J. Math. Mech. **15** (1966), pp. 953–970

On splines and their minimum properties

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Communicated by G. Birkhoff

0. Introduction. It is the purpose of this note to show that the several minimum properties of odd degree polynomial spline functions [4, 18] all derive from the fact that spline functions are representers of appropriate bounded linear functionals in an appropriate Hilbert space. (These results were first announced in *Notices, Amer. Math. Soc.*, **11** (1964) 681.) In particular, spline interpolation is a process of best approximation, i.e., of orthogonal projection, in this Hilbert space. This observation leads to a generalization of the notion of spline function. The fact that such generalized spline functions retain all the minimum properties of the polynomial splines, follows from familiar facts about orthogonal projections in Hilbert space.

1. Polynomial splines and their minimum properties. A *polynomial spline function*, $s(x)$, of degree $m \geq 0$, having the $n \geq 1$ *joints* $x_1 < x_2 < \cdots < x_n$, is by definition a real valued function of class $C^{(m-1)}(-\infty, \infty)$, which reduces to a polynomial of degree at most m in each of the $n+1$ intervals $(-\infty, x_1)$, (x_1, x_2), …, $(x_n, +\infty)$. The most general such function is given by

$$s(x) = \sum_{i=0}^{m} \alpha_i x^i + \sum_{j=1}^{n} \beta_j (x - x_j)_+^m \quad ,$$

where α_i, $i = 0, \ldots, m$, and β_j, $j = 1, \ldots, n$, are real numbers and

$$(x)_+^m = \begin{cases} x^m, & x \geq 0, \\ 0\ , & x < 0. \end{cases}$$

Specifically, let $m = 2k - 1$, and $n \geq k \geq 1$, and let S_0 denote the family of polynomial spline functions of odd degree m with joints $x_1, \ldots, x_n$, which reduce to polynomials of degree at most $k - 1$ in each of the two intervals $(-\infty, x_1)$ and (x_n, ∞). Equivalently, S_0 consists of all polynomial spline functions $s(x)$ of degree m with joints $x_1, \ldots, x_n$ which satisfy

$$\begin{aligned} &s^{(j)}(x_1) = s^{(j)}(x_n) = 0, \quad j = k, \ldots, 2k-2, \\ &s^{(2k-1)}(x) \equiv 0, \quad \text{all} \quad x \notin [x_1, x_n]. \end{aligned} \tag{1.1}$$

Hence, for $n = k$, S_0 consists just of the set $\{\pi_{k-1}\}$ of polynomials of degree at most $k-1$. Let $[a, b]$ be a finite interval containing all the joints $x_1, \ldots, x_n$ and consider S_0 as a subset of the class of functions [19]

$$F^{(k)}[a,b] = \Big\{ f(x) \mid f \in C^{(k-1)}[a,b], \quad f^{(k-1)} \text{absolutely continuous}, \quad f^{(k)} \in L^2[a,b] \Big\}. \tag{1.2}$$

The elements of S_0 have the following properties [4], [18]:

Interpolation property: Given $f \in F^{(k)}[a, b]$, there exists a unique element $s(x) \in S_0$ satisfying

$$s(x_i) = f(x_i), \quad i = 1, \ldots, n.$$

Denote this unique element by Pf.

[1] The work of R. E. Lynch was supported in part by the National Science Foundation through Grant GP–217 and by the Army Research Office(Durham) through Grant DA–ARO(D)–31–124–G388, at The University of Texas.

First minimum property: If $f \in F^{(k)}[a,b]$ and $s \in S_0$, then

$$\int_a^b \left[f^{(k)}(x) - s^{(k)}(x)\right]^2 dx \geq \int_a^b \left[f^{(k)}(x) - (Pf)^{(k)}(x)\right]^2 dx,$$

with equality if and only if $s = Pf + \pi_{k-1}$.

Second minimum property: If $f \in F^{(k)}[a,b]$, then

$$\int_a^b \left[f^{(k)}(x)\right]^2 dx \geq \int_a^b \left[(Pf)^{(k)}(x)\right]^2 dx,$$

with equality if and only if $f = Pf$.

A third minimum property concerns the linear approximation of a linear functional L on $F^{(k)}[a,b]$ of the form

$$Lf = \sum_{i=0}^{k-1} \int_a^b f^{(i)}(y) d\mu_i(y),$$

where the $\mu_i(x)$ are functions of bounded variation. For later reference, we denote by $\mathcal{L}^{(k)}$ the set of all such linear functionals.

Third minimum property: For all $L \in \mathcal{L}^{(k)}$, and all $f \in F^{(k)}[a,b]$, the best approximation, L^*f, in the sense of Sard [15] (see below in Section 2) to Lf by an expression of the form $\sum_{i=1}^n \alpha_i f(x_i)$ is given by operating with L on Pf; for short, $L^* = LP$. This approximation is exact for all $f \in S_0$.

2. Representers in Hilbert space and their minimum properties. In order to relate these minimum properties of the elements of S_0 to "familiar facts about orthogonal projections in Hilbert space," we need the following facts:

Theorem 2.1. *The linear space $F^{(k)}[a,b]$ is a Hilbert space with respect to the inner product*

$$(f,g) = \sum_{i=1}^k f(x_i)g(x_i) + \int_a^b f^{(k)}(y)g^{(k)}(y)dy, \quad \text{all} \quad f, \quad g \in F^{(k)}[a,b]. \tag{2.1}$$

This Hilbert space possesses a reproducing kernel [1], [3], *$K(x,y)$, which is*

$$\begin{aligned} K(x,y) = \sum_{i=1}^k c_i(x)c_i(y) + (-1)^k \Big\{ &(x-y)_+^{2k-1} \\ &+ \sum_{i=1}^k \sum_{j=1}^k (x_i - x_j)_+^{2k-1} c_i(x)c_j(y) \\ &- \sum_{i=1}^k \left[(x-x_i)_+^{2k-1} c_i(y) + (x_i - y)_+^{2k-1} c_i(x)\right] \Big\} \Big/ (2k-1)!, \end{aligned} \tag{2.2}$$

where

$$c_i(x) = \prod_{j=1, j\neq i}^k (x - x_j)/(x_i - x_j), \quad i = 1, \ldots, k.$$

For all $L \in \mathcal{L}^{(k)}$, for all $f \in F^{(k)}[a,b]$,

$$\begin{aligned} Lf =& \sum_{i=1}^k L(c_i)f(x_i) \\ &+ \frac{1}{(k-1)!}\int_a^b L_{(x)}\Big((x-y)_+^{k-1} - \sum_{i=1}^k c_i(x)(x_i - y)_+^{k-1}\Big) f^{(k)}(y)dy. \end{aligned} \tag{2.3}$$

Here and below, the subscript (x) [or (y)] indicates [5] that an operation is to be performed on a function of x [or y] for fixed y [or fixed x].

