $T^*T\sigma \in \operatorname{ran}\Lambda^*$. If also*

$$\langle\ ,\ \rangle_X : (f,g) \longmapsto \langle Tf, Tg\rangle_Y + \langle f, g\rangle_{\ker T}$$

with $\langle\ ,\ \rangle_{\ker T}$ the inner product on $\ker T$, then $\sigma \in \ker T + \operatorname{ran}\Lambda^*$.

Proof: For the first result, it follows by (3.3) that

$$T^*T\sigma = -\rho\,\Lambda_2^*\pi^*\pi(\Lambda_2\sigma - z_2) - \Lambda_1^* v,$$

implying that $T^*T\sigma$ is in the range of Λ^*.

For the second result, let $\sigma =: p + h$ in the orthogonal sum decomposition $(\ker T) \oplus (\ker T)^\perp$ of X. Then,

$$\langle T^*Th, \cdot\rangle_X = \langle Th, T\cdot\rangle_Y = \langle h, \cdot\rangle_{\ker T} + \langle Th, T\cdot\rangle_Y = \langle h, \cdot\rangle_X,$$

and so T^*T is the identity map on $(\ker T)^\perp$. Therefore,

$$\operatorname{ran}\Lambda^* \ni T^*T\sigma = T^*T(h+p) = T^*Th = h,$$

and so $\sigma = p + h \in \ker T + \operatorname{ran}\Lambda^*$. □

We will now assume that the map Λ is onto with a finite dimensional range Z. In this case, Λ^* is 1-1 (as stated in the proof below), and the map

$$\Psi := (\Lambda\Lambda^*)^{-1}\Lambda T^*T : X \longrightarrow Z \tag{3.5}$$

is well-defined. We decompose this map as follows:

$$\Psi =: (\Psi_1; \Psi_2) : X \longrightarrow Z_1 \times Z_2 : f \longmapsto (\Psi_1 f; \Psi_2 f).$$

(the map $\Lambda^*(\Lambda\Lambda^*)^{-1}\Lambda$ is the projector for spline interpolation restricted to $(\ker T)^\perp$, as shown in [BV93: equation (1.47)].)

Proposition 3.6. *Let $\sigma \in X$. Assume that $\Lambda : X \longrightarrow Z$ is onto, and that Z is finite-dimensional. Then, σ solves (1.1) iff $\Lambda_1\sigma = z_1$ and*

$$(\Psi_2 + \rho\,\pi^*\pi\Lambda_2)\sigma = \rho\,\pi^*\pi z_2, \tag{3.7}$$

with Ψ as defined in (3.5). Moreover, $v = -\Psi_1\sigma$ (with v as in (3.3)).

Proof: Since $\Lambda : X \longrightarrow Z$ is onto and Z is finite-dimensional, the map $\Lambda^* : Z \longrightarrow X$ is 1-1. Therefore, the map $\Lambda\Lambda^*$ is invertible. By (3.3),

$$T^*T\sigma + \rho\,(\pi\Lambda_2)^*\pi\Lambda_2\sigma + \Lambda_1^* v = \rho\,(\pi\Lambda_2)^*\pi z_2.$$

Equivalently,

$$T^*T\sigma + \rho\,\Lambda^*\begin{bmatrix} 0_1 \\ \pi^*\pi\Lambda_2\sigma \end{bmatrix} + \Lambda^*\begin{bmatrix} v \\ 0_2 \end{bmatrix} = \rho\,\Lambda^*\begin{bmatrix} 0_1 \\ \pi^*\pi z_2 \end{bmatrix}.$$

On multiplying through by $(\Lambda\Lambda^*)^{-1}\Lambda$, and with $\Psi = (\Psi_1; \Psi_2)$ as defined in (3.5), this reduces to

$$\begin{bmatrix} \Psi_1\sigma \\ \Psi_2\sigma \end{bmatrix} + \rho\begin{bmatrix} 0_1 \\ \pi^*\pi\Lambda_2\sigma \end{bmatrix} + \begin{bmatrix} v \\ 0_2 \end{bmatrix} = \rho\begin{bmatrix} 0_1 \\ \pi^*\pi z_2 \end{bmatrix}.$$

