External zonotopal algebra

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April 10, 2011

Key words. Multivariate polynomials, polynomial ideals, duality, grading, kernels of differential operators, polynomial interpolation, box splines, zonotopes, hyperplane arrangements, matroids, graphs, Hilbert series.

AMS subject classification. 13F20, 13A02, 16W50, 16W60, 47F05, 47L20, 05B20, 05B35, 05B45, 05C50, 52B05, 52B12, 52B20, 52C07, 52C35, 41A15, 41A63.

Abstract

We provide a general, unified, framework for *external zonotopal algebra*. The approach is critically based on employing simultaneously the two dual algebraic constructs and invokes the underlying matroidal and geometric structures in an essential way. This general theory makes zonotopal algebra an applicable tool for a larger class of polytopes.

1 Introduction

General. The most common methodology for constructing multivariate splines is via their definition as volume functions. In this approach, one begins with a linear map, usually a surjection

$$X : \mathbb{R}^N \to \mathbb{R}^n,$$

and continues by restricting this map to a special polyhedron $\mathbf{Z} \subset \mathbb{R}^N$. Most relevant to this paper is the theory of **box splines**, in which $\mathbf{Z}$ is chosen as the unit cube $[0,1]^N$. Two geometries underscore box spline theory: that of *zonotopes*, and the dual geometry of *hyperplane arrangements*. The theory continues with the association of the two geometries with corresponding dual algebraic structures, and culminates with a seamless cohesion of the geometry, the algebra, the spline function and pertinent combinatorial properties of the map X, where the latter viewed as a linear matroid.

Attempts to extend the aforementioned constructions beyond the original setup of box spline theory began in the mid 90's and reached their successful completion in [**?**]: that paper introduced a three-layer theory that was coined there *zonotopal algebra*, with the original box spline theory occupying the middle *central* layer. Two other algebraic constructions, over the same pair of dual geometries and related to the same matroid X, were newly introduced in [**?**]: an external theory and an internal theory. Further developments of zonotopal algebra were recorded in [**?**], [**?**] and [**?**]. We review below some of the pertinent constructions and results in those papers.

*Partially supported by the US National Science Foundation under Grant DMS-0604423.
†Supported by the US National Science Foundation under Grants DMS-0602837 and DMS-0914986 and by the National Institute of General Medical Sciences under Grant NIH-1-R01-GM072000-01.

Our paper is devoted to the external theory within zonotopal algebra, and solely focuses on the homogeneous, continuous setup (as [**?**, **?**, **?**] do). *Our goal is to provide a unifying theory that encompasses all the above-listed approaches and constructions.* We fix, as above, a linear $X : \mathbb{R}^N \to \mathbb{R}^n$, represent X as an $n \times N$ matrix (say, with respect to the standard bases in $\mathbb{R}^N$ and $\mathbb{R}^n$), and treat X also as the multiset of its columns. Zonotopal algebra, in each of its three layers, continues with the introduction of a pair of homogeneous polynomial spaces; the first is usually dubbed a "$\mathcal{P}$-space", is connected to the geometry of the zonotope and is explicit. The second is known as a "$\mathcal{D}$-space" and, as a rule, is defined implicitly as the joint kernel of a suitable set of differential operators, whose corresponding ideal of differential operators is labeled a "$\mathcal{J}$-ideal". The ideal $\mathcal{J}$ and its corresponding kernel $\mathcal{D}$ are associated with the geometry of the hyperplane arrangement.

Zonotopal algebra, central. Let us describe in further detail the setup. With the multiset $X \subset \mathbb{R}^n$ given and fixed, we associate every $x \in X$ (i.e., every column of the matrix X), with the linear form

$$p_x : \mathbb{R}^n \to \mathbb{R} \; : \; t \mapsto x \cdot t,$$

(with "$\cdot$" the standard inner product in $\mathbb{R}^n$) and the corresponding differential operator

$$p_x(D)$$

i.e., the directional derivative D_x in the x-direction. Given a (multi)subset $Z \subset X$, we further denote

$$p_Z := \prod_{x \in Z} p_x.$$

The central zonotopal algebra setup assumes then that X is of full rank n, and continues with a partition of 2^X into the collection of long subsets

$$L(X) := \{Z \subset X \mid \mathrm{rank}(X \backslash Z) < n\},$$

and its complementary collection of short subsets:

$$S(X) := 2^X \backslash L(X).$$

The central $\mathcal{P}$-space $\mathcal{P}(X)$ is defined with the aid of the short sets in $S(X)$:

$$\mathcal{P}(X) := \mathrm{span}\,\{p_Z : \; Z \in S(X)\}.$$

The long sets generate the $\mathcal{J}$-ideal:

$$\mathcal{J}(X) := \mathrm{Ideal}\{p_Z \mid Z \in L(X)\} = \mathrm{Ideal}\{p_Z \mid Z \cap B \neq \varnothing, \forall B \in \mathbb{B}(X)\},$$

with

$$\mathbb{B}(X) \tag{1.1}$$

the set of **bases** of X, i.e., subsets of X that form a basis for $\mathbb{R}^n$. The $\mathcal{D}$-space is then the kernel of $\mathcal{J}(X)$:

$$\mathcal{D}(X) := \{f \in \Pi \mid p(D)f = 0, \forall p \in \mathcal{J}(X)\} = \{f \in \Pi \mid p(D)f(0) = 0, \forall p \in \mathcal{J}(X)\},$$

with

$$\Pi = \mathbb{C}[t_1, \ldots, t_n]$$

the space of all polynomials in n variables. It is known, [**?**], that

$$\mathcal{P}(X) \oplus \mathcal{J}(X) = \Pi,$$

which is equivalent to the statement that the pairing

$$\langle \cdot, \cdot \rangle : \Pi \times \Pi \ : \ (p,q) \mapsto \langle p, q \rangle := p(D)q(0) \tag{1.2}$$

induces a linear bijection between $\mathcal{P}(X)$ and $\mathcal{D}(X)'$, i.e., every linear functional $\lambda \in \mathcal{D}(X)'$ is uniquely represented by some $p \in \mathcal{P}(X)$: $\lambda q = \langle p, q \rangle$, $q \in \mathcal{D}(X)$. Moreover, it is known, [**?**], [**?**], that

$$\dim \mathcal{P}(X) = \dim \mathcal{D}(X) = \#\mathbb{B}(X).$$

In the sequel we will also need the (multi)set

$$\mathbb{I}(X) \tag{1.3}$$

of all **independent subsets** of X (i.e., all subsets of the bases).

Connection with geometry and the least map. As said, two geometries underlie zonotopal algebra. We discuss here the connection of $\mathcal{D}(X)$ with hyperplane arrangements; cf. [**?**] and [**?**] for connections of $\mathcal{P}(X)$ and related spaces to zonotopes. One starts, [**?**], by associating each $x \in X$ with a constant $\lambda_x \in \mathbb{R}$. Set

$$q_x := p_x - \lambda_x \,. \tag{1.4}$$

Each $B \in \mathbb{B}(X)$ defines a *vertex* $\mathcal{V}(B) \in \mathbb{R}^n$, viz, the common zero of the polynomials $(q_x)_{x \in B}$. Assume that the map

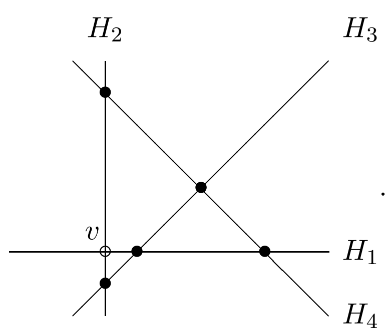
$$\mathcal{V} : \mathbb{B}(X) \to \mathbb{R}^n$$

is injective (which is the generic case in terms of the selection of (λ_x)), i.e., no point $v \in \mathbb{R}^n$ is a common zero for $n+1$ polynomials q_x, $x \in X$. The set $\mathcal{V}(\mathbb{B}(X))$ is then the vertex set of the hyperplane arrangement $\mathcal{H}(X)$ generated by the zero sets H_x of q_x, $x \in X$.

Example 1.1. *In case*

$$X = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 \end{pmatrix} =: (x_1, x_2, x_3, x_4)$$

the hyperplane arrangement $\mathcal{H}(X)$ *is as follows:*



Here, H_i *is the zero set of* q_{x_i}*, and the chosen constants are* $(0,0,1,5)$*. There are six vertices in* $\mathcal{V}(\mathbb{B}(X))$*. For example, the marked vertex* v *is* $\mathcal{V}(\{x_1, x_2\})$*.*

We apply then to the vertex set $\mathcal{V}(\mathbb{B}(X))$ the *least map* of [**?**]. The least map associates each finite $\Theta \subset \mathrm{IR}^n$ with a polynomial space $\Pi(\Theta)$ in the following manner. One defines first an exponential space

$$\mathrm{Exp}(\Theta) := \mathrm{span}\,\{e_\theta \mid \theta \in \Theta\}, \quad e_\theta : t \mapsto e^{\theta \cdot t}.$$

Each $f \in \mathrm{Exp}(\Theta)$ is an n-variate entire function hence admits an expansion

$$f = f_0 + f_1 + \ldots,$$

with f_j a homogeneous polynomial of degree j. Define

$$f_\downarrow := f_j, \quad j := \max\{j' \geq 0 \mid \ f_m = 0, \ \forall m < j'\}.$$

Then:

Theorem 1.2 ([**?**])**.** *With*

$$\Pi(\Theta) := \mathrm{span}\,\{f_\downarrow \mid \ f \in \mathrm{Exp}(\Theta)\},$$

the restriction map from IR^n *to* Θ*:* $p \mapsto p_{|\Theta}$ *is a bijection between* $\Pi(\Theta)$ *and* IR^Θ*. In particular,*

$$\dim \Pi(\Theta) = \#\Theta.$$

. □

The **least map** is then the association

$$\Theta \mapsto \Pi(\Theta),$$

with the polynomial space $\Pi(\Theta)$ known as the **least space** (of Θ).