This theorem is a special case of Theorem 3.1 and Lemma 3.1 proved below in Section 3.

Corollary. *The linear functionals L_i, $i = 1, \ldots, n$, given by*

$$L_i f = f(x_i), \quad i = 1, \ldots, n, \quad \text{all} \quad f \in F^{(k)}[a, b], \tag{2.4}$$

are bounded; their representers [3] span S_0.

Proof. A linear functional L on a real Hilbert space H with reproducing kernel $K(x, y)$ is bounded only if the function $\phi(x) = L_{(y)}K(x, y)$ is in H, and in that case, $\phi(x)$ is its representer, i.e.,

$$Lf = (f, \phi), \quad \text{all} \quad f \in H.$$

Since the linear functionals L_i, $i = 1, \ldots, n$, are linearly independent over $F^{(k)}[a, b]$, the corollary follows from the fact that, by (2.2), $K(x, x_i) \in S_0$, $i = 1, \ldots, n$.

Let now, more generally, $L_1, \ldots, L_n$ be n linearly independent bounded linear functionals over a real Hilbert space H, and let $S = \langle \phi_1, \ldots, \phi_n \rangle$ be the n–dimensional subspace of H spanned by the representers $\phi_1, \ldots, \phi_n$ of the L_i. Given $f \in H$, an element $g \in H$ is said to *interpolate f with respect to $L_1, \ldots, L_n$* if

$$L_i g = L_i f, \quad i = 1, \ldots, n.$$

Let P_s denote the orthogonal projection from H onto S, i.e., for $f \in H$, $P_s f$ is the unique best approximation to f by an element in S with respect to the norm in H. Then P_s satisfies [7]

$$(P_s f, h) = (P_s f, P_s h) = (f, P_s h), \quad \text{all} \quad f, \quad h \in H, \tag{2.5}$$

and

$$\|f\|^2 = \|f - P_s f\|^2 + \|P_s f\|^2, \quad \text{all} \quad f \in H. \tag{2.6}$$

By setting $h = \phi_i$ in(2.5), it follows that

$$L_i(P_s f) = L_i(f), \quad \text{all} \quad f \in H, \quad i = 1, \ldots, n. \tag{2.7}$$

This proves

Lemma 2.1. *Given $f \in H$, $P_s f$ is the unique element in S which interpolates f with respect to L_1, $\ldots$, L_n.*

For illustration, if H is $F^{(k)}[a, b]$ with inner product (2.1), and the L_i are given by (2.4), Lemma 2.1 states that the spline function $Pf \in S_0$, which, by definition, interpolates f at the points $x_1, \ldots, x_n$, is also the best approximation to f with respect to the norm

$$\|g\|^2 = \sum_{i=1}^{k} (g(x_i))^2 + \int_a^b [g^{(k)}(y)]^2 dy,$$

thus implying the first minimum property in Section 1.

An element $f \in H$ is in S if and only if $f = P_s f$. This may be put somewhat differently. Given $f \in H$, let $W_f = \{h \mid h \in H, \quad L_i(h) = L_i(f) \quad \text{for} \quad i = 1, \ldots, n\}$. Then, it follows from Lemma 2.1 that

$$P_s h = P_s f, \quad \text{all} \quad h \in W_f. \tag{2.8}$$

Hence one has from (2.6) that

$$\|P_s f\| \le \|h\|, \quad \text{all} \quad h \in W_f, \quad \text{with equality if and only if} \quad h = P_s f; \tag{2.9}$$

and we have proved the following

Lemma 2.2. *An element $f \in H$ is in S if and only if f is the element of minimal norm in the set W_f of all elements of H that interpolate f with respect to $L_1, \ldots, L_n$.*

Specifically, if H is $F^{(k)}[a,b]$ with inner product (2.1) and the L_i are given by (2.4), then for all $h \in W_f$,

$$\sum_{i=1}^{k} (h(x_i))^2 = \sum_{i=1}^{k} (f(x_i))^2,$$

and the second minimum property of polynomial splines follows.

Returning once more to the general situation, let L be a bounded linear functional on H with representer ϕ. We wish to approximate L by a linear functional of the form $\sum_{i=1}^{n} \alpha_i L_i$ in such a way that the norm

$$\|R\| = \sup_{\|f\| \leq 1} |Rf| \tag{2.10}$$

of the error functional $R = L - \sum_{i=1}^{n} \alpha_i L_i$ be minimized. Since $\|R\| = \|\phi - \sum \alpha_i \phi_i\|$, we have

Lemma 2.3. *The element $\overline{\phi} = P_S \phi$ is the representer of the unique best approximation $\overline{L}$ to L by a bounded linear functional of the form $\sum_{i=1}^{n} \alpha_i L_i$ with respect to the norm* (2.10).

In particular, one has by (2.5) that

$$\overline{L}(f) = (f, P_S \phi) = (P_S f, \phi) = L(P_S f), \quad \text{all} \quad f \in H. \tag{2.11}$$

Corollary 1. *Given $f \in H$, the value of the best approximation to L at f is equal to the value of L at the best approximation to f.*

It is this property which makes $\overline{L}$ so attractive for use in computational work: any computational scheme which solves the interpolation problem also solves the problem of computing the best approximation to *any* bounded linear functional.

