UNIVERSITY OF WISCONSIN-MADISON
CENTER FOR THE MATHEMATICAL SCIENCES

On ascertaining inductively the dimension of the joint kernel of certain commuting linear operators. II*

Carl de Boor, Amos Ron, and Zuowei Shen

Abstract: Given an index set $\mathbf{X}$, a collection $\mathbb{B}$ of subsets of $\mathbf{X}$, and a collection $(\ell_x : x \in \mathbf{X})$ of commuting linear maps on some linear space, the family of linear operators whose joint kernel $K = K(\mathbb{B})$ is sought consists of all $\ell_A := \prod_{a\in A} \ell_a$ with A any subset of $\mathbf{X}$ which intersects every $B \in \mathbb{B}$. It is shown that certain conditions on $\mathbb{B}$ and ℓ, used in [BRS] to obtain the inequality

$$\dim K(\mathbb{B}) \le \sum_{B\in\mathbb{B}} \dim K(\{B\}),$$

or the corresponding equality, can be weakened. For example, the additional assumption of equicardinality of the elements of $\mathbb{B}$ used there is dropped. However, the notion of 'placeability' introduced in [BRS] continues to play an essential role.

These results are then described in the rather different language employed in (the final version of) [DDM] to facilitate comparisons.

AMS (MOS) Subject Classifications: primary: 47A50; secondary: 05B353, 41A63, 35G05, 47D03.

Key Words and phrases: dimension of nullspaces, solvability, matroidal structure, placeable.

* This work was supported by the National Science Foundation under grants DMS-9000053, DMS-9102857, by the US Army Research Office under grants DAAL03-90-G-0090 and DAAH04-95-1-0089, and by the Alexander-von-Humboldt-Stiftung.

1. Introduction

Let $\mathbf{X}$ be an arbitrary finite set. We start with a collection of **bases** and mean by that any subset $\mathbb{B}$ of $2^{\mathbf{X}}$ with the property that, for any $A, B \in \mathbb{B}$, $A \subset B$ implies $A = B$. We denote by $\mathbb{I} = \mathbb{I}(\mathbb{B})$ the completion of $\mathbb{B}$ under subset formation, calling its elements the **independent** sets. We denote by $\mathbb{S} = \mathbb{S}(\mathbb{B})$ the completion of $\mathbb{B}$ under superset formation, calling its elements the **spanning** sets.

We also consider

$$R := \mathbb{R}[\mathbf{X}],$$

the ring of polynomials (with real coefficients) in the elements of $\mathbf{X}$. In particular, we can identify each subset A of $\mathbf{X}$ with the corresponding monomial $\prod_{x \in A} x$ (or else, more carefully, write m_A for it, as do [DDM]).

Let S be a real linear space, let

$$\ell : \mathbf{X} \to L(S) : x \mapsto \ell_x$$

be some map into the space of linear maps on S, and assume that its images commute. Then, the association

$$\mathbf{X}^\alpha \mapsto \prod_{x \in \mathbf{X}} \ell_x^{\alpha(x)}$$

extends, by linearity, to a ring-homomorphism

$$R \to L(S) : p \mapsto p(\ell).$$

Further, set

$$\mathbb{A} := \mathbb{A}(\mathbb{B}) := \{A \subset \mathbf{X} : \forall \{B \in \mathbb{B}\}\ A \cap B \neq \emptyset\}. \tag{1.1}$$

The joint kernel

$$K(\mathbb{B}) := \bigcap_{A \in \mathbf{A}} \ker \prod_{x \in A} \ell_x \tag{1.2}$$

is of interest. The specific goal is the identification of conditions under which the inequality

$$\dim K(\mathbb{B}) \leq \sum_{B \in \mathbb{B}} \dim K(\{B\}) \tag{1.3}$$

holds, as well as conditions under which there is equality here.

These questions have been studied by several authors (cf. [DM1], [DM2], [S], [JRS], [RJS], [DDM] and [BRS]) in the case that all elements of $\mathbb{B}$ have the same cardinality. In this note, we take up these questions without such assumption of **equicardinality**, as is also done in [DDM].

Specifically, we show (in Section 3) that the results of [BRS] concerning $\dim K(\mathbb{B})$ and involving something called the $\mathbb{E}$-condition there are valid even if $\mathbb{B}$ is not equicardinal. These results rely on induction on $\#\mathbb{B}$, relating the dimension of $K(\mathbb{B})$ to the dimension of $K(\mathbb{B}_{|x})$ and of $K(\mathbb{B}_{\backslash x})$. Here,

$$\mathbb{B}_{|x} := \{B \in \mathbb{B} : x \in B\}, \qquad \mathbb{B}_{\backslash x} := \{B \in \mathbb{B} : x \notin B\}.$$

More generally, for any subsets Y and Z of $\mathbf{X}$, we set

$$\mathbb{B}_{|Y} := \{B \in \mathbb{B} : Y \subset B\}, \qquad \mathbb{B}_{\backslash Z} := \{B \in \mathbb{B} : Z \cap B = \emptyset\} = \{B \in \mathbb{B} : B \subset (\mathbf{X} \backslash Z)\}.$$

Note that these two operations, of *restriction* and *deletion*, commute.

In the process, we abandon the $\mathbb{E}$-condition of [BRS] in favor of something we call the tree-condition which, in the equicardinal case, was shown in [BRS] to be equivalent to the $\mathbb{E}$-condition but which we now find much handier to use, even in the equicardinal case. The necessary details are developed in Section 2.

In Section 4, we compare the results of Section 3 with the related results in the final version of [DDM]. In particular, we relate our tree-condition to a basic condition used (but not named) in [DDM] which we dub here the shell-condition and which is shown in [DS] to be the most general condition under which one can hope to prove (1.3) inductively.