Applying the least map to the vertex set $\mathcal{V}(\mathbb{B}(X))$ of the hyperplane arrangement $\mathcal{H}(X)$ one obtains the following algebro-geometric interpretation of the equality $\dim \mathcal{D}(X) = \#\mathbb{B}(X)$:

Theorem 1.3 ([**?**])**.** $\Pi(\mathcal{V}(\mathbb{B}(X))) = \mathcal{D}(X)$*, for every generic choice of the constants* $(\lambda_x)_{x \in X}$*.*

. □

The duality between $\mathcal{P}(X)$ and $\mathcal{D}(X)$ then implies that $\mathcal{P}(X)$ interpolates correctly on $\mathcal{V}(\mathbb{B}(X))$; we explain and elaborate on this point in the sequel. In any event, the connection between the explicit $\mathcal{P}(X)$ and the explicit $\mathcal{V}(\mathbb{B}(X))$ can be established directly without a recourse to the implicit $\mathcal{D}(X)$, and is done as follows. We use here the notation

$$q_Z := \prod_{x \in Z} q_x, \quad Z \subset X.$$

Theorem 1.4 ([**?**])**.** [1] *Assuming the selection of constants* (λ_x) *above to be generic, the polynomials* $(q_{X \setminus B})_{B \in \mathbb{B}(X)}$ *form a Lagrange basis for* $\mathcal{P}(X)$ *with respect to the vertex set* $\mathcal{V}(\mathbb{B}(X))$*: given* $B \in \mathbb{B}(X)$*, the polynomial* $q_{X \setminus B}$ *vanishes at all points of* $\mathcal{V}(\mathbb{B}(X))$ *other than* $\mathcal{V}(B)$*.*

[1] The result as stated is straightforward once one knows that $\dim \mathcal{P}(X) = \#\mathbb{B}(X)$. However, the construction of the Lagrange basis was originally used in [**?**] to prove this dimension formula.

External zonotopal algebra. External zonotopal algebra (in its homogeneous continuous setup) deals with polynomial spaces that extend the (central) $\mathcal{P}$- and $\mathcal{D}$- spaces above. This is done, [**?**], by introducing a complementary set $Y \subset \mathbb{R}^n$ and ordering the elements of Y in some fixed way

$$Y = \{y_1, y_2, \ldots\}. \tag{1.5}$$

In [**?**] and [**?**] Y is a fixed, arbitrary, ordered basis for $\mathbb{R}^n$. In the present paper, Y is a (sufficiently long, see below) sequence of vectors in general position in $X \cup Y$: no vector $y \in Y$ is in the span of fewer than n vectors in $(X \cup Y)\backslash y$. We assume $X \cup Y$ to have full rank n, but make no such assumption on X.[2]

Whatever the choice of the complementary (ordered) matroid Y is, one continues by selecting suitably a subset $\mathbb{B}'$ from the basis set of the matroid $X \cup Y$:

$$\mathbb{B}' \subset \mathbb{B}(X \cup Y).$$

The selection is **external** whenever $\mathbb{B}(X) \subset \mathbb{B}'$. The corresponding $\mathcal{J}$-ideal (which is well-defined regardless whether $\mathbb{B}'$ is external or not) is then defined as

$$\mathcal{J}_{\mathbb{B}'} := \text{Ideal}\{p_Z \mid Z \subset X \cup Y,\ Z \cap B \neq \varnothing, \forall B \in \mathbb{B}'\}. \tag{1.6}$$

The corresponding $\mathcal{D}$-space

$$\mathcal{D}_{\mathbb{B}'}$$

is then defined as the *kernel* of $\mathcal{J}_{\mathbb{B}'}$, i.e., the space of all polynomials that are annihilated by all the differential operators induced by $\mathcal{J}_{\mathbb{B}'}$. Equivalently, $\mathcal{D}_{\mathbb{B}'}$ in the annihilator of $\mathcal{J}_{\mathbb{B}'}$ with respect to our pairing (**??**):

$$f \in \mathcal{D}_{\mathbb{B}'} \iff \langle f, \mathcal{J}_{\mathbb{B}'} \rangle = 0.$$

While we are interested in particular, structured, choices of $\mathbb{B}'$, we have the following unqualified estimate on $\dim \mathcal{D}_{\mathbb{B}'}$:

Theorem 1.5 ([**?**])**.** *For an arbitrary* $\mathbb{B}' \subset \mathbb{B}(X \cup Y)$,

$$\dim \mathcal{D}_{\mathbb{B}'} \geq \#\mathbb{B}'. \tag{1.7}$$

Note that in the central case, when $\mathbb{B}' = \mathbb{B}(X)$, there is an equality in (**??**). Indeed, we are only interested in this particular case:

Definition 1.6. *We say that the external selection* $\mathbb{B}(X) \subset \mathbb{B}' \subset \mathbb{B}(X \cup Y)$ *is* **coherent** *if*

$$\dim \mathcal{D}_{\mathbb{B}'} = \#\mathbb{B}'.$$

As said, [**?**] was the first to consider an external setup. It chose Y above to be an arbitrary (ordered) basis for $\mathbb{R}^n$, and defined a set injection

$$\text{ex} : \mathbb{I}(X) \to \mathbb{B}(X \cup Y),$$

via a greedy extension of each independent set to a basis using the elements of Y. The corresponding $\mathcal{D}$-space is then denoted there as $\mathcal{D}_+(X)$ and its corresponding ideal $\mathcal{J}_+(X)$. It is indeed proved in [**?**] that $\mathbb{B}' := \text{ex}(\mathbb{I}(X))$ is coherent:

$$\dim \mathcal{D}_+(X) = \#\mathbb{I}(X).$$

Subsequently the reference [**?**] generalized the above external setup by restricting the extension map ex to a subset $\mathbb{I}'$ of $\mathbb{I}(X)$ that satisfies an additional assumption:

[2] To be sure, a basis B is for $\mathbb{R}^n$; therefore, if $\text{rank}\, X < n$, we have $\mathbb{B}(X) = \varnothing$, ignoring the fact that X has an intrinsic rank and hence a possibly non-empty set of intrinsic bases.

Definition 1.7. *With X as above, let $\mathbb{I}' \subset \mathbb{I}(X)$. We say that $\mathbb{I}'$ is* **solid** *if, given any $I' \in \mathbb{I}'$ and $I \in \mathbb{I}(X)$,*

$$\operatorname{span} I' \subset \operatorname{span} I \implies I \in \mathbb{I}'.$$

[**?**] proved that $\mathbb{B}' := \mathrm{ex}(\mathbb{I}')$ is coherent, too, provided that $\mathbb{I}'$ is solid (in $\mathbb{I}(X)$).

Both references [**?**] and [**?**] build also suitable hyperplane arrangements, select a subset V of the vertex set of the arrangement and prove that their corresponding $\mathcal{D}$-space is the least space of the vertex set V. We refer to [**?**, **?**] for details.

$\mathcal{P}$-spaces. The original external version $\mathcal{P}_+(X)$ was introduced independently in [**?**] and [**?**]. It is defined as

$$\mathcal{P}_+(X) := \operatorname{span}\{p_Z \mid \; Z \subset X\}.$$

It is proved in [**?**] that $\mathcal{P}_+(X)$ and $\mathcal{D}_+(X)$ are dual[3] or, in other words, that

$$\mathcal{J}_+(X) \oplus \mathcal{P}_+(X) = \Pi.$$

This property definitely implies that $\dim \mathcal{P}_+(X) = \dim \mathcal{D}_+(X)$, hence

$$\dim \mathcal{P}_+(X) = \#\mathbb{I}(X).$$

In [**?**], a more general version is defined: one fixes $k \geq 0$, denotes by

$$\Pi_k$$

the space of all polynomials of degree $\leq k$ (in n variables), and defines

$$\mathcal{P}_{+k}(X) := \sum_{Z \subset X} p_Z \Pi_k.$$

The following can be deduced from [**?**]:

Theorem 1.8.

$$\dim \mathcal{P}_{+k}(X) = \sum_{I \in \mathbb{I}(X)} \binom{n+k-\#I}{k}.$$

The original external space $\mathcal{P}_+(X)$ thus corresponds to the case $k = 0$.