This equation, along with the interpolation condition $\Lambda_1\sigma = z_1$, establish (3.7). $\square$

Proposition 3.8. *Assume the hypotheses of Corollary 3.4 and Proposition 3.6, and that $\ker T \subset \operatorname{ran}\Lambda^*$. Let $V = (V_1, V_2)$ be the basis(-map) for $\operatorname{ran}\Lambda^*$ that is dual to Λ. Then, σ solves (1.1) iff $\sigma = V\alpha = V_1\alpha_1 + V_2\alpha_2$ for some $\alpha \in Z$ such that $\alpha_1 = z_1$ and*

$$(\Psi_2 V_2 + \rho\pi^*\pi)\alpha_2 = \rho\pi^*\pi z_2 - \Psi_2 V_1 z_1. \tag{3.9}$$

In this case, $v = -\Psi_1 V\alpha$ (with v as in (3.3)).

Proof: By Corollary 3.4, $\sigma \in \ker T + \operatorname{ran}\Lambda^*$, and since $\ker T \subset \operatorname{ran}\Lambda^*$, then $\sigma \in \operatorname{ran}\Lambda^*$. Since V is dual to Λ,

$$\Lambda_1\sigma = \Lambda_1 V\alpha = \Lambda_1(V_1\alpha_1 + V_2\alpha_2) = \alpha_1,$$

and likewise $\Lambda_2\sigma = \alpha_2$. By Proposition 3.6 it follows that $\Lambda_1\sigma = z_1 = \alpha_1$, $v = -\Psi_1\sigma = -\Psi_1 V\alpha$, and

$$\Psi_2(V_1 z_1 + V_2\alpha_2) + \rho\pi^*\pi\alpha_2 = \rho\pi^*\pi z_2,$$

which is equivalent to (3.9). $\square$

Note that the smoothing and interpolating parts of the characterizations in Propositions 3.6 and 3.8 are disjoint. As a consequence, v does not need to be calculated.

4. Piecewise polynomial splines and splines in tension

Let $m,\ n \in \mathbb{Z}_+$, and let ι be a subset of $n \times m$ with complement ι^c. Let

$$\begin{aligned}
X &:= L_2^{(m)}([0\,..1] \longrightarrow \mathbb{R}),\\
Y &:= L_2([0\,..1] \longrightarrow \mathbb{R}),\\
Z = Z_1 \times Z_2 &:= \mathbb{R}^{\iota} \times \mathbb{R}^{\iota^c} = \mathbb{R}^{m \times n},\\
T : X \longrightarrow Y &: f \longmapsto D^m f := f^{(m)},
\end{aligned}$$

with X the Sobolev space of functions $f : [0\,..1] \longrightarrow \mathbb{R}$ such that $f^{(m)}$ is in the Lebesgue space Y, and with inner products

$$\langle f, g\rangle_X := \underbrace{\sum_{j=1}^{m} f^{(j-1)}(0) \cdot g^{(j-1)}(0)}_{\langle f,g\rangle_{\ker T}} \ + \ \underbrace{\int_0^1 Tf(s) \cdot Tg(s)\,\mathrm{d}\,s}_{\langle Tf,Tg\rangle_Y} \tag{4.1}$$

and

$$\langle \alpha, \beta\rangle_{Z_2} := \sum_{(i,j)\in\iota^c} \alpha_{ij}\beta_{ij}.$$

In particular, $\langle\ ,\ \rangle_Y$ is a semi-norm on X, and the map T is bounded. Let $t = (t_i)$ be a sequence of **data sites** such that $0{=}t_1{<}t_2{<}\cdots{<}t_n{=}1$, and let

$$\begin{aligned}
\Lambda_1 &: X \longrightarrow Z_1 : f \longmapsto (f^{(j-1)}(t_i) : (i,j) \in \iota),\\
\Lambda_2 &: X \longrightarrow Z_2 : f \longmapsto (f^{(j-1)}(t_i) : (i,j) \in \iota^c).
\end{aligned} \tag{4.2}$$