The relevant argument in [DDM] starts with the observation that

$$K(\mathbb{B}) = \bigcap_{p\in I} \ker p(\ell),$$

with

$$I := I(\mathbb{B}) := \mathrm{ideal}(\mathbb{A}) \tag{1.4}$$

the ideal generated by the $A \in \mathbb{A}$. This makes it possible to exploit the isomorphism between $K(\mathbb{B})$ and the space $\hom(R/I, S)$ of R-homomorphisms. We recast our results into this language, in order to facilitate the comparison of results.

2. Placeability and the tree-condition

The concepts of placeability, the $\mathbb{E}$ set, the $\mathbb{E}$-condition, and the $\mathbb{E}$-tree were introduced, in that order, in [BRS] under the assumption that all the elements of $\mathbb{B}$ have the same cardinality. However, an examination of the development in [BRS] and of the proofs of the major results shows that the equicardinality of $\mathbb{B}$ is not used.

Definition 2.1. *We say that Y is* **placeable into** *B if $Y \cup C \in \mathbb{B}$ for some $C \subseteq B$. If Y is placeable into every $B \in \mathbb{B}$, then we say that Y is placeable (in $\mathbb{B}$), or,* **$\mathbb{B}$-placeable**.

We note that, in contrast to [BRS], we do not assume in this definition that $\#Y = \#(B\backslash C)$. E.g., with $\mathbb{B} = \{\{a\}, \{b, c\}\}$, a is $\mathbb{B}$-placeable.

Definition 2.2. *A* **$\mathbb{B}$-tree** *is any binary tree with the following properties:*

(i) The nodes are of the form $\mathbb{B}_{|Y\backslash Z}$ for certain Y, $Z \subset \mathbf{X}$ with $Y \cap Z = \emptyset$.

(ii) Each node is either a leaf, in which case it contains fewer than 2 elements, or else, it is the disjoint union of its two children, $\mathbb{B}_{|Y\cup b\backslash Z}$ and $\mathbb{B}_{|Y\backslash Z\cup b}$ (the latter may possibly be empty), for some $b \in \mathbf{X}\backslash(Y \cup Z)$.

(iii) $\mathbb{B}$ is the root of this tree.

The nonempty leaves of a $\mathbb{B}$-tree constitute the partition of $\mathbb{B}$ into its elements.

Definition 2.3. *A $\mathbb{B}$-tree is* **placeable** *if, for each of its nodes, the element used to split that node is placeable in that node.*

We say that $\mathbb{B}$ satisfies the **tree-condition** *if there is a placeable $\mathbb{B}$-tree.*

Since the tree-condition involves placeability within each node, any branch of a placeable tree is itself placeable. Hence, any node of a placeable $\mathbb{B}$-tree satisfies itself the tree-condition. Conversely, if b is placeable, and both $\mathbb{B}_{|b}$ and $\mathbb{B}_{\backslash b}$ satisfy the tree-condition, then so does $\mathbb{B}$.

A placeable $\mathbb{B}$-tree is what is called an $\mathbb{E}$-tree in [BRS] *except* for the more general definition 2.1 of placeability used here, and for the fact that empty nodes are allowed here. The latter is a convenience in certain proofs. However, we also have the following.

Proposition 2.4. *If $\mathbb{B} \neq \emptyset$ satisfies the tree-condition, then there is a placeable $\mathbb{B}$-tree without any empty nodes.*

Proof: Any $\mathbb{B}$-tree is necessarily finite since each node is associated with its distinct pair (Y, Z) of disjoint subsets of $\mathbf{X}$, and $\mathbf{X}$ is finite. Hence, it is sufficient to prove that the tree obtained from a placeable $\mathbb{B}$-tree by removal of an empty node is again a placeable $\mathbb{B}$-tree. But this is obvious since $\mathbb{B}_{|Y \cup b \backslash Z} = \mathbb{B}_{|Y \backslash Z}$ in case $\mathbb{B}_{|Y \backslash Z \cup b}$ is empty. $\square$

Proposition 2.5. *If $\mathbb{B}$ satisfies the tree-condition, then, for arbitrary $x \in \mathbf{X}$, so does $\mathbb{B}_{\backslash x}$.*

Proof: It is sufficient to show that the tree, obtained from the placeable $\mathbb{B}$-tree by removing all $B \in \mathbb{B}_{|x}$, is a placeable $\mathbb{B}_{\backslash x}$-tree. For this, consider the effect of this action on the node $\mathbb{B}_{|Y \backslash Z}$ of our $\mathbb{B}$-tree.

There is no difficulty with nodes which are entirely outside $\mathbb{B}_{|x}$ as they will be unchanged. There is also no difficulty with nodes which are reduced to fewer than 2 elements.

This leaves the nodes which will be reduced but still contain at least 2 elements. For this to happen, our node cannot be a leaf, i.e., it is being split by some node-placeable b, and we must have $x \in \mathbf{X} \backslash (Y \cup Z)$. If $x = b$, then the node is simply replaced by its right child, $\mathbb{B}_{|Y \backslash Z \cup x} = (\mathbb{B}_{\backslash x})_{|Y \backslash Z}$, and there is no problem. In the contrary case, we note that b is still placeable in the reduced node since there is, for every B in the reduced node $(\mathbb{B}_{\backslash x})_{|Y \backslash Z}$, some B' in the unreduced node of the form $B' = Y \cup b$ with $Y \subseteq B$ and, since $b \neq x$, this B' cannot contain x, hence is also in the reduced node. $\square$

We now relate the tree-condition to what was called the $\mathbb{E}$-condition in [BRS]. For this, we single out the following subset of $2^{\mathbf{X}}$.

Definition 2.6. *Let $\mathbb{F} = \mathbb{F}(\mathbb{B})$ be the collection of all those $C \subset \mathbf{X}$ satisfying the following two conditions:*
(a) C is $\mathbb{B}$-placeable;
(b) $\mathbb{B}_{|C}$ satisfies the tree-condition.