Two other papers introduce and study external $\mathcal{P}$-variants: [**?**], given a solid $\mathbb{I}' \subset \mathbb{I}(X)$, defines an intermediate

$$\mathcal{P}(X) \subset \mathcal{P}_{\mathbb{I}'} \subset \mathcal{P}_+(X)$$

and proves its duality with $\mathcal{D}_{\mathbb{I}'} := \mathcal{D}_{\mathrm{ex}\,\mathbb{I}'}$. Recently, Lenz, in [**?**], introduced a setup that generalizes [**?**] as well as [**?**]: given a nonnegative integer k and an upper set $J \subset \mathcal{L}(X)$, where $\mathcal{L}(X)$ is the lattice of flats of the matroid X, he defines

$$\mathcal{P}_{+k,J} := \sum_{Z \subset X} p_Z \Pi_{k+\epsilon(X \setminus Z)},$$

with ϵ the indicator function of J, and, for $Z \subset X$, $\epsilon(Z) := \epsilon(\operatorname{span} Z)$. He proved a suitable dimension formula for this $\mathcal{P}$-space.

[3] Note that $\mathcal{P}_+(X)$ depends only on X, while $\mathcal{D}_+(X)$ depends on the order basis Y, too. The duality is thus valid regardless of the way we choose Y.

Homogeneous basis for $\mathcal{P}(X)$ and Hilbert functions. There are no known explicit constructions of bases for $\mathcal{D}$-type spaces. In contrast, there are such basis constructions for the central $\mathcal{P}(X)$ and each of the external variants discussed above. These constructions allow one (in theory) to compute the Hilbert functions of those $\mathcal{P}$-spaces. The only "real" construction is the one that was given in [**?**] for the central $\mathcal{P}(X)$ and is done as follows. Given X as above, one fixes an arbitrary order $\prec$ on the elements of X. Then, given $B \in \mathbb{B}(X)$, one defines

$$X(B) := \{x \in X \backslash B \mid x \notin \ \operatorname{span}\{b \in B \mid b \prec x\}\}. \tag{1.8}$$

The cardinality of $X(B)$ is intimately connected to the *external activity* of B, which equals to $\#(X\backslash B) - \#X(B)$ (see, e.g., [**?**]).

Theorem 1.9. [**?**] *The polynomials*

$$p_{X(B)}, \quad B \in \mathbb{B}(X)$$

form a basis for $\mathcal{P}(X)$.

The construction of homogeneous bases for external $\mathcal{P}$-spaces is obtained as a variation of the above construction, using the following approach. Suppose that we have defined a $\mathcal{D}$-space $\mathcal{D}_{\mathbb{B}'}$, corresponding to the basis set $\mathbb{B}' \subset \mathbb{B}(X \cup Y)$, and a related $\mathcal{P}_{\mathbb{B}'}$ and proved a duality between the $\mathcal{D}$- and the $\mathcal{P}$- space. Now, necessarily,

$$\mathcal{P}_{\mathbb{B}'} \subset \mathcal{P}(X \cup Y).$$

Thus, we construct a homogeneous basis for $\mathcal{P}(X \cup Y)$ as above, and select the basis polynomials that correspond to $B \in \mathbb{B}'$. These polynomials are automatically linearly independent. Assuming that $\mathbb{B}'$ is coherent, we combine this coherence together with the assumed duality between $\mathcal{P}_{\mathbb{B}'}$ and $\mathcal{D}_{\mathbb{B}'}$ to conclude that

$$\dim \mathcal{P}_{\mathbb{B}'} = \dim \mathcal{D}_{\mathbb{B}'} = \#\mathbb{B}'.$$

Thus, the polynomials selected above will form a basis for $\mathcal{P}_{\mathbb{B}'}$ once we show that each of them actually lies in $\mathcal{P}_{\mathbb{B}'}$.

This approach was, at least implicitly, used in [**?**, **?**] for the construction of homogeneous bases for the external $\mathcal{P}$-spaces that were studied there. [**?**, **?**] used other methods since they introduced $\mathcal{P}$-spaces without corresponding $\mathcal{D}$-spaces.

Given any homogeneous polynomial space, $\mathcal{P}$, the construction of a homogeneous basis $(Q_B)_{B \in \mathbb{B}'}$, with $\mathbb{B}'$ some index set, allows one to compute the Hilbert function of that $\mathcal{P}$-space, i.e., the function

$$h_X : k \mapsto \dim(\mathcal{P} \cap \Pi_k^0),$$

with Π_k^0 the space of homogeneous polynomials of degree k. In the description above, the Hilbert function is combinatorial/matroidal:

$$h_X(k) = \#\{B \in \mathbb{B}' \mid \#(X(B)) = k\}. \tag{1.9}$$

Our setup. Our setup provides a general unified theory and analysis that captures all above-mentioned efforts as special cases. A key to our approach is the simultaneous development of the two types of spaces: $\mathcal{D}$- and $\mathcal{P}$- ones. Given our multiset X (which, in contrast with previous studies like the one in [**?**], is not assumed to be necessarily of full rank), we begin with an assignment

$$\kappa : 2^X \to \mathbb{N},$$

which is solid:

Definition 1.10. *An assignment κ as above is* **solid** *if, given $Z, Z' \subset X$, we have*

$$\operatorname{span} Z \subset \operatorname{span} Z' \implies \kappa(Z) \leq \kappa(Z').$$

Given a solid assignment κ, we define the $\mathcal{P}$-space as

$$\mathcal{P}_\kappa := \sum_{Z \subset X} p_{X \setminus Z} \Pi_{\kappa(Z)}.$$

In order to augment this definition with a corresponding $\mathcal{D}$-space, we choose $Y = \{y_1, y_2, \ldots\}$ to contain sufficiently many vectors in general position (in $X \cup Y$, cf. the discussion after (**??**)), and denote

$$Y_i := \{y_1, \ldots, y_i\}, \quad i > 0, \tag{1.10}$$

and $Y_i = \varnothing$ if $i \leq 0$. The associated basis set $\mathbb{B}' := \mathbb{B}_\kappa \subset \mathbb{B}(X \cup Y)$ is defined as follows:

$$\mathbb{B}_\kappa := \{B \in \mathbb{B}(X \cup Y) \mid \ B \cap Y \subset Y_{m(B \cap X)}\}, \tag{1.11}$$

where, for an independent $I \in \mathbb{I}(X)$,

$$m(I) := \kappa(I) + n - \#I.$$

It follows that each independent $I \subset X$ can be extended in $\binom{m(I)}{\kappa(I)}$ different ways to a basis in $\mathbb{B}_\kappa$, hence that

$$\#\mathbb{B}_\kappa = \sum_{I \in \mathbb{I}(X)} \binom{m(I)}{\kappa(I)}.$$

Example 1.11. *Let $X = \{x_1, x_2\} \subset \mathbb{R}^2$, where $x_1 = \binom{1}{0}$ and $x_2 = \binom{0}{1}$. Assume that κ is solid and that $\kappa(x_1) = \kappa(x_2)$. It then easily follows that*

$$\#\,\mathbb{B}_\kappa = \binom{2 + \kappa(\varnothing)}{2} + 2\kappa(x_1) + 3. \tag{1.12}$$

The $\mathcal{D}$-space $\mathcal{D}_\kappa$ is defined as

$$\mathcal{D}_\kappa := \mathcal{D}_{\mathbb{B}_\kappa} = \ker \mathcal{J}_{\mathbb{B}_\kappa},$$

where $\mathcal{J}_{\mathbb{B}_\kappa}$ is defined in (**??**) with respect to the choice $\mathbb{B}' = \mathbb{B}_\kappa$. As before, we associate each $z \in X \cup Y$ with a constant λ_z and assume the assignment to be generic. Every $B \in \mathbb{B}(X \cup Y)$ then corresponds to $\mathcal{V}(B) :=$ the common zero of the polynomials $(q_z)_{z \in B}$, and, by assumption, the map

$$\mathcal{V} : \mathbb{B}(X \cup Y) \to \mathbb{R}^n \quad : \quad B \mapsto \mathcal{V}(B)$$

is injective. We denote

$$V_\kappa := \mathcal{V}(\mathbb{B}_\kappa).$$

At this generality, we are able to prove only partial results:

Theorem 1.12. *Let κ be a solid assignment. Then:*

- *$\mathbb{B}_\kappa$ is coherent, i.e.,* $\dim \mathcal{D}_\kappa = \#\mathbb{B}_\kappa$. *Furthermore,* $\Pi(V_\kappa) = \mathcal{D}_\kappa$.
- $\mathcal{P}_\kappa + \mathcal{J}_\kappa = \Pi$.

- *$\mathcal{P}_\kappa$ contains a Lagrange basis for V_κ: for each $v \in V_\kappa$ there exists $L_v \in \mathcal{P}_\kappa$, such that L_v vanishes on $V_\kappa \backslash v$, but not at v.*

A few remarks are then in order:

1. We provide an explicit construction of the aforementioned Lagrange basis.

2. The second result in the above theorem implies that $\dim \mathcal{P}_\kappa \geq \dim \mathcal{D}_\kappa$. Simple examples show that this inequality can be sharp.