For $f \in S$, $P_S f = f$; therefore, by (2.11), $\overline{L}(f) = L(f)$, for $f \in S$.

Corollary 2. *The approximation $\overline{L}$ to L is exact for $f \in S$.*

Once again, let H be, in particular, $F^{(k)}[a,b]$ with inner product (2.1), and let the L_i be given by (2.4). Then Corollaries 1 and 2 of Lemma 2.3 imply the third minimum property of polynomial splines. To see this, we need to recall the definition of L^*, the best approximation to L in the sense of Sard [15], [18] by a linear combination of the L_i.

Since $n \geq k$, it is possible to choose numbers $\alpha_1, \ldots, \alpha_n$ so that $Rf = 0$ whenever $f \in \{\pi_{k-1}\}$. The set

$$\mathcal{M} = \left\{ L' = \sum_{i=1}^{k} \alpha_i L_i \mid Lf = L'f \quad \text{for} \quad f \in \{\pi_{k-1}\} \right\}$$

is therefore not empty. By Peano's theorem [3], [11], or by (2.3) in Theorem 2.1, we can write Rf as

$$Rf = (L - L')f = \int_a^b K(t) f^{(k)}(t) dt, \quad \text{all} \quad L' \in \mathcal{M},$$

where

$$K(t) = R_{(x)} (x - t)_+^{k-1} / (k-1)!,$$

provided that $L \in \mathcal{L}^{(k)}$. With this, L^* is defined as the unique element in $\mathcal{M}$ which minimizes

$$\int_a^b \big(K(t)\big)^2 dt.$$

But this is just the demand that L^* minimize $\|L - L'\|$ over all $L' \in \mathcal{M}$. It is now immediate that $\overline{L}$ and L^* agree. First, the assumption that $L \in \mathcal{L}^{(k)}$ is sufficient (though not necessary) to insure that $L_{(y)}K(x,y) \in F^{(k)}[a,b]$, so that $\overline{L}$ is defined. Also, by Corollary 2 of Lemma 2.3, $\overline{L} \in \mathcal{M}$. Hence, since $\overline{L}$ minimizes $\|L - L'\|$ over all $L' = \sum \alpha_i L_i$, we have $\overline{L} = L^*$, and the third minimum property follows.

Finally, we mention some estimates for the error $\overline{R}f = Lf - \overline{L}f$. Since

$$Lf - \overline{L}f = (\phi - \overline{\phi}, f) = \big(\phi, (I - P_S)f\big),$$

use of Schwarz's inequality gives

$$|Lf - \overline{L}f| \le \|\phi - \overline{\phi}\| \, \|f\|, \quad \text{and} \quad |Lf - \overline{L}f| \le \|\phi\| \, \|f - P_S f\|. \tag{2.12}$$

But since, by (2.5), $((I - P_S)\phi, f) = ((I - P_S)\phi, (I - P_S)f)$, we have the better estimate

$$|Lf - \overline{L}f| \le \|\phi - \overline{\phi}\| \, \|f - P_S f\|. \tag{2.13}$$

Hence if $\|f\| \le r$, which implies $\|f - P_S f\| \le (r^2 - \|P_S f\|^2)^{1/2}$, then

$$\overline{L}f - \|\phi - \overline{\phi}\| \, (r^2 - \|P_S f\|^2)^{1/2} \le Lf \le \overline{L}f + \|\phi - \overline{\phi}\| \, (r^2 - \|P_S f\|^2)^{1/2}. \tag{2.14}$$

The importance of the fact that this estimate depends only on the bound r and the numbers $L_i(f)$, $i = 1, \ldots, n$, and is optimal with respect to this information, is rightfully stressed in [5].

3. The Hilbert space $F^{(k)}[a,b]$. The linear space $F^{(k)}[a,b]$ can be made into a Hilbert space in various ways, thus providing various classes of functions which, due to the fact that they are representers of suitable linear functionals, have all the minimum properties of polynomial splines.

Specifically, let M be a k–th order ordinary linear differential operator in normal form,

$$M = (d^k/dx^k) + \sum_{i=0}^{k-1} a_i(x)(d^i/dx^i), \tag{3.1}$$

and let $L_1, \ldots, L_k$ be k linear functionals. Under suitable conditions on the $a_i(x)$ and the L_i,

$$(\mathrm{e}, f) = \sum_{i=1}^{k} L_i(\mathrm{e})L_i(f) + \int_a^b (M\mathrm{e})(y)(Mf)(y)dy, \quad \text{all } \mathrm{e}, f \in F^{(k)}[a,b], \tag{3.2}$$

is an inner product defined on $F^{(k)}[a,b]$, which makes $F^{(k)}[a,b]$ into a Hilbert space with reproducing kernel.

This is proved in the following theorem, which provides facts necessary to define and describe generalized splines and their minimum properties.

Theorem 3.1. *Let M be any k–th order ordinary linear differential operator in normal form, (3.1), where $k \ge 1$ and and $a_i \in C[a,b]$, $i = 0, \ldots, k-1$. Let $\mathcal{N}(M)$ denote the k–dimensional linear subspace of all functions f in $C^{(k)}[a,b]$ for which $Mf = 0$. Let $L_1, \ldots, L_k$ be any set of k linear functionals in $\mathcal{L}^{(k)}$, which is linearly independent over $\mathcal{N}(M)$. Then $F^{(k)}[a,b]$ is a Hilbert space with respect to the inner product (3.2), and has a reproducing kernel. This reproducing kernel, K, is given by*

$$K(x,y) = \sum_{i=1}^{k} c_i(x)c_i(y) + \int_a^b G(x,t)G(y,t)dt, \quad x, y \in [a,b], \tag{3.3}$$

where $c_i, \ldots, c_k$ is the dual basis to $L_1, \ldots, L_k$ in $\mathcal{N}(M)$, and $G(x,y)$ is the Green's function for the differential equation $(Mf)(x) = e(x)$ with $L_i(f) = 0$, $i = 1, \ldots, k$.