Corollary 2.7. *If $C \in \mathbb{F}(\mathbb{B})$, then, for arbitrary $x \in \mathbf{X} \backslash C$ with $\mathbb{B}_{\backslash x} \neq \emptyset$, $C \in \mathbb{F}(\mathbb{B}_{\backslash x})$.*

Proof: Since C is placeable, there exists, for arbitrary $B \in \mathbb{B}$, some $B' = C \cup Y \in \mathbb{B}$ with $Y \subset B$. If now $B \in \mathbb{B}_{\backslash x}$, then $x \notin Y$, while $x \notin C$ by assumption, hence $x \notin B'$, i.e., $B' \in \mathbb{B}_{\backslash x}$. Thus, C is $\mathbb{B}_{\backslash x}$-placeable. Further, $\mathbb{B}_{\backslash x|C}$ satisfies the tree-condition by Proposition 2.5 since $B_{|C}$ satisfies the tree-condition. $\square$

Now recall the following definition from [BRS] which is stated and used there only for equicardinal $\mathbb{B}$ but makes sense for more general $\mathbb{B}$ since the cardinality of the elements of $\mathbb{B}$ is not referred to in the definition.

Definition 2.8. *Let $\mathbb{E} = \mathbb{E}(\mathbb{B})$ be the collection of all $C \in \mathbb{I}$ which are either in $\mathbb{B}$ or else there is some $b \in \mathbf{X} \backslash C$, called a* **$\mathbb{B}$-extender for** *$C$, which satisfies the following two conditions:*
(i) $C \cup b \in \mathbb{E}$;
(ii) if $\mathbb{B}_{\backslash b} \neq \emptyset$, then $C \in \mathbb{E}(\mathbb{B}_{\backslash b})$.

It can be shown that $\mathbb{F} = \mathbb{E}$. However, in the sequel, we only need the fact that the set $\mathbb{F}$ has all the properties of the set $\mathbb{E}$.

Theorem 2.9. *Every $C \in \mathbb{F}(\mathbb{B})$ is in $\mathbb{I}$, and is either in $\mathbb{B}$, or else there exists some $b \in \mathbf{X} \backslash C$ which satisfies the following two conditions:*
(i) $C \cup b \in \mathbb{F}$;

(ii) *if* $\mathbb{B}_{\backslash b} \neq \emptyset$, *then* $C \in \mathbb{F}(\mathbb{B}_{\backslash b})$.

Proof: Since any element in $\mathbb{F}(\mathbb{B})$ is $\mathbb{B}$-placeable, $\mathbb{F} \subset \mathbb{I}$.

For an arbitrary $C \in \mathbb{F}(\mathbb{B})\backslash\mathbb{B}$, let b be the $\mathbb{B}_{|C}$-placeable element which splits the node $\mathbb{B}_{|C}$ in the placeable $\mathbb{B}_{|C}$-tree. Then, since C is $\mathbb{B}$-placeable and b is $\mathbb{B}_{|C}$-placeable, $C \cup b$ is $\mathbb{B}$-placeable. Also, $\mathbb{B}_{|C\cup b}$, as a node in the placeable $\mathbb{B}_{|C}$-tree, satisfies the tree-condition. This verifies that $C \cup b \in \mathbb{F}$, hence proves (i).

Assertion (ii) follows from Corollary 2.7. □

3. Dimension estimates

In this section, we point out that certain results in [BRS] concerning the dimension of the kernel $K(\mathbb{B})$ defined in (1.2) are valid in the present more general context of a set of bases of arbitrary cardinality. To avoid repetition, we merely state here the results in question and point out where, if at all, in the proofs given in [BRS] the present more general setup requires modifications.

Here is the basic result.

Theorem 3.1 (see [BRS: Theorem 2.16]). *If* y *is* $\mathbb{B}$*-placeable, then*

$$\dim K(\mathbb{B}) \le \dim K(\mathbb{B}_{|y}) + \dim K(\mathbb{B}_{\backslash y}), \tag{3.2}$$

with equality if and only if ℓ_y *maps* $K(\mathbb{B})$ *onto* $K(\mathbb{B}_{\backslash y})$.

Its proof requires no modification, except for the fact that it relies on the following Lemma whose proof, however, requires no modification, either.

Lemma 3.3 (see [BRS: Lemma 2.15]). *Let* $Y \subseteq \mathbf{X}$, *and set* $\mathbb{Y} := \mathbb{A}(\{Y\})$. *Then,* $\mathbb{A}(\mathbb{B}_{|Y}) \supseteq \mathbb{A}(\mathbb{B}) \cup \mathbb{Y}$, *with equality if and only if* Y *is* $\mathbb{B}$*-placeable. In the latter case,*

$$K(\mathbb{B}_{|Y}) = K(\mathbb{B}) \cap \bigcap_{y \in Y} \ker \ell_y.$$

Equality in (3.2) is equivalent to the equation

$$\ell_y ? = f$$

having solutions in $K(\mathbb{B})$ for any $f \in K(\mathbb{B}_{\backslash y})$. For this reason, our ℓ-conditions, i.e., the conditions imposed on ℓ under which we can derive useful statements concerning $\dim K(\mathbb{B})$, are connected to the solvability of systems of the form

$$(C, \varphi): \quad \ell_c ? = \varphi_c, \quad c \in C, \tag{3.4}$$

with $C \subseteq \mathbf{X}$ and φ a map into S and defined (at least) on C. For their statement, we need the following definition.

Definition ([BRS]). *We call the system* (C, φ)
(i) **special**, *or, more explicitly,* $\mathbb{B}$**-special** *if* $\varphi_c \in K(\mathbb{B}_{\backslash c})$, *all* $c \in C$;
(ii) **compatible** *if* $\ell_c \varphi_b = \ell_b \varphi_c$ *for all* $c, b \in C$;
(iii) **independent**, *resp.* **basic**, *if* $C \in \mathbb{I}$, *resp.* $C \in \mathbb{B}$.

Solvability condition 3.5 (see [BRS: 3.2]). *Any special compatible basic system is solvable.*

The main result, Theorem 3.7 below, relies on the following.