3. The third result implies that $\dim \mathcal{P}_\kappa \geq \# V_\kappa = \# \mathbb{B}_\kappa$. This inequality follows also from the second result, since (**??**),

$$\dim \mathcal{D}_\kappa \geq \# \mathbb{B}_\kappa$$

even without the solid assumption.

4. We also identify in $\mathcal{P}_\kappa$ a family of $\# \mathbb{B}_\kappa$ linearly independent homogeneous polynomials. That construction not only reproves the inequality $\dim \mathcal{P}_\kappa \geq \# \mathbb{B}_\kappa$, but also provides a lower bound on the values assumed by the Hilbert function of $\mathcal{P}_\kappa$.

Stronger results are obtained once we make an additional assumption:

Definition 1.13. *We say that an assignment κ is* **incremental** *if, for every $Z \subset X$ and $x \in X$,*

$$\kappa(Z \cup x) \leq \kappa(Z) + 1.$$

Indeed, we obtain a complete theory for assignments that are both solid and incremental:

Theorem 1.14. *Assume the assignment κ to be solid and incremental. Set $X' := X \cup Y$, and, for $I \in \mathbb{I}(X)$, $X'_I := X \cup Y_{m(I)}$. Then*

- *The polynomials*

$$q_{(X'_{B \cap X}) \backslash B}, \quad B \in \mathbb{B}_\kappa$$

  form an inhomogeneous basis for $\mathcal{P}_\kappa$. In particular,

$$\dim \mathcal{P}_\kappa = \# \mathbb{B}_\kappa.$$

- *The polynomials*

$$p_{X'(B)}, \quad B \in \mathbb{B}_\kappa$$

  form a homogeneous basis for $\mathcal{P}_\kappa$.[4]

It follows from this result that the Lagrange basis in Theorem **??** is also a basis for $\mathcal{P}_\kappa$. Also, we can now conclude that

$$\mathcal{J}_\kappa \oplus \mathcal{P}_\kappa = \Pi,$$

or in other words that $\mathcal{P}_\kappa$ and $\mathcal{D}_\kappa$ are dual to each other.

Finally, the construction of a homogeneous basis for $\mathcal{P}_\kappa$ leads to a combinatorial formula for the Hilbert function h_κ of $\mathcal{P}_\kappa$, which, due to the duality between $\mathcal{P}_\kappa$ and $\mathcal{D}_\kappa$, is also the Hilbert function of $\mathcal{D}_\kappa$: for $j \geq 0$ we have

$$h_\kappa(j) = h_X(j) + \sum \binom{j - \# X(I) + n - \# I - 1}{n - \# I - 1}, \tag{1.13}$$

where the sum runs over all $I \in \mathbb{I}(X) \backslash \mathbb{B}(X)$ for which $j - \kappa(I) \leq \#(X(I)) \leq j$, and with h_X the Hilbert function of $\mathcal{P}(X)$ (cf. (**??**)).

[4] The notation $X'(B)$ is defined in (**??**), with X there replaced by X' here.

Proof. Given $j \geq 0$, we need to count the number of polynomials $p_{X'(B)}$ in the homogeneous basis for $\mathcal{P}_\kappa$ that are of degree j (cf. Theorem **??**). In other words, we need to find out the number

$$\#\{B \in \mathbb{B}_\kappa \mid \#X'(B) = j\}.$$

Since

$$h_X(j) = \#\{B \in \mathbb{B}(X) \mid \#X'(B) = j\},$$

we need only to focus on $\mathbb{B}_\kappa \backslash \mathbb{B}(X)$. To this end, we write $B = I \cup J \in \mathbb{B}_\kappa \backslash \mathbb{B}(X)$ with $I \subset X$ and $J \subset Y$; also, let y_k be the maximal element of J. Then $X'(B) = X(I) \cup (Y_k \backslash J)$. Since we need to have $\#X'(B) = j$, it is necessary that $0 \leq j - \#X(I) \leq \kappa(I)$. Once our I is fixed, y_k, the last element of J, has to satisfy that $k = j - \#X(I) + n - \#I$. Then, we can freely choose the remaining $n - \#I - 1$ elements from Y_{k-1}. This validates the given formula. $\square$

Example 1.15. *Consider $X = \{x_1, x_2, x_3\}$, where $x_1 = \binom{1}{0}$ and $x_2 = x_3 = \binom{0}{1}$. Then, we have $X(\varnothing) = \{x_1, x_2, x_3\}$, $X(\{x_1\}) = \{x_2, x_3\}$, $X(\{x_2\}) = \{x_1\}$, $X(\{x_3\}) = \{x_1, x_2\}$, $X(\{x_1, x_3\}) = \{x_2\}$ and $X(\{x_1, x_2\}) = \varnothing$. Assume $\kappa(\varnothing) = \kappa(\{x_1\}) = 1$ and $\kappa(\{x_2\}) = \kappa(\{x_3\}) = 2$. Then for $j = 4$, the independent sets in the sum (**??**) are $\varnothing$ and $\{x_3\}$, and we have $h_\kappa(4) = 3$. For $j < 4$, one finds out that $h_\kappa(j) = j + 1$, hence that $\Pi_3 \subset \mathcal{P}_\kappa$. Note that $\dim \mathcal{P}_\kappa = \#\mathbb{B}_\kappa = 13$ here.*

2 Construction and analysis of $\mathcal{D}_\kappa$

The main objective in this section is to show that the space $\mathcal{D}_\kappa$ is coherent, whenever κ is solid. Thus, the main result in this section is the following:

Theorem 2.1. *$\mathbb{B}_k$ is coherent for all solid assignments κ:*

$$\dim \mathcal{D}_\kappa = \# \mathbb{B}_\kappa .$$

Recall that the lower bound $\dim \mathcal{D}_\kappa \geq \# \mathbb{B}_\kappa$ is valid, Theorem **??**, without any conditions or assumptions on $\mathbb{B}_\kappa$. The solid assumption on κ, thus, leads to a matching upper bound. In proving this matching bound, we will invoke the notion of *placability*:

Definition 2.2. *Let X be a matroid and $\varnothing \neq \mathbb{B}' \subset \mathbb{B}(X)$.*

1. *Given $x \in X$, the actions of* **deletion** *of x and* **restriction** *to x decompose $\mathbb{B}'$ into*

$$\mathbb{B}'_{\backslash x} := \{B \in \mathbb{B}' \mid x \notin B\}, \quad \textit{and} \quad \mathbb{B}'_{/x} := \{B \in \mathbb{B}' \mid x \in B\}.$$

2. *An element $x \in X$ is* **placable** *in $\mathbb{B}'$ if for each $B \in \mathbb{B}'$, there exists an element $a \in B$ such that $(\{x\} \cup B) \backslash \{a\} \in \mathbb{B}'$.*

3. *A (placable)* **split** *of $\mathbb{B}'$ is a set partition $\mathbb{B}'_{/x} \sqcup \mathbb{B}'_{\backslash x}$ by a placable element x such that both $\mathbb{B}'_{/x}, \mathbb{B}'_{\backslash x} \neq \varnothing$.*

4. *We say that $\mathbb{B}'$ is* **placible** *if one of the following two conditions holds:*

    (a) *$\mathbb{B}'$ is a singleton.*

    (b) *There exists $x \in X$ which is placable in $\mathbb{B}'$, for which $\mathbb{B}'_{/x}$ and $\mathbb{B}'_{\backslash x}$ are, each, non-empty and placible.*

Note that Part 4 of the above definition is inductive; this inductive definition is valid, since we assume both $\mathbb{B}'_{/x}$ and $\mathbb{B}'_{\backslash x}$ to be nonempty.

The following is known:

Lemma 2.3 ([?]). *Let $\mathbb{B}' \subset \mathbb{B}(X)$. If $\mathbb{B}'$ is placible, then* $\dim \mathcal{D}_{\mathbb{B}'} \leq \#\mathbb{B}'$.

Thus, in view of the above lemma, the inequality $\dim \mathcal{D}_\kappa \leq \# \mathbb{B}_\kappa$ will follow once we show that $\mathbb{B}_\kappa$ is placible, as we do now.

First, recall from (**??**) that

$$\mathbb{B}_\kappa := \{B \in \mathbb{B}(X \cup Y) \mid \ B \cap Y \subset Y_{m(B\cap X)}\}.$$

Given two disjoint subsets, A, C, of X, we denote

$$\mathbb{B}_{\kappa,A,C} := \{B \in \mathbb{B}_\kappa \mid B \cap A = \varnothing, C \subset B\}.$$

Notice that it is possible that another pair A', C' defines the same set: $\mathbb{B}_{\kappa,A',C'} = \mathbb{B}_{\kappa,A,C}$. Assume in the following proposition that A and C are maximal. It then follows, since $C \subset B$, for each $B \in \mathbb{B}_{\kappa,A,C}$, that $\operatorname{span} C \subset A \cup C$.