Since $t_i < t_{i+1}$ for $i{=}1{:}n{-}1$, the map $\Lambda : X \longrightarrow Z$ is onto. Finally, let

$$\pi : Z_2 \longrightarrow Z_2 : a \longmapsto (\sqrt{w_{ij}}\, a_{ij} : (i,j) \in \iota^c)$$

for **weights** $w_{ij} \geq 0$. Then, the map π corresponds to a diagonal matrix, and so $\pi^* = \pi$. Let $W_2 := \pi^*\pi = \pi^2$. Problem (1.1) then reduces to the problem

$$\operatorname*{minimize}_{\substack{f^{(j-1)}(t_i)=z_{ij}\\ (i,j)\in\iota}} \int_0^1 |f^{(m)}(s)|^2\ \mathrm{d}\,s \ + \sum_{(i,j)\notin\iota} w_{ij}\,|f^{(j-1)}(t_i) - z_{ij}|^2. \tag{4.3}$$

It is a standard result that $\operatorname{ran}\Lambda^* = \$_{2m,t}$, the space of piecewise polynomials on $[0\,..1]$ of order $2m$ and with $m-1$ continuous derivatives at the "breakpoints" t_i (as discussed in [K99a], for example). In particular, $\ker T \subset \operatorname{ran}\Lambda^*$, and so by Corollary 3.4 solutions are in $\operatorname{ran}\Lambda^*$. Let

$$\operatorname{jmp}_{t_i} : f \longmapsto \lim_{u\downarrow t_i} f(u) - \lim_{u\uparrow t_i} f(u),$$

with $f^{(j-1)}(0^-) := 0 =: f^{(j-1)}(1^+)$ for $j{=}1{:}m$.

Lemma 4.4 ([K99a: Lemma 5.3]). *Let* $f \in \operatorname{ran} \Lambda^*$. *Then*

$$\Psi f = (\Lambda\Lambda^*)^{-1} \Lambda T^* T f = ((-1)^{m+j-1} \mathrm{jmp}_{t_i} f^{(2m-j)} : i{=}1{:}n,\ j{=}1{:}m).$$

Proof: In [K99a] it was shown by integration by parts that

$$\langle Tf, Tg \rangle_{Z_2} = \sum_{i=1}^{n} \sum_{j=1}^{m} (-1)^{m-j+1} \left(\mathrm{jmp}_{t_i} f^{(2m-j)}\right) \cdot g^{(j-1)}(t_i) =: \langle \mathrm{jmp}_t f, \Lambda g \rangle_{Z_2}$$

for all $g \in X$. On passing to the adjoints, $T^*Tf = \Lambda^* \mathrm{jmp}_t f$. Since Λ maps X onto the finite-dimensional space Z, then Λ^* is 1-1, and so $\Lambda\Lambda^*$ is invertible. Hence, on multiplying through by $(\Lambda\Lambda^*)^{-1}\Lambda$, we have that $(\Lambda\Lambda^*)^{-1}\Lambda T^*Tf = \mathrm{jmp}_t f$. That is, jmp_t is the map Ψ defined in (3.5). □

By (3.7)

$$\Psi_2 \sigma = -\rho W_2 (\Lambda_2 \sigma - z_2), \tag{4.5}$$

or, with $\sigma = V\alpha$ with V the basis(-map) for $\operatorname{ran} \Lambda^*$ that is dual to Λ, it follows by (3.9) that

$$(\Psi_2 V_2 + \rho W_2)\alpha_2 = \rho W_2 z_2 - \Psi_2 V_1 z_1. \tag{4.6}$$

In particular,

$$\mathrm{jmp}_{t_i} \sigma^{(2m-j)} = (-1)^{m+j} \rho\, w_{ij} (\sigma^{(j-1)}(t_i) - z_{ij}) \tag{4.7}$$

when $(i,j) \notin \iota$. Here, this dual basis V is the "piecewise Hermite" basis for $\$_{2m,t}$.

Example 4.8 (best interpolation). $w_{ij} = 0$ *for* $(i,j) \in \iota^c$.