Proof. The assumptions on the coefficients of the differential operator M are sufficient to insure [2, p. 117] that $\mathcal{N}(M)$ is indeed of dimension k. Moreover, for each $x \in [a,b]$ and each $f \in F^{(k)}[a,b]$ there exists a unique element $s \in \mathcal{N}(M)$ such that

$$s^{(j)}(x) = f^{(j)}(x), \quad j = 0, \ldots, k-1.$$

Denote this element by $Q^{(x)}f$ and define a function h on $[a,b] \times [a,b]$ by

$$h(x,y) = Q^{(y)}{}_{(x)}(x-y)^{k-1}/(k-1)!. \tag{3.4}$$

Then

$$M_{(x)}h(x,y) = 0, \quad \text{all } x, y \in [a,b], \tag{3.5}$$

$$\partial^j h(x,y)/\partial x^j|_{x=y} = \partial_{j,k-1}, \quad j = 0, \ldots, k-1, \tag{3.5$'$}$$

and [2, p. 89] for all $f \in C^{(k)}[a,b]$,

$$f(x) = (Q^{(a)}f)(x) + \int_a^x h(x,y)(Mf)(y)dy, \quad \text{all } x \in [a,b]. \tag{3.6}$$

Thus, the function

$$g(x,y) = \begin{cases} h(x,y), & x \geq y, \\ 0 \quad , & x < y, \end{cases} \tag{3.7}$$

is just the Green's function for the initial value problem

$$Mf = \mathrm{e}, \quad f^{(j)}(a) = 0, \quad j = 0, \ldots, k-1.$$

For the proof of Theorem 3.1, we need to know that (3.6) holds not only for $f \in C^{(k)}[a,b]$, but also for all $f \in F^{(k)}[a,b]$.

Lemma 3.1. *Under the hypotheses of Theorem 3.1, the identy*

$$f(x) = (Q^{(a)}f)(x) + \int_a^b g(x,y)(Mf)(y)dy \tag{3.8}$$

is valid for all $f \in F^{(k)}[a,b]$ and all $x \in [a,b]$. More generally, for any linear functional $L \in \mathcal{L}^{(k)}$,

$$Lf = L(Q^{(a)}f) + \int_a^b L_{(x)}g(x,y)(Mf)(y)dy, \quad \text{all } f \in F^{(k)}[a,b]. \tag{3.9}$$

We were unable to find a reference for this lemma. We defer its proof to the Appendix and continue with the proof of Theorem 3.1.

By hypothesis, the linear functionals L_i, $i = 1, \ldots, k$, form a maximal linearly independent set over $\mathcal{N}(M)$. There exist, therefore, k functions, $c_i \in \mathcal{N}(M)$, $i = 1, \ldots, k$, such that

$$L_i c_j = \delta_{ij}, \quad i, j = 1, \ldots, k.$$

With these, define a projection operator P from $F^{(k)}[a,b]$ onto $\mathcal{N}(M)$ by

$$(Pf)(x) = \sum_{i=1}^{k} (L_i f) c_i(x), \quad \text{all } f \in F^{(k)}[a,b]. \tag{3.10}$$

Then, in particular, $Pf = f$ for $f \in \mathcal{N}(M)$, and, by Lemma 3.1,

$$\begin{aligned}(Pf)(x) &= \big(P(Q^{(a)}f)\big)(x) + P\int_a^b g(x,y)(Mf)(y)dy \\ &= (Q^{(a)}f)(x) + \int_a^b P_{(x)}g(x,y)(Mf)(y)dy, \quad \text{all } f \in F^{(k)}[a,b],\end{aligned}$$

where, by Lemma 3.1, the interchange of integration and the operator P is justified by the assumption that the L_i are in $\mathcal{L}^{(k)}$. Therefore, with the definition

$$G(x,y) = (I-P)_{(x)}g(x,y), \quad x,y \in [a,b], \tag{3.11}$$

it follows that

$$f(x) = (Pf)(x) + \int_a^b G(x,y)(Mf)(y)dy, \quad x \in [a,b], \quad \text{all } f \in F^{(k)}[a,b]. \tag{3.12}$$

Hence, if for some $\mathrm{e} \in L^2[a,b]$ and some $f_0 \in \mathcal{N}(M)$,

$$f(x) = f_0(x) + \int_a^b G(x,y)\mathrm{e}(y)dy, \tag{3.13}$$

then

$$f \in F^{(k)}[a,b], \quad L_i f = L_i f_0, \quad i = 1,\ldots,k, \quad \text{and} \quad Mf = \mathrm{e} \quad \text{a.e.},$$

which also shows that G is the Green's function for $Mf = \mathrm{e}$, $L_i f = 0$, $i = 1, \ldots, k$. With these facts established, all assertions of the theorem follow.

First, one checks that (3.2) indeed defines an inner product, i.e., a symmetric positive definite bilinear form on $F^{(k)}[a,b]$.

Secondly, $F^{(k)}[a,b]$ is complete with respect to the norm $\|f\| = (f,f)^{1/2}$. For if $\{f_j\}_1^\infty$ is a Cauchy sequence in $F^{(k)}[a,b]$ with respect to this norm, then $\{L_i f_j\}_{j=1}^\infty$ is a Cauchy sequence of real numbers, so $\gamma_i = \lim_{j\to\infty}(L_i f_j)$ exists, $i = 1, \ldots, k$; furthermore, $\{Mf_j\}_{j=1}^\infty$ is a Cauchy sequence in $L^2[a,b]$ which, therefore, converges in L^2 to some $\mathrm{e} \in L^2[a,b]$. But then,

$$f(x) = \sum_{i=1}^k \gamma_i c_i(x) + \int_a^b G(x,y)\mathrm{e}(y)dy$$

is the limit point of the sequence $\{f_j\}_1^\infty$ in $F^{(k)}[a,b]$.