Proposition 2.4. *Assume that κ is solid. Then each element $x \in X\backslash(A \cup C)$ is placable in $\mathbb{B}_{\kappa,A,C}$.*

Proof. Let $x \in X\backslash(A \cup C)$ and $B \in \mathbb{B}_{\kappa,A,C}$. We need to show that we can replace some element of B by x to obtain another basis in $\mathbb{B}_\kappa$. This is trivial if $x \in B$. So we assume that B contains C and is disjoint of A, x, and (due to the maximality of A) $\operatorname{span} C\backslash C$. Denote $I := B \cap X$ and $J := B \cap Y$. There are two cases to consider:

1. $x \notin \operatorname{span}(I)$. In this case, we replace the last element y of J with x. We claim that

   $$B' := (I \cup \{x\}) \cup (J\backslash\{y\}) := I' \cup J' \in \mathbb{B}_{\kappa,A,C}\,.$$

   First, it is clear that $I' \cap A = \varnothing$, since $I \cap A = \varnothing$, and $x \notin A$. Also, $C \subset I'$, since $C \subset I$. Therefore, we only need to show that $J' \subset Y_{m(I')}$. We know, by assumption, that $J \subset Y_{m(I)}$. Since y is the *last* element of J, we conclude that $J' \subset Y_{m(I)-1}$. However,

   $$m(I) - 1 = \kappa(I) + \operatorname{corank} I - 1 = \kappa(I) + \operatorname{corank} I' \leq \kappa(I') + \operatorname{corank} I' = m(I'),$$

   with the inequality following from the solid property of κ. Consequently, $J' \subset Y_{m(I')}$ as required.

2. $x \in \operatorname{span} I$. Since we assume that $x \notin A$, and A is maximal, we have $x \notin \operatorname{span} C$. So there exists $a \in I\backslash C$ such that, with $I' := \{x\} \cup I\backslash\{a\}$, $\operatorname{span} I = \operatorname{span} I'$. We now claim that

   $$B' := I' \cup J \in \mathbb{B}_\kappa\,.$$

   Here all the requisite conditions are immediate. First, $I' \cap A = \varnothing$, since $I \cap A = \varnothing$, and $x \notin A$. Second, $C \subset I'$ since $a \notin C$ and $C \subset I$. Finally, since $\operatorname{span} I = \operatorname{span} I'$, and since κ is solid, we must have $\kappa(I') = \kappa(I)$. Since also $\#I' = \#I$, we conclude that $m(I) = m(I')$. Therefore, the required inclusion $J \subset Y_{m(I')}$ follows from the assumed inclusion $J \subset Y_{m(I)}$.

□

Now, we can build a binary tree whose root is $\mathbb{B}_\kappa$, and with each branching of a node done by deletion/restriction using some element $x \in X$. Obviously, every node in such tree is of the form $\mathbb{B}_{\kappa,A,C}$. Let us assume that the branching of the node $\mathbb{B}_{\kappa,A,C}$ is done by an element $x \in X\backslash(A\cup C)$. Such element was just proved to be placable in $\mathbb{B}_{\kappa,A,C}$. The maximality assumption on A, C easily leads to the conclusion that the split is non-trivial. We can continue branching the nodes of the tree as much as it is possible. Obviously, we will have to stop only when $X \subset A\cup C$, i.e., $X\backslash A \subset C$. Since we assume $\mathbb{B}_{\kappa,A,C} \neq \varnothing$, it must be the case that $C = I$ is some independent set and $A = X\backslash I$. So this node corresponds to the set $\mathrm{ex}(I)$, i.e., bases in $\mathbb{B}_\kappa$ which extend I using elements of $Y_{m(I)}$. If $I \in \mathbb{B}(X)$, we are done since the node is a singleton, and the same applies if $\kappa(I) = 0$. Otherwise, every $y \in Y_{m(I)}$ is placable in every subset of $\mathrm{ex}(I)$, as one easily verifies. Thus, we can split $\mathrm{ex}(I)$ successively using elements of $Y_{m(I)}$ until $\mathrm{ex}(I)$ is completely split to singletons.

Thus, we have shown that $\mathbb{B}_\kappa$ is placible. *Consequently, we can invoke Lemma* **??** *to obtain Theorem* **??**.

Next, we return our attention to the inhomogeneous polynomials q_z, $z \in X \cup Y$, the associated hyperplane arrangement $\mathcal{H}(X \cup Y)$, and the bijection $\mathcal{V}$ from $\mathbb{B}(X \cup Y)$ onto the vertex set V_κ of $\mathcal{H}(X \cup Y)$ (cf. the discussion around (**??**)). Let $\Pi(V_\kappa)$ be the least space of V_κ (cf. Theorem **??**). Now,

$$\dim \Pi(V_\kappa) = \#V_\kappa = \#\mathbb{B}_\kappa = \dim \mathcal{D}_\kappa,$$

with the last equality implied by Theorem **??**. However, [**?**], we (always, i.e., even in the absence of the solid property of κ) have that

$$\Pi(V_\kappa) \subset \mathcal{D}_\kappa\,.$$

Therefore:

Corollary 2.5. $\Pi(V_\kappa) = \mathcal{D}_\kappa$, *where* $V_\kappa = \mathcal{V}(B_\kappa)$.

Example 2.6 (continuation of Example **??**)**.** *Let* $X = \{x_1, x_2\} \subset \mathbb{R}^2$, *where* $x_1 = \binom{1}{0}$ *and* $x_2 = \binom{0}{1}$. *Choose* $\kappa(\varnothing) = 1$, $\kappa(x_1) = \kappa(x_2) = 1$, *and* $\kappa(\{x_1, x_2\}) = 2$. *It is trivial to check that this* κ *is solid. We want to find* $\mathbb{B}_\kappa$ *and* $V_\kappa = \mathcal{V}(B_\kappa)$ *in this example.*

Recall that $m(I) = \kappa(I) + 2 - \#I$. *So* $m(\varnothing) = 3$ *and* $m(x_1) = m(x_2) = m(\{x_1, x_2\}) = 2$. *Therefore it suffices for* Y *to have 3 elements:* $Y = \{y_1, y_2, y_3\}$. *By the definition of* $\mathbb{B}_\kappa$ *in (**??**), we have*

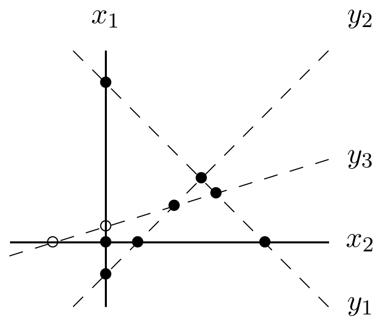
$$\mathbb{B}_\kappa = \mathrm{ex}(\varnothing) \cup \mathrm{ex}(\{x_1\}) \cup \mathrm{ex}(\{x_2\}) \cup \mathrm{ex}(\{x_1, x_2\}),$$

with

$$\begin{aligned}
&\mathrm{ex}(\varnothing) = \{\{y_1, y_2\}, \{y_1, y_3\}, \{y_2, y_3\}\},\\
&\mathrm{ex}(\{x_i\}) = \{\{x_i, y_1\}, \{x_i, y_2\}\},\ i = 1, 2,\\
&\mathrm{ex}(\{x_1, x_2\}) = \{\{x_1, x_2\}\}.
\end{aligned}$$

In particular, $\#\mathbb{B}_\kappa = 8$ *which matches (**??**) in Example* **??**.

The associated hyperplane arrangement $\mathcal{H}(X \cup Y)$ *is depicted in the following figure and* $V_\kappa = \mathcal{V}(B_\kappa)$ *corresponds to the vertices of the arrangement that are marked solid (viz. all vertices but the intersections of* $\{x_1, y_3\}$ *and* $\{x_2, y_3\}$*).*



3 Construction and analysis of $\mathcal{P}_\kappa$

Recall from the introduction the definition of the polynomials spaces Π and Π_k, $k \geq 0$, and the definition of $\mathcal{P}_\kappa$:

$$\mathcal{P}_\kappa := \mathcal{P}_\kappa(X) := \sum_{Z \subset X} p_{X \backslash Z} \Pi_{\kappa(Z)}. \tag{3.1}$$

One of our primary aims is to establish, under some conditions on κ, a duality between $\mathcal{D}_\kappa$ and $\mathcal{P}_\kappa$. Thus, we need to have

$$\dim \mathcal{P}_\kappa = \dim \mathcal{D}_\kappa = \#\mathbb{B}_\kappa,$$

with the left equality necessary for the duality and the right one our requirement of coherence.

Example 3.1 (Continuation of Example **??**). *Let $X = \{x_1, x_2\} \subset \mathbb{R}^2$, where $x_1 = \binom{1}{0}$ and $x_2 = \binom{0}{1}$. Let $\kappa(x_1) = \kappa(x_2) = k$, $\kappa(\varnothing) = j$ with $j \leq k$ and $\kappa(\{x_1, x_2\}) = \ell$ with $\ell \geq k$. One can check that κ is solid. As in Example **??**, we have*

$$\#\mathbb{B}_\kappa = \binom{j+2}{2} + 2\binom{k+1}{1} + 1.$$

In this example, we will compute $\mathcal{P}_\kappa$ explicitly, and compare its dimension with $\#\mathbb{B}_\kappa$.