Proposition 3.6 (see [BRS: Proposition 3.18]). *If the solvability condition 3.5 holds, then, any special compatible system (C, φ) with $C \in \mathbb{F}$ can be extended to a special compatible basic system, hence has solutions in $K(\mathbb{B})$.*

To be sure, [BRS: Proposition 3.18] refers to the set $\mathbb{E}$ rather than the set $\mathbb{F}$. However, the proof uses only those properties of $\mathbb{E}$ which we showed in Theorem 2.9 to hold for $\mathbb{F}$, hence goes through without explicit change in the present more general context.

Theorem 3.7 (see [BRS: Theorem 3.19]).

(a) Assume that the solvability condition 3.5 holds. Then, for any $y \in \mathbb{F}$, ℓ_y maps $K(\mathbb{B})$ onto $K(\mathbb{B}_{\backslash y})$, and

$$\dim K(\mathbb{B}) = \dim K(\mathbb{B}_{|y}) + \dim K(\mathbb{B}_{\backslash y}). \tag{3.8}$$

(b) Assume that $\mathbb{B}$ satisfies the tree-condition. Then

$$\dim K(\mathbb{B}) \leq \sum_{B \in \mathbb{B}} \dim K(\{B\}),$$

with equality in case the solvability condition 3.5 holds.

Proof: The proof for (a) in [BRS: Theorem 3.19] uses the Proposition 3.6, and, in that connection, makes use of the fact that $\{y\} \in \mathbb{F}$ implies that $\{y\}$ is placeable (which is part of the definition of $\mathbb{F}$ here), and then relies on Theorem 3.1.

The statement of (b) in [BRS: Theorem 3.19] assumes that $\mathbb{B}$ satisfies a certain condition (the $\mathbb{E}$-condition) and then relies on a certain proposition to deduce from this the existence of a placeable element b for which both $\mathbb{B}_{|b}$ and $\mathbb{B}_{\backslash b}$ also satisfy that $\mathbb{E}$-condition, thus making a proof by induction on $\#\mathbb{B}$ possible. In the present context, the existence of a placeable b for which both $\mathbb{B}_{|b}$ and $\mathbb{B}_{\backslash b}$ satisfy again the tree-condition is an immediate consequence of the assumption that $\mathbb{B}$ satisfies the tree-condition. The only possible hitch in the remainder of the proof as given in [BRS] is the fact that, in order to apply the induction hypothesis, we must be sure that both $\mathbb{B}_{|b}$ and $\mathbb{B}_{\backslash b}$ are strictly smaller than $\mathbb{B}$. While the tree-condition does not directly guarantee this, Proposition 2.4 ensures that we may always assume that our placeable $\mathbb{B}$-tree has no empty nodes. With these modifications, the proof for (b) in [BRS: Theorem 3.19] applies verbatim to the proof of (b) here. □

4. Relation to [DDM]

[BRS] was written without knowledge of the final version of [DDM] which differed in significant detail from the preliminary version available to the authors of [BRS]. In particular, in the present context, the final version has the following general results.

Let

$$I := I(\mathbb{B}) := \text{ideal}(\mathbb{A}) \tag{4.1}$$

be the ideal in $R = \mathbb{R}[\mathbf{X}]$ generated by the $A \in \mathbb{A}$. Then

$$I(\mathbb{B}) = \cap_{B \in \mathbb{B}} P_B,$$

with P_B the prime ideal generated by (the elements of) B. Further,

$$K(\mathbb{B}) = \bigcap_{p\in I} \ker p(\ell) =: \ker I,$$

hence $K(\mathbb{B})$ is isomorphic to the linear space $\hom(R/I, S)$ of R-homomorphisms between the R-modules R/I and S, the latter an R-module via

$$R \times S \to S : (p, s) \mapsto p \cdot s := p(\ell)s.$$

Indeed, take $F : \hom(R/I, S) \to \ker I : f \mapsto 1_f := f(1 + I)$. $f \in \hom(R/I, S) \implies f(p + I) = f(p(1+I)) = p \cdot f(1+I) = p \cdot 1_f$, hence $f \in \hom(R/I, S)$ is entirely determined by 1_f; in particular, $f = g$ if and only if $1_f = 1_g$. Conclusion: F is 1-1. Also, for all $p \in I$, $0 = p \cdot 1_f$, hence $1_f \in \ker I$, i.e., F maps $\hom(R/I, S)$ 1-1 into $\ker I$. If $s \in \ker I$, then $f_s : (p + I) \mapsto p \cdot s$ is well defined since $p \in I$ implies $p \cdot s = 0$, hence $f_s \in \hom(R/I, S)$ and $F(f_s) = s$; thus F is onto.

[DDM] make use of the fact that any exact sequence

$$0 \to N' \to N \to N'' \to 0 \tag{a(4.2)}$$

between R-modules induces a corresponding exact sequence

$$\begin{aligned} 0 \to \hom(N'', S) \hookrightarrow \hom(N, S) \to \hom(N', S) \to \\ \mathrm{Ext}^1(N'', S) \to \mathrm{Ext}^1(N, S) \to \mathrm{Ext}^1(N', S) \to \cdots \end{aligned} \tag{4.2b}$$

for the corresponding spaces of R-homomorphisms to some fixed R-module S and their Ext^1-spaces.

In particular, for any particular $r \in R\backslash I$, we have the exact sequence

$$0 \to R/(I{:}r) \xrightarrow{i} R/I \xrightarrow{j} R/(I + Rr) \to 0,$$

in which

$$I{:}r := \{p \in R : rp \in I\},$$

$$i : y + (I{:}r) \mapsto yr + I, \qquad j : z + I \mapsto z + (I + Rr).$$

Hence, correspondingly,

$$\dim \hom(R/I, S) \le \dim \hom(R/(I{:}r), S) + \dim \hom(R/(I + Rr), S) \tag{4.3}$$

with equality if and only if $\mathrm{Ext}^1(R/(I + Rr), S) = 0$.