In this case, (1.1) reduces to the problem

$$\underset{f \in \Lambda_1^{-1}\{z_1\}}{\text{minimize}} \int_0^1 |f^{(m)}(s)|^2 \,\mathrm{d}\, s,$$

and $\sigma = V\alpha$ solves (1.1) iff $\alpha_1 = \Lambda_1 \sigma = z_1$ and $\Psi_2 V\alpha = \Psi_2 \sigma = 0$. That is, σ is characterized by the equations $\sigma^{(j-1)}(t_i) = z_{ij}$ if $(i,j) \in \iota$ and $\mathrm{jmp}_{t_i} \sigma^{(2m-j)} = 0$ if $(i,j) \in \iota^c$.

Example 4.9 (smoothing). $Z_2 = Z$.

Problem (1.1) reduces to

$$\underset{f \in X}{\text{minimize}} \int_0^1 |f^{(m)}(s)|^2 \,\mathrm{d}\, s \;+\; \rho \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij} \, |f^{(j-1)}(t_i) - z_{ij}|^2.$$

By (3.9),

$$(\Psi V + \rho W_2)\alpha = \rho W_2 z.$$

This reduces to the standard problem of smoothing (as in [R71], for example) when $w_{ij} = 0$ for $j > 1$ and $w_{ij} > 0$ for $j = 1$. In particular, there is no difficulty with zero weights in this formulation, unlike the standard characterizations given in the literature that involve the inverse of the weights. Moreover, with the derivative functionals, this generalizes the usual problem of smoothing to "Hermite smoothing".

Example 4.10 (the ν-spline). *$Z_1 = (z_{ij} : j = 1)$ and $Z_2 = (z_{ij} = 0 : j > 1)$, and $\rho = 1$.*

Problem (1.1) reduces to

$$\underset{f(t_i)=z_{i1}}{\text{minimize}} \int_0^1 |f^{(m)}(s)|^2 \,\mathrm{d}\, s \;+\; \sum_{i=1}^{n}\sum_{j=2}^{m} w_{ij}\, |f^{(j-1)}(t_i)|^2.$$

The solutions to this problem are ν-splines (see [N74]), generalized here to $m > 2$. The weights w_{ij} for $j > 1$ are termed "tension parameters". In particular, by (4.7),

$$\mathrm{jmp}_{t_i} \sigma^{(2m-j)} = (-1)^{m+j} \rho\, w_{ij} (\sigma^{(j-1)}(t_i) - z_{ij})$$

for $j > 1$, which generalizes the property

$$\sigma''(t_i^+) - \sigma''(t_i^-) = \nu_i\, \sigma'(t_i)$$

for ν-splines given in [N74] for $m = 2$, where here $\nu_i = w_{i2}$. Hence, ν-splines are a special case of mixed splines.

Example 4.11 (limits of smoothing splines and the ν-spline). *$Z_2 = Z$ and $w_{ij} \longrightarrow \infty$ for some ij.*

As in Example 4.9,

$$(\Psi V + \rho\, W_2)\alpha = \rho\, W_2 z.$$

Assume that W_1 consists of those w_{ij} such that $w_{ij} \longrightarrow \infty$, and that Λ_i, Z_i, W_i, etc., are defined similarly. Then,

$$\Psi_1 V \alpha + \rho\, W_1 \alpha_1 = \rho\, W_1 z_1,$$
$$\Psi_2 V \alpha + \rho\, W_2 \alpha_2 = \rho\, W_2 z_2.$$

On replacing the first of these equations by

$$W_1^{-1}(\Psi_1 V \alpha) + \rho\, \alpha_1 = \rho\, z_1,$$

and passing to the limit $w_{ij} \longrightarrow \infty$ for w_{ij} in W_1, the limiting case of the first equation is