*By (**??**), we have $\mathcal{P}_\kappa = \operatorname{span}\{\Pi_k p_{x_1}, \Pi_k p_{x_2}, \Pi_\ell, \Pi_j p_X\}$. There are three cases:*

1. *If $\ell > k+1$, we have $\mathcal{P}_\kappa = \Pi_\ell$ hence $\dim \mathcal{P}_\kappa = \binom{\ell+2}{2}$; since we assume $j \leq k < \ell - 1$, it is easy to see that we get here $\dim \mathcal{P}_\kappa > \#\mathbb{B}_\kappa$.*

2. *If $\ell \leq k+1$ and $j \leq k-1$, we have $\mathcal{P}_\kappa = \Pi_{k+1}$, hence $\dim \mathcal{P}_\kappa = \binom{k+3}{2} = \binom{k+1}{2} + 2\binom{k+1}{1} + 1$; consequently, $\dim \mathcal{P}_\kappa \geq \#\mathbb{B}_\kappa$ with equality if and only if $j = k-1$.*

3. *If $\ell \leq k+1$ but $j = k$, we have*

$$\mathcal{P}_\kappa = \Pi_{k+1} + \operatorname{span}\{p_{x_1}^{m+1} p_{x_2}^{k-m+1} \mid m = 0, \ldots, k\}.$$

Therefore,

$$\dim \mathcal{P}_\kappa = \binom{k+3}{2} + k + 1 = \#\mathbb{B}_\kappa.$$

Note that the inequality $\dim \mathcal{P}_\kappa \geq \#\mathbb{B}_\kappa$ is valid in each of the above three cases. Our results in this section make clear that this is not an accident, and is due to the fact that κ is solid. At the same time, this example clearly shows that the solid assumption alone does guarantee our desired equality. *To this end, we will revisit the case here in Example **??**, and will study closely the situations when equality holds.*

. $\square$

As we just said, the lower bound on $\dim \mathcal{P}_\kappa$ that was observed in the example above is true, in general, for every solid assignment κ:

Theorem 3.2. *Assume κ to be solid. Then:*

$$\dim \mathcal{P}_\kappa \geq \dim \mathcal{D}_\kappa = \#\mathbb{B}_\kappa.$$

The equality $\dim \mathcal{D}_\kappa = \#\mathbb{B}_\kappa$ was proved in Theorem **??**. We need thus to prove the inequality assertion. We provide below three complementary proofs, each revealing a different property of $\mathcal{P}_\kappa$.

3.1 First proof of Theorem ??: embedding $\mathcal{D}'_\kappa$ in $\mathcal{P}_\kappa$

The inequality $\dim \mathcal{P}_\kappa \geq \dim \mathcal{D}_\kappa$ follows (directly) from the following stronger result (cf. the discussion above Theorem **??** for the definition of the ideal $\mathcal{J}_\kappa$):

Proposition 3.3. *Assume κ to be solid. Then:*

$$\mathcal{J}_\kappa + \mathcal{P}_\kappa = \Pi.$$

Proof of Proposition **??**. Set

$$A := \mathcal{J}_\kappa + \mathcal{P}_\kappa\,.$$

Let $X' \subset X$, and $Y' \subset Y_{s(X',Y')}$, with

$$s(X',Y') := n - \operatorname{rank} X' + \#Y' - 1.$$

(Note that the definition makes sense even when $\operatorname{rank} X' = n$: we have then $s(X',Y') < \#Y'$, which merely forces Y' to be empty.) We claim that, for an arbitrary polynomial f, the product

$$F := f\, p_{X\backslash X'}\, p_{Y'}$$

lies in A. Choosing $X' := X$ and $Y' := \varnothing$, we will then obtain the desired result, since f is arbitrary.

In order to prove that $F \in A$, we first fix X' and assume Y' to be "large enough": $\#Y' > \kappa(X')$. We claim that in this case $F \in \mathcal{J}_\kappa$, which will follow once we prove that

$$B \cap (Y' \cup X\backslash X') \neq \varnothing,$$

for every $B \in \mathbb{B}_\kappa$. To this end, we assume that $B \cap X\backslash X' = \varnothing$, and examine $J := B \cap Y'$. Then $B\backslash J \subset X'$, and since κ is solid, $\kappa(B\backslash J) \leq \kappa(X')$. Also, since $B \in \mathbb{B}_\kappa$, $J \subset Y_{m(B\backslash J)}$, where

$$m(B\backslash J) = \#J + \kappa(B\backslash J) \leq \#J + \kappa(X') \leq \#J + \#Y' - 1.$$

But we also have

$$s(X',Y') \leq \#J + \#Y' - 1,$$

because $n - \#J \leq \operatorname{rank} X'$. We conclude that Y' as well as J are both subsets of $Y_{\#J+\#Y'-1}$, implying that these two sets intersect.

Thus, it remains to show that $F \in A$ when $\#Y' \leq \kappa(X')$. Note that the number of pairs X', Y' for which $\#Y' \leq \kappa(X')$ (and in addition $X' \subset X$, $Y' \subset s(X',Y')$) is finite. We will thus prove that $F \in A$ by descending induction on $\#Y' + \#(X\backslash X')$.

Now, let X' and Y' be as above. Choose a basis $I \subset X'$ for $\operatorname{span} X'$, and let $J := Y_{s(X',Y')+1} \backslash Y'$. Then $B := I \cup J$ is a basis for $\mathbb{R}^n$. Therefore, we can write

$$f = c + \sum_{b \in B} p_b \, f_b,$$

with c some scalar and $(f_b)_{b \in B}$ some polynomials. Therefore

$$F = c \, p_{X \backslash X'} \, p_{Y'} + \sum_{b \in B} p_b \, p_{X \backslash X'} \, p_{Y'} \, f_b.$$

Note that $p_{X \backslash X'} \, p_{Y'} \in \mathcal{P}_\kappa \subset A$, since $\#Y' \leq \kappa(X')$, by assumption. We will use our induction hypothesis to show that each of the summands

$$p_b \, p_{X \backslash X'} \, p_{Y'} \, f_b, \quad b \in B$$

lies in A, too. There are two cases to consider:

1. $b \in X'$. In this case, with $X'' := X' \backslash b$, we need to check that $Y' \subset Y_{s(X'',Y')}$, and then the induction will apply. However, $s(X'', Y') = n - \operatorname{rank} X'' + \#Y' - 1 \geq n - \operatorname{rank} X' + \#Y' - 1 = s(X', Y')$. Therefore, $Y' \subset Y_{s(X',Y')} \subset Y_{s(X'',Y')}$.

2. $b \in Y$. In this case, with $Y'' := Y \cup \{b\}$, we need to show that $Y'' \subset Y_{s(X',Y'')}$. However, we have that $s(X', Y'') = s(X', Y') + 1$, and thus $b \in J \subset Y_{s(X',Y')+1} = Y_{s(X',Y'')}$. Hence the induction hypothesis applies here as well.

This completes the inductive step, hence the proof that $F \in A$, hence the proof of Proposition **??**, hence the first proof of Theorem **??**. □

3.2 Second proof of Theorem ??: homogeneous basis

As noted in the Introduction, we can attempt to construct a homogeneous basis for a subspace of $\mathcal{P}_\kappa$ by adapting the basis construction for $\mathcal{P}(X)$ from [**?**].

Since in our case $\mathbb{B}_\kappa \subset \mathbb{B}(X')$, $X' := X \cup Y$, we first follow [**?**] and construct a homogenous basis

$$p_{X'(B)}, \quad B \in \mathbb{B}(X')$$

for $\mathcal{P}(X')$, as in Theorem **??**. In the actual construction, we need to order the vectors in X': We choose any order on X, retain the given order on Y, and insist that $x \prec y$ for every $x \in X$ and $y \in Y$.

The polynomials

$$p_{X'(B)}, \quad B \in \mathbb{B}_\kappa$$

are trivially linearly independent. Theorem **??** will then follow once we show that each one of them lies in $\mathcal{P}_\kappa$. So, fix $B \in \mathbb{B}_\kappa$. Then, with $I := X \cap B$, $B \backslash I \subset Y_{m(I)}$. The definition of $X'(B)$ clearly shows that $X'(B)$ contain no vectors that are larger than the maximal vector in B. Therefore, $X'(B) \cap Y \subset Y_{m(I)}$. Now,

$$\#(X'(B) \cap Y) \leq m(I) - \#(B \backslash I) = \kappa(I).$$

Since $I \subset X \backslash X'(B) =: Z$ and κ is solid, we have that

$$\kappa(I) \leq \kappa(Z),$$

and hence

$$p_{X'(B)} = p_{X'(B) \cap X} \, p_{X'(B) \cap Y} \in p_{X'(B) \cap X} \, \Pi_{\kappa(I)} \subset p_{X \backslash Z} \, \Pi_{\kappa(Z)} \subset \mathcal{P}_\kappa \, .$$

Thus, the linearly independent polynomials

$$p_{X'(B)}, \quad B \in \mathbb{B}_\kappa$$

lie in $\mathcal{P}_\kappa$, and Theorem **??** follows:

$$\dim \mathcal{P}_\kappa \geq \# \mathbb{B}_\kappa.$$

3.3 Third proof of Theorem ??: Lagrange basis

We retain our assumption that κ is solid, and recall the definition of the inhomogeneous polynomials q_x, $x \in X \cup Y$, together with the assignment

$$\mathcal{V} : \mathbb{B}_\kappa \to \mathbb{R}^n$$

that assigns to each basis the common zero of the polynomials q_x, $x \in B$. Also, $V_\kappa := \mathcal{V}(\mathbb{B}_\kappa)$. We will show that $\mathcal{P}_\kappa$ contains a Lagrange basis with respect to V_κ:

Proposition 3.4. *Assume that κ is solid and let V_κ be as above. For every $B \in \mathbb{B}_\kappa$, there exists $L_B \in \mathcal{P}_\kappa$ such that $L_B(\mathcal{V}(B)) \neq 0$, while L_B vanishes on $V_\kappa \backslash \mathcal{V}(B)$.*

Obviously, the above Lagrange polynomials are linearly independent, and therefore Proposition **??** implies that

$$\dim \mathcal{P}_\kappa \geq \# V_\kappa = \# \mathbb{B}_\kappa,$$

providing thereby another proof to Theorem **??**.