In order to relate this to the results in [BRS] and in Section 3, we introduce the following

Placeable-Split Condition 4.4. *$\mathbb{B}$ is the disjoint union of two nontrivial subsets, $\mathbb{B}'$ and $\mathbb{B}''$, and there is some $L \in \mathbb{A}(\mathbb{B}')$ which meets none of the $B \in \mathbb{B}''$, yet all its elements are placeable into every $B \in \mathbb{B}''$. We call $\mathbb{B}'$ the* **left part**, *and $\mathbb{B}''$ the* **right part**, *of such a split.*

If $\mathbb{B}$ satisfies the tree-condition (see Definition 2.2), then the placeable-split-condition is satisfied with $\mathbb{B}'' = \mathbb{B}_{\backslash b}$, where b is the placeable element used to split the root node of the placeable $\mathbb{B}$-tree. In this case, we can choose $L = \{b\}$.

The weakest condition imposed on $\mathbb{B}$ in [DDM] is the following (see [DDM: (7.6)]):

Shell-Condition. *There is an ordering* $(B_1, B_2, \ldots)$ *of the elements of* $\mathbb{B}$ *so that, for all* $i < j$, *there exists some* $b \in B_i$ *and some* $i' < j$ *for which* $\{b\} = B_{i'} \backslash B_j$.

If $\mathbb{B}$ satisfies the shell-condition, then the placeable-split condition is satisfied with $\mathbb{B}'' = \{B\}$ and B the last term in the ordering which figures in the shell-condition. In this case, the set L consists of all the elements of $\mathbf{X}\backslash B$ which are placeable into B (of which, by assumption, each $B' \in \mathbb{B}' = \mathbb{B}\backslash\{B\}$ contains at least one).

Lemma 4.5. *If* $\mathbb{B}$ *satisfies the placeable-split condition, then, with* $\mathbb{B}'$, $\mathbb{B}''$, L *as in 4.4 and* $r := L$,

$$I(\mathbb{B}) + rR = I(\mathbb{B}'), \quad I{:}r = I(\mathbb{B}''),$$

hence

$$0 \to R/I(\mathbb{B}'') \to R/I(\mathbb{B}) \to R/I(\mathbb{B}') \to 0.$$

Proof: We begin with the claim that

$$L = \bigcap_{A \in \mathbb{A}(\mathbb{B}')\backslash\mathbb{A}(\mathbb{B})} A. \tag{4.6}$$

Indeed, any $A \in \mathbb{A}(\mathbb{B}')\backslash\mathbb{A}(\mathbb{B})$ meets every $B' \in \mathbb{B}'$ but fails to meet some $B \in \mathbb{B}''$. On the other hand, every $b \in L$ is placeable into this B, hence, there exists some $B_b \in \mathbb{B}$ with $B_b\backslash B = \{b\}$. Since L fails to meet any element of $\mathbb{B}''$ yet contains b, it follows that $B_b \in \mathbb{B}'$, hence B_b is met by A, yet no element of B_b other than b can be in A (since A fails to meet B). Consequently, $b \in A$, and, as $b \in L$ was arbitrary here, we conclude that $L \subset A$ for any $A \in \mathbb{A}(\mathbb{B}')\backslash\mathbb{A}(\mathbb{B})$, hence L must lie in their intersection. On the other hand, since L itself is in $\mathbb{A}(\mathbb{B}')\backslash\mathbb{A}(\mathbb{B})$, the intersection cannot be bigger than L.

It follows that

$$\mathrm{ideal}(\mathbb{A}(\mathbb{B}), L) \supset \mathrm{ideal}(\mathbb{A}(\mathbb{B}')),$$

while the converse inclusion is trivial (since $\mathbb{A}(\mathbb{B}) \cup \{L\} \subset \mathbb{A}(\mathbb{B}')$).

This proves that $I(\mathbb{B}') = I(\mathbb{B}) + rR$, with $r = L$.

On the other hand, with this choice for r,

$$I{:}r = I(\mathbb{B}'') = \bigcap_{B \in \mathbb{B}''} P_B,$$

since $r \in \cap_{B' \in \mathbb{B}'} P_{B'}$, while, for any $B \in \mathbb{B}''$, $L \cap B = \emptyset$, hence $rp \in P_B$ if and only if $p \in P_B$. □

In view of (4.3), the lemma implies the following.

Corollary 4.7. *Under the placeable-split condition 4.4,*

$$\dim K(\mathbb{B}) \leq \dim K(\mathbb{B}') + \dim K(\mathbb{B}''), \tag{4.8}$$

with equality if and only if $\mathrm{Ext}^1(R/I(\mathbb{B}'), S) = 0$.

If the shell-condition holds, then $\mathbb{B}''$ is a singleton, while $\mathbb{B}'$ satisfies the shell-condition, hence the induction argument is obvious.

The induction argument is just as obvious when the tree-condition holds, since then both $\mathbb{B}'$ and $\mathbb{B}''$ satisfy the tree-condition.

Under either condition, induction gives the basic inequality

$$\dim K(\mathbb{B}) \le \sum_{B\in\mathbb{B}} \dim K(\{B\}). \tag{4.9}$$

The tree-condition is explicitly stated in terms of a particular $\mathbb{B}$-tree. Also the shell-condition leads to a tree whose nodes consist of subsets of $\mathbb{B}$, with each nonterminal node the disjoint union of its two children. However, the resulting tree is a very simple tree, since each nonterminal node has at least one leaf among its two children.

Finally, at the root level, both the tree-condition and the shell-condition are special cases of the placeable-split condition. This suggests the following more general tree-condition:

Split-Tree-Condition. *There is a* **placeable-split tree**, *i.e., a binary tree with the following properties:*
(i) *The nodes are subsets of* $\mathbb{B}$.
(ii) *Each node is either a leaf, in which case it contains exactly one element, or else its two children provide a placeable split for it.*
(iii) $\mathbb{B}$ *is the root of the tree.*

Proposition 4.10. *If* $\mathbb{B}$ *satisfies the split-tree condition, then (4.9) holds.*

It is obvious that $\mathbb{B}$ satisfies the split-tree-condition if and only if there is a placeable split for it, with both parts satisfying the split-tree-condition.