Finally, let K be the function on $[a,b]\times[a,b]$ defined by (3.3). Then, by the remarks following (3.12), $K(x,y)$, as a function of x, is in $F^{(k)}[a,b]$ for each $y \in [a,b]$, and

$$\begin{aligned}&M_{(y)}K(x,y) = G(x,y), \quad x,y \in [a,b], \\ &L_{i(y)}K(x,y) = c_i(x), \quad i = 1,\ldots,k, \quad x \in [a,b].\end{aligned} \tag{3.14}$$

Therefore, using (3.12) once more, we get

$$\begin{aligned}f(x) &= \sum_{i=1}^k c_i(x)(L_i f) + \int_a^b G(x,y)(Mf)(y)dy \\ &= \sum_{i=1}^k L_{i(y)}K(x,y)L_i f + \int_a^b M_{(y)}K(x,y)(Mf)(y)dy \\ &= \big(K(x,y), f(y)\big)_{(y)}, \quad x \in [a,b], \quad \text{all } f \in F^{(k)}[a,b].\end{aligned}$$

But this shows that K is the reproducing kernel for $F^{(k)}[a,b]$, which concludes the proof.

Remark. Lemma 3.1 has as a consequence many classical integral representations for the error of formulas for interpolation, quadrature, and numerical differentiation, e.g. those of Peano [11], [12], Radon [13], Remes [14], Milne [10], Golomb and Weinberger [5], Weinberger [23], and others. Explicitly, if $n+1$ linear functionals L_0, …, L_n from $\mathcal{L}^{(k)}$ are given, and if the error functional, $R = L_0 - \sum_{i=1}^{n} \alpha_i L_i$, vanishes on the null space $\mathcal{N}(M)$ of the ordinary k–th order linear differential operator M, (3.1), then

$$Rf = \int_a^b R_{(x)} g(x,y)(Mf)(y)dy, \quad \text{all } f \in F^{(k)}[a,b], \tag{3.15}$$

where $g(x,y)$ is the Green's function associated with the initial value problem

$$(Mf)(x) = \mathrm{e}(x), \quad f^{(j)}(a) = 0, \quad j = 0, \ldots, k-1.$$

By Schwarz's inequality, this results in the error estimate

$$|Rf|^2 \le \int_a^b [R_{(x)} g(x,y)]^2 dy \int_a^b [(Mf)(y)]^2 dy, \tag{3.16}$$

which corresponds to (2.12) and supplies the motivation for Sard's proposal to choose the approximation $\sum_{i=1}^{n} \alpha_i L_i$ to L_0 in such a way that $\int_a^b [R_{(x)} g(x,y)]^2 dy$ be minimized.

4. Polynomial spline functions. As an illustration for the use of Theorem 3.1 and Lemmas 2.1–2.3 and their corollaries, we consider once more polynomial splines. Accordingly, we choose $M = (d^k/dx^k)$, so that $\mathcal{N}(M) = \{\pi_{k-1}\}$. In this case, $g(x,y)$ in (3.7) is

$$g(x,y) = (x-y)_+^{k-1}/(k-1)! \tag{4.1}$$

so that the reproducing kernel (see (3.3) and (3.14)) is indeed given by (2.2), provided $L_i f = f(x_i)$, $i = 1$, …, k. By the corollary to Theorem 2.1, the space S_0 is spanned by the representers of the linear functionals L_i, …, L_n, given by $L_i f = f(x_i)$, $i = 1$, …, n.

As noted above, because of the condition (1.1), the elements of S_0 restricted to the interval $[x_1, x_n]$ do not constitute all polynomial spline functions of degree $2k-1$ with joints x_2, …, x_{n-1} in that interval. But the subspace S_1 spanned by the representers ϕ_1, …, ϕ_m of L_1, …, L_m does have this property in case the L_i are given by

$$\begin{aligned} L_i f &= f^{(i-1)}(x_1), && i = 1, \ldots, k, \\ L_{k-1+i} f &= f(x_i), && i = 2, \ldots, n-1, \\ L_{m+1-i} f &= f^{(i-1)}(x_n), && i = 1, \ldots, k, \end{aligned}$$

where, as before, $a \le x_1 < \cdots < x_n \le b$, while only $n \ge 2$. So $m = 2k + n - 2$.

We omit the straightforward verification of this fact and note only that with this choice for the L_i, Lemma 2.1 implies the remark following Lemma 2 of [4] as well as Theorem 2 of [4]; Lemma 2.2 implies Lemma 2' of [4].

The first two minimum properties of polynomial splines were first pointed out explicitly* by Walsh, Ahlberg, and Nilson [21] in the special case $k = 2$ of cubic *periodic* splines. In the remainder of this section, we derive the three minimum properties for periodic polynomial splines of degree $2k-1$. The statement of the first two will be somewhat more general than as given in [22].

The linear functionals

$$L_1 f = f(a), \quad L_i f = f^{(i-2)}(b-) - f^{(i-2)}(a+), \quad i = 2, \ldots, k,$$

are linearly independent over $\{\pi_{k-1}\}$, and are in $\mathcal{L}^{(k)}$. Their dual basis c_1, …, c_n in $\{\pi_{n-1}\}$ can be computed

* All the minimum properties as listed in Section 1 are contained implicitly already in [5].

recursively by

$$c_1(x) \equiv 1, \quad c_i(x) = d_i(x) - d_i(a), \quad i = 2, \dots, k,$$

where $d_1(x) \equiv 1/(b-a)$,

$$d_{i+1}(x) = \int_a^x d_i(y)dy - \frac{1}{b-a}\int_a^b \int_a^x d_i(y)dydx, \quad i = 1, \dots, k-1.$$