Before we embark on the proof of the Proposition, we mention the following simple fact:

Lemma 3.5. *Assume that κ is solid, let $Z \subset X$ and let f be a polynomial of degree no more than $\kappa(X \backslash Z)$. Then $q_Z f \in \mathcal{P}_\kappa$.*

Proof. Expanding q_Z, we have that $q_Z f$ is a linear combination of $p_{Z'} f$ for some $Z' \subset Z$. Since κ is solid, $\deg(f) \leq \kappa(X \backslash Z) \leq \kappa(X \backslash Z')$, so we have that $p_{Z'} f \in \mathcal{P}_\kappa$. Therefore, $q_Z f \in \mathcal{P}_\kappa$ □

Our next task is to construct the aforementioned Lagrange basis. So, we fix $B \in \mathbb{B}_\kappa$, and denote by $v \in V_\kappa$ the corresponding vertex $v = \mathcal{V}(B)$. Given $x \in X \cup Y$, we have that $q_x(v) = 0$ if and only if $x \in B$. With the above B in hand, we denote

$$X_B := X \cup Y_{m(X \cap B)},$$

and

$$L_B := q_{X_B \backslash B}\, \ell_B =: Q_B\, \ell_B$$

with ℓ_B a linear polynomial that we define in the sequel. Assuming that we make sure that $\ell_B(v) \neq 0$, it is clear that $L_B(v) \neq 0$. Our goal is to show, then, that $L_B(v') = 0$, for every $v' \in V_\kappa \backslash v$, and that $L_B \in \mathcal{P}_\kappa$.

Our first observation is that Q_B above already vanishes at "most" of the points in V_κ. Indeed, each $B' \in A := \{B' \in \mathbb{B}_\kappa : Q_B(\mathcal{V}(B')) \neq 0\}$ has the following form:

$$B' = I' \cup J \cup \{y_{m(I)+1}, \dots, y_{m(I')}\}, \tag{3.2}$$

where $J = B \cap Y$ and $I' = B' \cap X \subset B \cap X = I$ with $\kappa(I') = \kappa(I)$. This is implied by the following lemma.

Lemma 3.6. *$B' \in A$ if and only if the following four conditions hold:*

(i) $I' \subset I$,

(ii) $B' \cap Y_{m(I)} = J$,

(iii) $\kappa(I') = \kappa(I)$, *and*

(iv) $Y \cap B' \backslash B = Y_{m(I')} \backslash Y_{m(I)}$.

Proof of Lemma **??***:* First, it is easy to see that $B' \in A$ iff, with $v' := \mathcal{V}(B')$, $q_{X \backslash B}(v') \neq 0$ and $q_{Y_{m(I)} \backslash B}(v') \neq 0$ iff $I' \subset I$ (Condition (i)) and $B' \cap Y_{m(I)} \subset B \cap Y = J$ (half of Condition (ii)). We claim that this implies Condition (ii). In fact, if $J' := B' \cap Y_{m(I)}$ is a proper subset of $J = B \backslash I$, then the maximal possible cardinality of B' is

$$\#I' + \#J' + (m(I') - m(I)) = \#I' + \#J' + \#I - \#I' + k(I') - k(I) \leq \#I + \#J' < n,$$

since $I' \subset I$ and κ is solid. Therefore $B' \in A$ iff (i) and (ii) hold.

Next we want to show that (iii) is implied by (i) and (ii). By (i) and the fact that κ is solid, we have $\kappa(I') \leq \kappa(I)$. Therefore,

$$m(I') - m(I) = \#(I \backslash I') + \kappa(I') - \kappa(I) \leq \#(I \backslash I'),$$

with equality if and only if $\kappa(I) = \kappa(I')$. However, (ii) implies that the set $Y_{m(I')} \backslash Y_{m(I)}$ contains exactly $\#(I \backslash I')$ elements of B' hence is of cardinality $\geq \#(I \backslash I')$, and (iii) thus follows.

Last, we show that condition (iv) is implied by the other three: the argument in the previous paragraph shows that $m(I') - m(I) = \#(I \backslash I')$, and that B' contains exactly $\#(I \backslash I')$ vectors from $Y_{m(I')} \backslash Y_{m(I)}$, so (iv) follows. $\square$

So we have a bijection between A and the subsets of I' of I that satisfy $\kappa(I') = \kappa(I)$. In that bijection, I' is extended to $B' \in A$ via (**??**).

We now need to define ℓ_B in a way that it vanishes on $\mathcal{V}(A) \backslash v$. In view of the above bijection, we choose a *proper* subset I' of I for which $\kappa(I) = \kappa(I')$, extend it to B' as above, and verify that our soon-to-be-defined ℓ_B vanishes at $v' := b(B')$. Since I' is a proper subset of I, it follows that $m(I') > m(I)$, hence that the vector $y' := y_{m(I)+1}$ lies in B'. Thus, $q_{y'}(v') = 0$, and hence the polynomial

$$Q_B \, q_{y'}$$

vanishes on $V_\kappa \backslash v$. However, this polynomial may not be in $\mathcal{P}_\kappa$. We, therefore, write $q_{y'}$ as the sum $\ell_B + (q_{y'} - \ell_B)$, with ℓ_B a linear polynomial that is chosen so that: (i) $Q_B \, \ell_B \in \mathcal{P}_\kappa$, and (ii) $q_{y'} - \ell_B$ vanishes on $\mathcal{V}(A)$. Condition (ii) will imply that ℓ_B vanishes on $\mathcal{V}(A) \backslash v$, hence $Q_B \ell_B$ is the sought-for Lagrange polynomial.

To this end, we write $y' = \sum_{x \in B} a(x) x$, for some coefficients $(a(x))_{x \in B}$, and claim first that, if $x \in J$, or, alternatively, if $x \in I'' := \{x \in I : \ \kappa(I \backslash x) < \kappa(I)\}$, then, in each case, q_x vanishes on $\mathcal{V}(A)$. Once we prove it, we define

$$\ell_B := q_{y'} - \sum_{x \in I'' \cup J} a(x) q_x,$$

and conclude that ℓ_B vanishes on $\mathcal{V}(A) \backslash v$. Moreover, since $p_{y'} = \sum_{x \in B} a(x) p_x$, we have that

$$\ell_B = \sum_{x \in I \backslash I''} a(x) p_x - \lambda_{y'} + \sum_{x \in I'' \cup J} a(x) \, \lambda_x = \sum_{x \in I \backslash I''} a(x) q_x - \lambda_{y'} + \sum_{x \in B} a(x) \, \lambda_x \, .$$

Therefore, $Q_B \ell_B$ is a linear combination of Q_B, and $Q_B q_x$, $x \in I \backslash I''$.

Now, Q_B itself lies in $\mathcal{P}_\kappa$: it is the product of q_Z, $Z := X \backslash I$ by a polynomial P of degree $m(I) - (n - \#I) = \kappa(I)$, hence lies in $\mathcal{P}_\kappa$ by Lemma **??**. As to $Q_B q_x$, we can write it as the product $q_{Z \cup x} P$, with Z, P as above. Now, $X \backslash (Z \cup x) = I \backslash x$, and since we assume that $\kappa(I \backslash x) = \kappa(I)$, we still have that $\deg P = \kappa(I \backslash x)$, hence by Lemma **??**, we have $q_{Z \cup x} P \in \mathcal{P}_\kappa$. In conclusion, $Q_B \ell_B \in \mathcal{P}_\kappa$.

So, it remains to show that q_x vanishes on $\mathcal{V}(A)$, whenever $x \in J \cup I''$. If $x \in J$, then trivially, q_x does so, since $J = B \backslash I$ is a common subset for all the bases in A (cf. (ii) in Lemma **??**). Otherwise, $x \in I$, and $\kappa(I \backslash x) < \kappa(I)$. Now, if $q_x(\mathcal{V}(B')) \neq 0$ for some $B' \in A$, then $x \notin I' := B' \cap X$. However, by property (i) of A, $I' \subset I$, and we conclude that $I' \subset I \backslash x$, and, since κ is solid, that $\kappa(I') \leq \kappa(I \backslash x) < \kappa(I)$, in contradiction to (ii) of Lemma **??**. So, q_x vanishes on $\mathcal{V}(A)$ for every $x \in J \cup I''$ and our proof is complete.