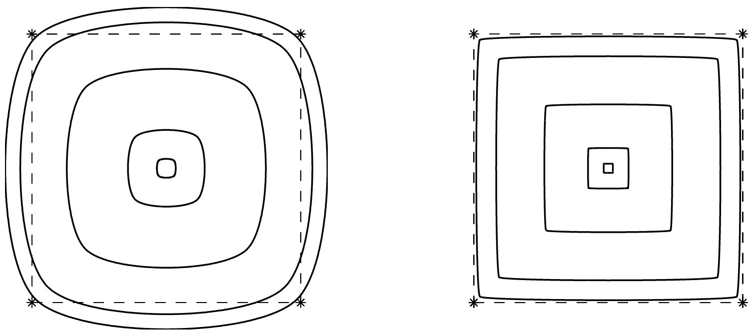
$$\rho\, \alpha_1 = \rho\, z_1,$$

i.e., $\alpha_1 = z_1$ when $\rho > 0$. Indeed, since $\sigma := V\alpha$ solves (4.3) for fixed w, then

$$w_{ij} |\alpha_{ij} - z_{ij}|^2 \le \|T\sigma\|_Y^2 + \sum_{i,j} w_{ij}\, |\alpha_{ij} - z_{ij}|^2 \le \|Tf\|_Y^2$$

with f the interpolating spline, and so $w_{ij}|\alpha_{ij} - z_{ij}|^2$ is bounded, and $|\alpha_{ij} - z_{ij}| \longrightarrow 0$ as $w_{ij} \longrightarrow \infty$. Hence, smoothing splines converge to mixed interpolating-smoothing splines when $w_{ij} \longrightarrow \infty$ for some ij. In particular, when $w_{ij} \longrightarrow \infty$ for $j = 1$ and $z_{ij} = 0$ for $j > 1$, the corresponding smoothing splines converge to a ν-spline.

This convergence is illustrated in Figure 4.12 (a) and (b). Since the effect of the tension parameter in the ν-spline is particularly striking for curves, we prescribe the data z_{ij} in $\mathrm{I\!R}^2$, and solve the above linear systems with 2 right hand sides. In this case, the functions f are vector-valued maps $f : [0\,..1] \longrightarrow \mathrm{I\!R}^2$. In each of the sequences in Figure 4.12, the weights w_{i1} range from 50 to $31,250$ ($\approx \infty$), while w_{i2} is set to 20 and 400, respectively. Moreover, $z_{i2} = 0$ for all i, and so for large w_{i1}, the weights w_{i2} are close to the tension parameters of the corresponding ν-splines. The sharper corners (smaller tangent vectors) in (b) correspond to the larger tension parameters. To obtain the C^1 periodic curves, we let $\Lambda_1 : f \longmapsto (f(t_n) - f(t_1); f'(t_n) - f'(t_1))$, $z_{n1}{=}z_{n2}{=}(0,0)$, and $\Lambda_2 : f \longmapsto (f^{(j-1)}(t_i) : i{=}1{:}n{-}1,\ j = 1, 2)$.



(a) $w_{i2} = 20$ (b) $w_{i2} = 400$

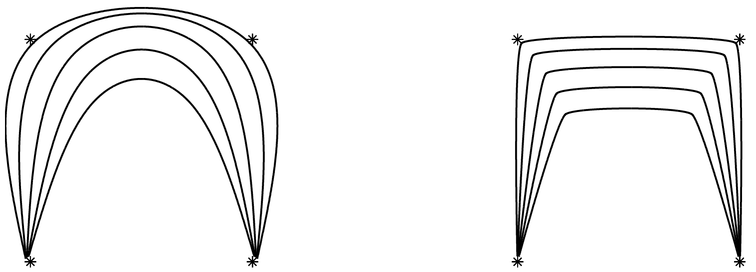
Figure 4.12. Convergence of smoothing splines to a ν-spline.