Proposition 4.11. *The split-tree-condition is equivalent to the shell-condition.*

Proof: We already observed that the shell-condition implies the split-tree condition. For the converse, let $(B_1, B_2, \ldots)$ be the ordering of the elements of $\mathbb{B}$ as they appear, from left to right, on a placeable-split tree. If $i < j$, let $\widehat{\mathbb{B}}$ be the root of the smallest subtree containing both leaves, B_i and B_j. Then, necessarily, B_i is in the left part and B_j is in the right part of the split of $\widehat{\mathbb{B}}$, hence B_i must contain some element, b say, not in any element of the right part, which is $\widehat{\mathbb{B}}$-placeable into B_j. This implies that there exists $B \in \widehat{\mathbb{B}}$ with $B\backslash B_j = \{b\}$, and, since $b \in B \in \widehat{\mathbb{B}}$, B must be in the left part. In particular, we must have $B = B_{i'}$ for some $i' < j$. □

Proposition 4.12. *The tree-condition is strictly stronger than the shell-condition.*

Proof: We already observed that the tree-condition implies the split-tree condition. On the other hand, the converse does not hold in general, since $\mathbb{B}$ may satisfy the shell-condition even though none of the elements of $\mathbf{X}$ is even placeable, as the following example illustrates. □

Example 4.13. Here is an example of a $\mathbb{B}$ which is even equicardinal and which satisfies the shell-condition (for the given ordering of its elements), but has *no* placeable element, hence, *a fortiori*, does not satisfy the tree-condition. (The set obtained by leaving off the last element in the given sequence does satisfy the tree-condition.)

In studying this example, it may be helpful to call $b \in \mathbf{X}$ **left-placeable into** $B \in \mathbb{B}$ if, with $B = B_j$, there exists $k < j$ so that $B_k\backslash B = \{b\}$. In these terms, the shell-condition says that, for every $i < j$, B_i contains some element left-placeable into B_j.

The example uses $\mathbf{X} = 1234567 := \{1, 2, 3, 4, 5, 6, 7\}$. Each item in the listing below is of the form (B, L, U), with B the particular element of $\mathbb{B}$, L the set of elements left-placeable into B, and U the set of elements that cannot be placed into B. If (B_j, L_j, U_j) denotes the jth term, then the shell-condition is equivalent to having

$$B_i \cap L_j \neq \emptyset, \quad \forall i < j,$$

while the tree-condition would imply that

$$\cup_j U_j \neq \mathbf{X}.$$

The U_j's given in the second row suffice to verify violation of this condition.

Here is the list:

$$(123, , 7),\ (234, 1,),\ (245, 3, 6),\ (135, 2,),\ (145, 23, 6),$$

$$(236, 14, 57),\ (136, 25, 47),\ (457, 12, 36),\ (157, 34, 26),\ (247, 35, 16).$$

□

As to the *history* of the shell-condition, it appears first (and unnamed) in [DDM: (7.6)]. We gave it here this particular name since, as observed by R. Simon [Si], the condition is equivalent to the shellability of the simplicial complex

$$\Delta_{\backslash \mathbb{B}} := \{A \subset \mathbf{X} : \exists\{B \in \mathbb{B}\}\ A \cap B = \emptyset\}.$$

Given that the split-tree-condition is equivalent to the shell-condition, it is natural to ask why one would ever consider stronger conditions, such as the tree-condition. One reason is the following. While every placeable-split tree for $\mathbb{B}$ provides the basic *in*equality (4.9), a proof of *equality* depends, offhand, on the particulars of the placeable-split tree used since there will be equality in (4.9) if and only if there is equality at every nonterminal node $\widehat{\mathbb{B}}$ of that tree. Equality at the node $\widehat{\mathbb{B}}$ is equivalent to having $\mathrm{Ext}^1(R/I(\widehat{\mathbb{B}}'), S) = 0$, and (sufficient) conditions for the latter may well depend on the details of $\widehat{\mathbb{B}}$.

To make this point, we now relate quickly the results from [DDM] concerning equality in (4.9) and urge the reader to compare these with our results, especially Theorem 3.7.

[DDM] define an R-module M to be **$\mathbb{B}$-pure** if there exists a **filtration**, i.e., an increasing sequence

$$0 = M_0 \subset \cdots \subset M_l = M$$

of submodules, so that, for each i, M_i/M_{i-1} is isomorphic to R/P_{B_i} for some $B_i \in \mathbb{B}$. Further, for any $B \in \mathbb{B}$, P_B is a prime ideal, hence the number

$$m_B^M := \#\{i : B_i = B\}$$

is independent of the particular filtration used. In particular, they remark that the R-module

$$M(\mathbb{B}) := R/I(\mathbb{B})$$

is $\mathbb{S}(\mathbb{B})$-pure, and that

$$m_B^{M(\mathbb{B})} = 1, \quad \forall B \in \mathbb{B}. \tag{4.14}$$

A submodule N of a $\mathbb{B}$-pure module M appears in such a filtration if and only if both, N itself and the factor module M/N, are $\mathbb{B}$-pure. Any such submodule of a $\mathbb{B}$-pure module is called a **$\mathbb{B}$-submodule**. Finally, if, for every $\mathbb{B}$-submodule N of the $\mathbb{B}$-pure module M, the restriction-homomorphism

$$\hom(M, S) \to \hom(N, S)$$

is onto, then M is called **$(\mathbb{B}, S)$-injective**.