4 Incremental assignments

Assuming that the assignment κ is solid, we have proved that

$$\dim \mathcal{P}_\kappa \geq \dim \mathcal{D}_\kappa = \#\mathbb{B}_\kappa.$$

Moreover, the three different proofs for the inequality above that were presented in §3 show that:

Corollary 4.1. *Let κ be a solid assignment and assume that $\dim \mathcal{P}_\kappa = \#\mathbb{B}_\kappa$. Then:*

- *$\mathcal{P}_\kappa$ and $\mathcal{D}_\kappa$ are dual to each other:*
$$\mathcal{P}_\kappa \oplus \mathcal{J}_\kappa = \Pi.$$
- *The homogeneous basis that was constructed in §3.2 is a basis for $\mathcal{P}_\kappa$.*
- *The Lagrange basis that was constructed in §3.3 is a basis for $\mathcal{P}_\kappa$.*[5]

We will show in this section that the equality

$$\dim \mathcal{P}_\kappa = \#\mathbb{B}_\kappa \tag{4.1}$$

is valid once we assume κ to be (solid and) incremental.

Example 4.2 (Continuation of Example **??**)**.** *We revisit the analysis made in Example **??** of $\mathcal{P}_\kappa$. In the setup of that example, we already showed that $\dim \mathcal{P}_\kappa \geq \#\mathbb{B}_\kappa$, which must be the case since κ in that example is solid. Further, the example identifies exactly the cases when equality holds: $\dim \mathcal{P}_\kappa = \dim \mathcal{D}_\kappa$ if and only if $\ell \in \{k, k+1\}$ and $j \in \{k-1, k\}$. It is easy to check that these are exactly the cases when the solid assignment κ is incremental. Thus, for the simple setup of Example **??**, the incrementality of κ is equivalent to the equality (**??**).*

In order to prove that (**??**) holds, we revisit the Lagrange basis that was constructed in §3.3, and that, so far, is only known to be a basis for a subspace of $\mathcal{P}_\kappa$. We will show below that, once κ is assumed to be incremental, a slightly simpler version of this basis can be proved to span the entire $\mathcal{P}_\kappa$ space. To this end, we retain the notations from §3.3, and in particular the set

$$X_B := X \cup Y_{m(B \cap X)}.$$

[5] Since $\mathcal{D}_\kappa$ was proved to be equal to $\Pi(V_\kappa)$, then, once we know that $\mathcal{P}_\kappa$ is dual to $\mathcal{D}_\kappa$, the existence of a Lagrange basis for $\mathcal{P}_\kappa$ follows. However, §3.3 provides an explicit construction of that basis.

The Lagrange basis in §3.3 was indexed by $\mathbb{B}_\kappa$, with the basis polynomial that corresponds to $B \in \mathbb{B}_\kappa$ taking the form of the product of

$$Q_B := q_{X_B \backslash B} \tag{4.2}$$

and a carefully chosen linear polynomial ℓ_B. It is shown in the proof of Proposition **??** that the polynomials Q_B, $B \in \mathbb{B}_\kappa$, lie, each, in $\mathcal{P}_\kappa$. The following theorem claims much more:

Theorem 4.3. *Assume that κ is incremental. Then the polynomials $(Q_B)_{B\in\mathbb{B}_\kappa}$ form a basis for $\mathcal{P}_\kappa$.*

Proof of Theorem **??**. Since we already know that $\dim \mathcal{P}_\kappa \geq \#\mathbb{B}_\kappa$, and since we have exactly $\#\mathbb{B}_\kappa$ functions in the polynomial set $(Q_B)_{B\in\mathbb{B}_\kappa}$, we just need to prove that those polynomials span $\mathcal{P}_\kappa$. Let us denote by $\mathcal{Q}$ their linear span. We need to prove that, for every $Z \subset X$,

$$p_{X\backslash Z}\Pi_{\kappa(Z)} \subset \mathcal{Q}.$$

We first prove that for $I \in \mathbb{I}(X)$,

$$q_{X\backslash I}\Pi_{\kappa(I)} \subset \mathcal{Q}. \tag{4.3}$$

We prove this result by induction on $\#I$. Denote $Y_I := Y_{m(I)} \cup I$. By our assumption on Y, the vectors in the set Y_I are in general position; also $\#(Y_I) = \kappa(I)+n-\#I+\#I = \kappa(I)+n$. Therefore, the polynomials

$$q_W, \quad W \subset Y_I,\ \#W = \kappa(I)$$

form a basis for $\Pi_{\kappa(I)}$ (since they are linearly independent: they form a Lagrange basis over the vertices of the arrangement associated with Y_I). Therefore, once we show that

$$q_{X\backslash I}q_W \in \mathcal{Q},$$

for every W as above, we will conclude that (**??**) holds. Now, if $W \subset Y$ (which is the only case if $I = \varnothing$) then, with $J := Y_{m(I)}\backslash W$, we have that $B := I \cup J \in \mathbb{B}_\kappa$, and that

$$q_{X\backslash I}q_W = Q_B \in \mathcal{Q}.$$

This completes the proof of (**??**) for the initial case of the induction ($I = \varnothing$). For all other I, we need to consider also the case when $W \not\subset Y$. In that case, we write

$$q_{X\backslash I}q_W = q_{X\backslash I}q_{W\cap X}q_{W\backslash X},$$

and set $I' := I\backslash(W \cap X)$. Then, $\#I - \#I' = \#(W \cap X)$, and we conclude from the incremental property of κ that

$$\kappa(I') \geq \kappa(I) - \#(W \cap X).$$

On the other hand, $\#(W\backslash X) = \#W - \#(W \cap X) = \kappa(I) - \#(W \cap X)$. Consequently,

$$\deg q_{W\backslash X} \leq \kappa(I').$$

Thus,

$$q_{X\backslash I}q_W = q_{X\backslash I'}q_{W\backslash X} \in q_{X\backslash I'}\Pi_{\kappa(I')} \subset \mathcal{Q},$$

with the last inclusion by the induction hypothesis (which we are allowed to invoke since I' is a proper subset of I). This completes the proof of (**??**).

Next, let $Z \subset X$, not necessarily independent. We want to show that

$$q_{X \backslash Z} \Pi_{\kappa(Z)} \subset \mathcal{Q}\,. \tag{4.4}$$

In order to prove the above, we consider Z as a matroid, and let $I \in \mathbb{B}(Z) \subset \mathbb{I}(X)$. Since $\operatorname{span} I = \operatorname{span} Z$ and κ is solid, $\kappa(I) = \kappa(Z)$. Hence, by (**??**),

$$q_{X \backslash Z} q_{Z \backslash I} \Pi_{\kappa(Z)} = q_{X \backslash I} \Pi_{\kappa(I)} \subset \mathcal{Q}\,.$$

This implies that

$$q_{X \backslash Z} \Pi_{\kappa(Z)} \operatorname{span}\{q_{Z \backslash I} : I \in \mathbb{B}(Z)\} \subset \mathcal{Q}\,.$$

However, by [**?**], the polynomials

$$q_{Z \backslash I}, \quad I \in \mathbb{B}(Z)$$

form a basis for the central space $\mathcal{P}(Z)$, hence we conclude that

$$q_{X \backslash Z} \Pi_{\kappa(Z)}\, \mathcal{P}(Z) \subset \mathcal{Q}\,.$$

Since $\mathcal{P}(Z)$ always contains the constants, we obtain (**??**).

Finally, we prove that, for every $Z \subset X$,

$$p_{X \backslash Z} \Pi_{\kappa(Z)} \subset \mathcal{Q}\,. \tag{4.5}$$

That will imply that $\mathcal{P}_\kappa \subset \mathcal{Q}$, and will complete the proof of the theorem. We prove (**??**) by induction on $\#(X \backslash Z)$, with the initial case $X = Z$ being trivial since for this case there is no difference between $q_{X \backslash Z}$ and $p_{X \backslash Z}$, hence (**??**) is implied here by (**??**). Now, assume that $Z \neq X$, and write

$$p_{X \backslash Z} = q_{X \backslash Z} + \sum_{Z'} a(Z') p_{X \backslash Z'}, \tag{4.6}$$

with Z' ranging over all the proper supersets of Z. The induction hypothesis implies that $p_{X \backslash Z'} \Pi_{\kappa(Z')} \subset \mathcal{Q}$, for each Z' as above. Since $\kappa(Z') \geq \kappa(Z)$, we have that

$$p_{X \backslash Z'} \Pi_{\kappa(Z)} \subset \mathcal{Q},\ Z' \supset Z,\ Z' \neq Z.$$

Since (**??**) shows that $q_{X \backslash Z} \Pi_{\kappa(Z)} \subset \mathcal{Q}$, we conclude from (**??**) that $p_{X \backslash Z} \Pi_{\kappa(Z)} \subset \mathcal{Q}$. □

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