$F^{(k)}[a,b]$ is, therefore, a Hilbert space with respect to the inner product (3.2) (with $M = d^k/dx^k$), and has a reproducing kernel, $K(x,y)$, given by

$$\begin{aligned} K(x,y) =& \sum_{i=1}^{k} c_i(x)c_i(y) + \gamma(x,y), \\ \gamma(x,y) =& (-1)^k \Bigg\{ (x-y)_+^{2k-1}/(2k-1)! \\ & - \sum_{i=2}^{k} \Big[(b-y)^{2k-i+1}c_i(x) - (x-a)^{2k-i+1}c_i(y)\Big]/(2k-i+1)! \\ & - \sum_{i=2}^{k}\sum_{j=2}^{k}(b-a)^{2k-i-j+3}c_i(x)c_j(y)/(2k-i-j+3)! \Bigg\}. \end{aligned} \tag{4.2}$$

Let $L_{k+1}f = f^{(k-1)}(b-) - f^{(k-1)}(a+)$, and set $c_{k+1}(x) = L_{k+1(y)}K(x,y)$. The closed subspace

$$F_p^{(k)}[a,b] = \big\{ f \in F^{(k)}[a,b] \mid L_i f = 0, \quad i = 2, \dots, k+1 \big\}$$

of $F^{(k)}[a,b]$ is also a Hilbert space with respect to (3.2), and one checks that $K_p(x,y)$, given by

$$\begin{aligned} K_p(x,y) &= K(x,y) - \sum_{i=2}^{k+1} c_i(x)c_i(y) \\ &= 1 - c_{k+1}(x)c_{k+1}(y) + \gamma(x,y), \end{aligned} \tag{4.3}$$

is its reproducing kernel. On $F_p^{(k)}[a,b]$, (3.2) simplifies to

$$(\mathrm{e}, f) = \mathrm{e}(a)f(a) + \int_a^b \mathrm{e}^{(k)}(y)f^{(k)}(y)dy, \quad \text{all } \mathrm{e}, f \in F_p^{(k)}[a,b]. \tag{4.4}$$

Let $a = x_1 < x_2 < \cdots < x_n = b$, $n \geq 2$. Then, the linear functionals N_1, …, N_{n-1}, given by $N_i f = f(x_i)$, $i = 1$, …, $n-1$, are linearly independent and bounded over $F_p^{(k)}[a,b]$. Their representers, $K_p(x, x_i)$, $i = 1$, …, $n-1$, span therefore an $(n-1)$–dimensional subspace, S_2, of $F_p^{(k)}[a,b]$. By (4.2) and (4.3), S_2 consists of piecewise polynomial functions in $C^{(2k-2)}[a,b]$ of degree $2n-1$ with joints at x_1, …, x_n. Also, along with any other element of $F_p^{(k)}[a,b]$, any $s \in S_2$ satisfies

$$s^{(j)}(a+) = s^{(j)}(b-), \quad j = 0, \dots, k-1.$$

But, in fact, if $d \in [a,b]$ and $\phi_d(x) = K_p(x,d)$, then also

$$\phi_d^{(j)}(a+) = \phi_d^{(j)}(b-), \quad j = k, \dots, 2k-2, \tag{4.5}$$

which implies that S_2 is the set of periodic polynomial splines with period $(b-a)$ of degree $2k-1$ with joints x_1, …, x_{n-1} (considered as functions defined on $[a,b]$ only).

To verify (4.5), we observe that, in the notation of Theorem 3.1 (see (3.10), (3.11)),

$$
\begin{aligned}
P_{(y)}G(x,y) &= P_{(y)}(I-P)_{(x)}g(x,y) = (I-P)_{(x)}P_{(y)}g(x,y) \\
&= \sum_{i=1}^{k}(I-P)_{(x)}\big(L_{i(y)}g(x,y)\big)c_i(y).
\end{aligned}
\tag{4.6}
$$

Since in this particular case, $g(x,y) = (x-y)_+^{k-1}/(k-1)!$, we have

$$
L_{1(y)}g(x,y) \equiv 0, \quad L_{i(y)}g(x,y) = -(x-a)^{k-i+1}/(k-i+1)!, \quad i = 2,\dots,k,
$$

hence

$$
L_{i(y)}g(x,y) \in \{\pi_{k-1}\}, \quad i = 1,\dots,k,
$$

so that the coefficient of $c_i(y)$ in (4.6) vanishes indentically for each $i = 1, \dots, k$. Hence $P_{(y)}G(x,y) \equiv 0$ on $[a,b]\times[a,b]$. But as $(\partial^k/\partial x^k)K(x,y) \equiv G(x,y)$, this implies that $\psi_d^{(j)}(a+) = \psi_d^{(j)}(b-)$, $\quad j = k, \dots, 2k-2$, where $\psi_d(x) = K(x,d)$. Hence, as $\phi_d^{(k)}(x) \equiv \psi_d^{(k)}(x)$ + constant, while

$$
\phi_d^{(j)}(x) \equiv \psi_d^{(j)}(x), \quad j = k+1,\dots,2k-2,
$$

equation (4.5) follows.

We proved

Theorem 4.1. *Let S_2 be the set of polynomial spline functions of degree $2k-1$ with joints $x_1, \dots, x_{n-1}$, where $a = x_1 < \cdots < x_n = b$, which are periodic with period $(b-a)$. For $i = 1, \dots, n-1$, let $\phi_i(x)$ be the representer in $F_p^{(k)}[a,b]$ with respect to (4.4) of the linear functional N_i, given by $N_i f = f(x_i)$. Then S_2, considered as a subset of $F_p^{(k)}[a,b]$, is spanned by $\phi_1, \dots, \phi_{n-1}$.*

Corollary. *For all $f \in F_p^{(k)}[a,b]$,* (i) *there exists a unique element $s_f \in S_2$ satisfying $s_f(x_i) = f(x_i)$, $i = 1, \dots, n-1$;* (ii) *for all $s \in S_2$,*

$$
\int_a^b \left[f^{(k)}(x) - s^{(k)}(x)\right]^2 dx \ge \int_a^b \left[f^{(k)}(x) - s_f^{(k)}(x)\right]^2 dx
$$

with equality if and only if $s(x) - s_f(x) \equiv$ constant; (iii)

$$
\int_a^b [f^{(k)}(x)]^2 dx \ge \int_a^b [s_f^{(k)}(x)]^2 dx
$$

with equality if and only if $f = s_f$; (iv) *if L is any linear functional in $\mathcal{L}^{(k)}$, and $\overline{L}$ is that element of the set*

$$
\Big\{N = \sum_{i=1}^{n-1}\alpha_i N_i \mid (L-N)(1) = 0, \quad \alpha_i \text{ real}, \quad i = 1,\dots,n-1\Big\}
$$

which minimizes

$$
\int_a^b \left[(L-N)_{(y)}(x-y)_+^{k-1}/(k-1)!\right]^2 dx,
$$

then $\overline{L}f = Ls_f$.