Lemma 4.15 (see [DDM: Lemma 7.2]). *Assume that* $\dim \hom(I/P_B, S) < \infty$ *for all* $B \in \mathbb{B}$. *Then, a* $\mathbb{B}$*-pure module* M *is* $(\mathbb{B}, S)$*-injective if and only if*

$$\dim \hom(M, S) = \sum_{B \in \mathbb{B}} m_B^M \cdot \dim \hom(R/P_B, S). \tag{4.16}$$

Proposition 4.17 ([DDM: Prop. 7.1]). *If* M *is* $\mathbb{B}$*-pure, then*

$$\dim \hom(M, S) \in \sum_{B \in \mathbb{B}} m_B^M \cdot (\dim \hom(I/P_B, S) - [0, \dim \operatorname{Ext}^1(M, S_{\mathbb{B}})]).$$

Moreover, if $\operatorname{Ext}^1(R/P_B, S_{\mathbb{B}}) = 0$ *for every* $B \in \mathbb{B}$, *then* $\operatorname{Ext}^1(M, S) = 0$ *for every* $\mathbb{B}$*-pure module* M.

Here,

$$S_{\mathbb{B}} := \{ s \in S : \exists\{n \in \mathbb{N}\} \, \forall\{p \in I(\mathbb{B})\} \;\; p^n \cdot s = 0 \} \;=\; \bigcup_{n=1}^{\infty} \bigcap_{A \in \mathbf{A}} \ker A(\ell)^n.$$

Also, note that, by (4.14), all the m_B^M appearing here equal 1 in the only case of interest here, namely when $M = M(\mathbb{B}) = R/I(\mathbb{B})$.

This leaves the question under what conditions a module is $\mathbb{B}$-pure. This question is taken up in [DS] whose purpose it is to characterize a more general property introduced there called *cleanness*. In the present context, the result of [DS] of immediate interest (see, also, [DDM: (7.6)]) is the following (note that the prime ideal in R generated by the elements of some $A \subset \mathbf{X}$, denoted in the present paper and in [DDM] by P_A, is denoted in [DS] by $P_{\mathbf{X} \backslash A}$).

Proposition 4.18 ([DS]). *The* R*-module* $M(\mathbb{B}) = R/I(\mathbb{B})$ *is* $\mathbb{B}$*-pure if and only if* $\mathbb{B}$ *satisfies the shell-condition.*

By Corollary 4.7, the condition $\operatorname{Ext}^1(R/I(\mathbb{B}'), S) = 0$ is equivalent to having equality in the inequality

$$\dim K(\mathbb{B}) \le \dim K(\mathbb{B}') + \dim K(\mathbb{B}'')$$

obtainable for placeable splits $\mathbb{B} = \mathbb{B}' \cup \mathbb{B}''$. Any time we have a placeable-split tree with all leaves singletons, we get (4.9). Consequently, if we actually get equality in (4.9), then there has to be equality at all the nodes in *any* available placeable-split tree.

In particular, we know that the tree-condition gives such a placeable-split tree, and, if also the solvability condition 3.5 holds, then Theorem 3.7 ensures equality in (4.9). It follows that, under these assumptions, we must have $\operatorname{Ext}^1(R/I(\widehat{\mathbb{B}}'), S) = 0$ for many $\widehat{\mathbb{B}}$ (namely for all the nodes $\widehat{\mathbb{B}}$ involving some restriction). However, we will not get $\operatorname{Ext}^1(R/I(\mathbb{B}), S) = 0$ under the solvability condition 3.5, as shown by the following example.

Example. Let $X := \{1, 2\}$, $\mathbb{B} := \{X\}$, $\ell_1 = \ell_2 := \frac{d}{dx}$, and the linear space S be the space Π of all univariate polynomials. Then, $\mathbb{B}$ satisfies the tree-condition. Since $K(\mathbb{B}_{\backslash 1}) = K(\mathbb{B}_{\backslash 2}) = \{0\}$, also the solvability condition 3.5 holds. However, the system

$$\ell_1 ? = 1$$
$$\ell_2 ? = 2$$

has no solution in S, even though it is compatible. This implies, by [DDM: Remark 4.1], that

$$\operatorname{Ext}^1(R/I(\mathbb{B}), S) \neq 0.$$

□

However, even under the split-tree-condition, the conclusion $\mathrm{Ext}^1(R/I(\mathbb{B}), S) = 0$ is immediate if the following stronger solvability condition holds, which, by [DDM: Remark 4.1], is equivalent to the assumption that $\mathrm{Ext}^1(R/I(\{B\}), S) = 0$ for all $B \in \mathbb{B}$.

Solvability condition 4.19. *Any compatible basic system is solvable (note that being 'special' is not part of the assumption).*

For, in this case, we have $\mathrm{Ext}^1(R/I(\widehat{\mathbb{B}}), S) = 0$ for all *leaves* $\widehat{\mathbb{B}}$ of the given placeable-split tree. Hence, assuming by induction that, for a given *node* $\widehat{\mathbb{B}}$ of the tree, we have

$$\mathrm{Ext}^1(R/I(\widehat{\mathbb{B}}'), S) = 0 = \mathrm{Ext}^1(R/I(\widehat{\mathbb{B}}''), S),$$

we conclude from (4.2b) (true since (4.2a) holds) that also

$$\mathrm{Ext}^1(R/I(\widehat{\mathbb{B}}), S) = 0,$$

which finishes the inductive proof of the following.

Proposition 4.20. *If* $\mathbb{B}$ *satisfies the split-tree-condition, and* $\mathrm{Ext}^1(R/I(\{B\}), S) = 0$ *for all* $B \in \mathbb{B}$*, then* $\mathrm{Ext}^1(R/I(\widehat{\mathbb{B}}), S) = 0$ *for any node* $\widehat{\mathbb{B}}$ *in any split-placeable tree for* $\mathbb{B}$.

5. An example: box splines

The topic discussed in the present article was motivated by studies in spline theory, particularly, in box spline theory. In this section, we briefly describe the pertinent spline problem, and discuss the relevance of the results of [BRS], [DDM] and the present article to that problem.