Example 4.13 (limits of near-interpolants and the ν-spline). *$Z_2 = Z$ and $\varepsilon_{ij} \longrightarrow 0$ for some ij.*

As stated in (1.3), near-interpolants solve the problem

$$\text{minimize } \Big\{ \int_0^1 |f^{(m)}(s)|^2 \,\mathrm{d}\, s : |f^{(j-1)}(t_i) - z_{ij}| \le \varepsilon_{ij},\ i{=}1{:}n,\ j{=}1{:}m \Big\}.$$

As shown in [K99b], solutions to this problem solve the problem of smoothing for weights w_{ij} that correspond to the Lagrange multipliers associated to the constraints, hence these weights depend on ε_{ij}. In particular, $|f^{(j-1)}(t_i) - z_{ij}| \longrightarrow 0$ when $\varepsilon_{ij} \longrightarrow 0$, in which case either $w_{ij} \longrightarrow \infty$ or $|f^{(j-1)}(t_i) - z_{ij}| = 0$ for small ε_{ij}. Hence, when $\varepsilon_{ij} \longrightarrow 0$ for $j = 1$ and $z_{ij} = 0$ for $j > 1$, the corresponding near-interpolants converge to a ν-spline. This is illustrated in Figure 4.14. Here, the data points $z_{i1} \in \mathbb{R}^2$ are at the corners of the unit cube, and the prescribed tangents z_{i2} are set to zero. In both of the curve sequences in (a) and (b), the tolerances ε_{i1} range from 0.4 to 0.025 at the upper corners, and they are fixed at 0.025 at the end points. Since ε_{i2} is larger in (a) than in (b), the weights w_{i2} of the corresponding smoothing splines are larger in (b) than in (a). These weights are close to the tension parameters of a ν-spline when ε_{i1} are small.



(a) $\varepsilon_{i2} = 5$ (b) $\varepsilon_{i2} = 0.5$

Figure 4.14. Convergence of near-interpolants to a ν-spline.

Example 4.15 (Splines in tension (with smoothing)).

Schweikert's spline in tension ([Se66]) is the solution to (1.1) when $T : f \longmapsto D^2 f + \alpha D f$ and $\Lambda_1 : f \longmapsto f(t_i)$, and with no smoothing term, as was shown by Nielson in [N74]. Here, we include the smoothing term in (1.1) to obtain

$$\text{(4.16)} \qquad \underset{\substack{f^{(j-1)}(t_i)=z_{ij}\\ (i,j)\in\iota}}{\text{minimize}} \int_0^1 |f''(s)+\alpha\, f'(s)|^2 \,\mathrm{d}\, s \;+\; \sum_{(i,j)\notin\iota} w_{ij}\, |f^{(j-1)}(t_i)-z_{ij}|^2.$$

(In [P78], Pruess studied splines in tension with smoothing, but for functionals of the form $f \longmapsto f(t_i)$ only). The maps Λ_1 and Λ_2 are as defined in (4.2), and ι is a subset of $n \times 2$. The inner product on X is taken as in (4.1) with $m = 2$ and with

$$\ker T = \{a + b\, e^{-\alpha(\cdot)} \,:\, a,\, b \in \mathrm{I\!R}\},$$

the space of those $f \in X$ that satisfy the homogeneous differential equation $f'' + \alpha f' = 0$. Solutions to (4.16) have the form

$$\text{(4.17)} \qquad a_i + b_i(\cdot - t_i) + c_i e^{\alpha(\cdot - t_i)} + d_i e^{-\alpha(\cdot - t_i)}$$

on each interval $(t_i \,..\, t_{i+1})$. In particular, $\ker T \subset \operatorname{ran}\Lambda^*$, and so it follows by Corollary 3.4 that solutions σ are in $\operatorname{ran}\Lambda^*$. The dual basis V described in Proposition 3.8 is the "piecewise exponential Hermite basis" (see [NF84], equations (2.2) and (2.3)). Here, with the Hermite-type functionals, the curves are generally C^1, rather than C^2. In particular, the characterization given in Proposition 3.8 applies. Finally,

$$\Psi\sigma = (\Lambda\Lambda^*)^{-1}\Lambda T^* T\sigma = (\mathrm{jmp}_{t_i}(\sigma''' - \alpha^2\sigma'),\ -\mathrm{jmp}_{t_i}(\sigma'' + \alpha\sigma') : i{=}1{:}n),$$

as can be derived by integration by parts (as in [N74]), and $\|T\sigma\|_Y^2 = \langle \Psi\sigma, \Lambda\sigma\rangle_{Z_2}$.

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