5. General spline functions. We have proved Theorem 3.1 in all its generality since, practical considerations aside, there seems to be nothing inherently special about the use of the seminorm

$$
\int_a^b [f^{(k)}(x)]^2 dx \tag{5.1}
$$

as compared with the seminorm

$$\int_a^b [Mf(x)]^2 dx, \tag{5.2}$$

where M is the k–th order ordinary differential operator (3.1). Moreover, there is no mathematical reason to single out the point functionals

$$L_i(f) = f(x_i), \quad i = 1, \ldots, n, \tag{5.3}$$

from the more general set of bounded linear functionals over the Hilbert space $F^{(k)}[a, b]$. It would, therefore, seem acceptable to define a general set of spline functions S on $[a, b]$ belonging to the differential operator M and the set $L_1, \ldots, L_n$ of linearly independent bounded linear functionals over $F^{(k)}[a, b]$ as the n–dimensional subspace S spanned by functions $\phi_1, \ldots, \phi_n$ which are the representers of the linear functionals $L_1, \ldots, L_n$ with respect to the inner product (3.2). (M. Atteia in a note to appear in Comptes Rendus Acad. Sci. Paris has gone even farther and given a definition of "spline function" entirely in the setting of abstract Hilbert space.) According to Section 2, these splines retain all the (appropriately worded) minimum properties of polynomial splines.

But in order not to dilute the notion of spline function too much, we prefer to follow Greville's definition of general spline function [6].

Definition. Let S be an m–dimensional subspace of $C^{(m)}[a, b]$, where $[a, b]$ is a finite interval and let $a \leq x_1 < \cdots < x_n \leq b$. A real valued function f on $[a, b]$ is a *spline function* with respect to S of order m with joints $x_1, \ldots, x_n$, provided that $f \in C^{(m-2)}[a, b]$ and f coincides in each interval (a, x_1), (x_1, x_2), $\ldots$, (x_n, b) with some element in S.

If the coefficients a_i of the differential operator M in (3.1) are such that $a_i \in C^{(i)}[a, b]$, $i = 0, \ldots, k-1$, (i.e., if the coefficients are smoother than required in Theorem 3.1), then M possesses an adjoint differential operator M^* given by

$$M^* = (-1)^k (d^k/dx^k) + \sum_{i=0}^{k-1} (-1)^i (d^i a_i(x)/dx^i), \tag{5.4}$$

which, after carrying out the differentiation and rearranging the terms, can be written as

$$M^* = (-1)^k \Big[(d^k/dx^k) + \sum_{i=0}^{k-1} b_i(x)(d^i/dx^i) \Big], \tag{5.4$'$}$$

for appropriate $b_i \in C^{(i)}[a, b]$. As we now show, in this case the representers of the point functionals (5.3) in the Hilbert space $F^{(k)}[a, b]$ with inner product (3.2) are spline functions of order $2k$ with respect to the null space $\mathcal{N}(M^*M)$ of M^*M.

Let $u_1, \ldots, u_k$ be a basis in $\mathcal{N}(M)$. The function $h(x, y)$ which satisfies (3.5)–(3.5') can be written as

$$h(x, y) = \sum_{i=1}^{k} v_i(y) u_i(x),$$

where the coefficients v_i are the solution of the system

$$\sum_{i=1}^{k} v_i(y) d^j u_i/dx^j = \delta_{j,k-1} \quad \text{for } x = y \quad \text{and } j = 0, \ldots, k-1.$$

Consequently [8, p. 78], the functions v_i form a basis for $\mathcal{N}(M^*)$. Hence as a function of y, $h(x, y) \in \mathcal{N}(M^*)$. It then follows that as a function of y, $G(x, y)$ in (3.11) satisfies $M^*_{(y)} G(x, y) = 0$, $y \neq x$. But then, because of (3.14),

$$M^*_{(y)} M_{(y)} K(x_j, y) = 0, \quad y \neq x_j, \quad j = 1, \ldots, n,$$

which shows that in the intervals (a, x_1), (x_1, x_2), …, (x_n, b),

$$K(x_j, y) \in \mathcal{N}(M^*M), \quad j = 1, \ldots, n.$$

Since as a function of y, $G(x, y) \in C^{k-2}$, it follows from (3.14) that as a function of y, $K(x, y) \in C^{2k-2}$. Finally, since $K(x, y) \equiv K(y, x)$, the representers ϕ_i of the point functionals (5.3) are, $\phi_i(x) = K(x_i, x)$, which completes the demonstration.

Remark added in 1976. The argument just given assumes implicitly that the coefficients a_j are smoother than explicitly assumed. That being so, it seems strange that the smoothness of the adjoined coefficients, b_j, does not seem to play a role in the next theorem.

Theorem 5.1. *Let M be any k–th order ordinary linear differential operator in normal form, (3.1), where $k \geq 1$, $a_i \in C^i[a, b]$, $i = 0$, …, k, and $[a, b]$ is a finite interval. Let $a \leq x_1 < x_2 < \cdots < x_n \leq b$, with $n \geq k$ and assume that the first k of the n linear functionals L_1, …, L_n given by (5.3) are linearly independent over $\mathcal{N}(M)$. Then the space S spanned by the representers ϕ_i of the L_i with respect to the inner product (3.2) consists of all spline functions with respect to $\mathcal{N}(M^*M)$ of order $2k$ with joints x_1, …, x_n. For all $f \in F^{(k)}[a, b]$, there exists a unique element $s \in S$ denoted by Pf which interpolates f with respect to L_1, …, L_n. For all $f \in F^{(k)}[a, b]$ and all $h \in S$,*

$$\int_a^b \Big[(Mf)(y) - (M(Pf))(y)\Big]^2 dy \leq \int_a^b \Big[(Mf)(y) - (Mh)(y)\Big]^2 dy$$

with equality if and only if $h - Pf \in \mathcal{N}(M)$. For all $f \in F^{(k)}[a, b]$ and all $h \in F^{(k)}[a, b]$ interpolating f with respect to L_1, …, L_n,

$$\int_a^b [M(Pf)(y)]^2 dy \leq \int_a^b [(Mh)(y)]^2 dy$$

with equality if and only if $h = Pf$. For all $L \in \mathcal{L}^{(k)}$, Sard's best formula $L^* = \sum_{i=1}^n \alpha_i L_i$ for L (with respect to M) satisfies $L^* f = LPf$, for all $f \in F^{(k)}[a, b]$.

Appendix

Proof of Lemma 3.1. The assumptions on the differential operator M are sufficient [8, pp. 72–75] to insure that the function h defined in (3.4) has all partial derivatives with respect to x up to and including the k–th, and that these partial derivatives are continuous on $[a, b] \times [a, b]$ in x and y jointly. Therefore, there exists a constant N such that

$$|\partial^j h(x, y)/\partial x^j| \leq N, \quad \text{all } x, y \in [a, b], \quad j = 0, \ldots, k.$$

Furthermore, if $\mathrm{e} \in C[a, b]$, then, using Leibniz's rule [9] and (3.5'), one computes

$$\frac{d^j}{dx^j} \int_a^x h(x, y)\mathrm{e}(y) dy = \delta_{kj}\mathrm{e}(x) + \int_a^x \frac{\partial^j}{\partial x^j} h(x, y)\mathrm{e}(y) dy, \quad j = 1, \ldots, k.$$

Let $\mathrm{e} \in L^2[a, b]$, then there exists a sequence $\{\mathrm{e}_i\}_1^\infty$, $\mathrm{e}_i \in C[a, b]$, such that as $i \to \infty$,

$$\|\mathrm{e} - \mathrm{e}_i\|_2 = \Big[\int_a^b |\mathrm{e}(y) - \mathrm{e}_i(y)|^2 dy\Big]^{1/2} \to 0.$$

Set

$$f_i(x) = \int_a^x h(x, y)\mathrm{e}_i(y) dy, \quad i = 1, 2, \ldots,$$

and

$$s_j(x) = \delta_{kj}\mathrm{e}(x) + \int_a^x \frac{\partial^j}{\partial x^j} h(x, y)\mathrm{e}(y) dy \quad j = 0, \ldots, k.$$

Note that s_k is defined only almost everywhere. We have for all $x \in [a,b]$, $j = 0, \ldots, k$, and $i = 1, 2, \ldots,$

$$\Big| \int_a^x \frac{\partial^j}{\partial x^j} h(x,y)\mathrm{e}(y)dy - \int_a^x \frac{\partial^j}{\partial x^j} h(x,y)\mathrm{e}_i(y)dy \Big|$$

$$\begin{aligned} &\leq \int_a^x \Big| \frac{\partial^j}{\partial x^j} h(x,y) \Big| \ |\mathrm{e}(y) - \mathrm{e}_i(y)| dy \\ &\leq \left[\int_a^x \Big| \frac{\partial^j}{\partial x^j} h(x,y) \Big|^2 dy \right]^{1/2} \|\mathrm{e} - \mathrm{e}_i\|_2 \\ &\leq (b-a)^{1/2} N \|\mathrm{e} - \mathrm{e}_i\|_2. \end{aligned}$$

Therefore, $f_i^{(j)} \to s_j$ uniformly on $[a,b]$, for $j = 0, \ldots, k-1$, and $f_i^{(k)} \to s_k$ in the L^2 norm. Hence setting $f = s_0$, we have $f^{(j)} = s_j$, $j = 0, \ldots, k$, where for $j = k$, the equality again holds only almost everywhere. In particular, $f^{(j)}(a) = 0$, $j = 0, \ldots, k-1$, and

$$\begin{aligned} (Mf)(x) &= \sum_{i=0}^{k-1} a_i(x) f^{(i)}(x) + f^{(k)}(x) \\ &= \int_a^x M_{(x)} h(x,y)\mathrm{e}(y)dy + \mathrm{e}(x) = \mathrm{e}(x). \end{aligned}$$

Therefore, if $f \in F^{(k)}[a,b]$, then

$$f(x) - (Q^{(a)} f)(x) = \int_a^x h(x,y) M(f - Q^{(a)} f)(y) dy = \int_a^x h(x,y)(Mf)(y) dy,$$

which proves the first part of Lemma 3.1.

Let L be the linear functional on $F^{(k)}[a,b]$ defined by

$$Lf = \int_a^b f^{(j)}(x) d\mu(x),$$

where μ is a function of bounded variation and $0 \leq j \leq k-1$. If

$$f(x) = \int_a^x h(x,y)\mathrm{e}(y)dy,$$

where $\mathrm{e} \in L^2[a,b]$, then by the above and by Fubini's Theorem,

$$\begin{aligned} Lf &= \int_a^b \int_a^x \frac{\partial^j}{\partial x^j} h(x,y)\mathrm{e}(y) dy d\mu(x) \\ &= \int_a^b \int_y^b \frac{\partial^j}{\partial x^j} h(x,y) d\mu(x) \mathrm{e}(y) dy \\ &= \int_a^b L_{(x)} g(x,y)\mathrm{e}(y) dy, \end{aligned}$$

which implies the second part of Lemma 3.1.

Remark added in 1976. To be sure, $r(y) := L_{(x)} g(x,y)$ is, at times, only defined almost everywhere, but equals $\int_y^b g(x,y) d\mu(x)$ a.e.

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