A **box spline** is a compactly supported piecewise-polynomial function in $d \geq 1$ variables. It is defined with respect to a spanning multiset of *directions*

$$\Xi \subset \mathbb{R}^d \backslash 0$$

as the function M_Ξ whose Fourier transform has the form

$$\widehat{M_\Xi}(\omega) = \prod_{\xi \in \Xi} \int_0^1 e^{-it\xi\cdot\omega}\,\mathrm{d}t.$$

Our present interest is in the space

$$\Pi_\Xi$$

of all *polynomials* that are writeable as a (necessarily, infinite) linear combination of the integer translates of M_Ξ. Specifically, we seek a formula for

$$\dim \Pi_\Xi$$

that invokes directly the structure of the set Ξ without requiring subtle information about the underlying box spline M_Ξ. We are guided by the following characterization of Π_Ξ from [RS]; in the case when $\Xi \subset \mathbb{Z}^d$, this result is due to [BH].

Result 5.1. *With Ξ and Π_Ξ as above, let ℓ_ξ, $\xi \in \Xi$, be the directional derivative in the direction ξ, and let*

$$\ell_A := \prod_{\xi \in A} \ell_\xi, \quad A \subset \Xi.$$

Then, $p \in \Pi_\Xi$ if and only if

$$\ell_A p = 0, \quad \text{all} \;\; A \in \mathbb{A}(\Xi) \; := \; \{A \subset \Xi : \Pi_{\Xi \backslash A} = \{0\}\}.$$

The characterization of Π_Ξ given in the above result may be useful only if one is able to decide whether, for any $X \subset \Xi$, the equality $\Pi_X = \{0\}$ holds (or not). For that, the following (essentially well-known) result is useful:

Result 5.2. *Let X be a multiset of directions. Then $\Pi_X = \{0\}$ if and only if there exists an integer $\alpha \in \mathbb{Z}^d \backslash 0$ such that*

$$\xi \cdot \alpha \notin \mathbb{Z} \backslash 0, \quad \text{all} \;\; \xi \in X.$$

In order to connect the present setting with the general development detailed in the previous sections, we make the following suggestive definitions:

Definition 5.3. *Let Ξ be a direction set. We say that $X \subset \Xi$* **spans** *if $\Pi_X \neq \{0\}$. A minimally spanning set is a* **basis**. *The set of all bases in Ξ is denoted by $\mathbb{B}_\Xi$.*

In terms of this definition, we realize that

$$\Pi_\Xi = K(\mathbb{B}_\Xi).$$

The ultimate question we consider here is whether or not an equality of form

$$\dim K(\mathbb{B}_\Xi) = \sum_{B \in \mathbb{B}_\Xi} \dim K(B),$$

whose validity is established (under various assumptions) in the previous sections, is valid in the present setup as well.

The following remark is helpful in this regard: in case $\Xi \subset \mathbb{Z}^d$, Result 5.2 easily shows that $X \subset \Xi$ spans if and only if that set spans $\mathbb{R}^d$ in the standard linear algebra sense. Hence, in that case only, a basis is a linear algebra basis. In the general case, a set $B \subset \Xi$ is a basis only if its rank is d, i.e., only if it contains a linear algebra basis. From that, one easily concludes that, for every basis $B \in \mathbb{B}_\Xi$, $K(B)$ consists of the constants only and thus

$$\dim K(B) = 1, \quad \text{all} \;\; B \in \mathbb{B}_\Xi.$$

This means that the equality we seek is of the form

$$\dim K(\mathbb{B}_\Xi) = \#\mathbb{B}_\Xi. \tag{5.4}$$

We pause momentarily for a quick review of the history of this problem. Equality in (5.4) was proved in [DM1] for the case when $\Xi \subset \mathbb{Z}^d \backslash 0$, and this result was the one that initiated the interest in this entire subject. Next, [BR] proved the general inequality

$$\dim K(\mathbb{B}_\Xi) \geq \#\mathbb{B}'_\Xi, \tag{5.5}$$

with $\mathbb{B}'_\Xi$ the bases in $\mathbb{B}_\Xi$ whose cardinality is exactly d. In fact, though bases of cardinality $> d$ exist (already for $d = 2$; cf. Example 3.12 of [RS]) such examples are the exception rather than the rule, and the analysis of such cases seems extremely difficult; for example, in terms of the notions of the present paper, the existence of such large bases makes the special solvability assumption hard to deal with. For these reasons, we assume here and hereafter that all bases in $\mathbb{B}_\Xi$ have a cardinality d. (Alternatively, we exclude from $\mathbb{B}_\Xi$ all bases with cardinality $> d$, and estimate the dimension of the subspace $K(\mathbb{B}_\Xi)$ of Π_Ξ). With this assumption in hand, we already have the lower bound estimate (5.5), hence seek only a matching upper bound. We mention in passing that the proof of (5.5) in [BR] is done with the aid of exponential box spline theory combined with some ideal theory basics, and is entirely disjoint from the approaches described in [DDM], [BRS] or the present paper. Indeed, in these latter approaches, the equality (5.4) is established by proving first that $\dim K(\mathbb{B}_\Xi) \leq \#\mathbb{B}'_\Xi$.

The required equality in (5.4) was already proved in section 4 of [BRS], under an assumption on the structure of the set $\mathbb{B}_\Xi$ that is stronger than the d-cardinality assumption we adopt here. Though we do not spell out here the nature of this stronger assumption, we do remark that (a): that assumption is *equivalent* to d-cardinality if $d \leq 3$; and (b): that assumption is shown in [BRS] to imply the $\mathbb{E}$-condition, or, in the present terms, the tree-condition, hence (by (b) of Theorem 3.7)) the desired equality (5.4) holds for this case.

However, for $d = 4$, we present below an example that violates the structural assumption of [BRS]. It is natural then to ask the following: assuming attention is restricted, as already mentioned, only to bases with cardinality d, does the set $\mathbb{B}_\Xi$ necessarily satisfy the tree-condition? We answer below that question to the *negative*. A subsequent question is then whether $\mathbb{B}_\Xi$ must satisfy the shell condition. The answer, unfortunately, is still *negative*. In fact, in the counterexample we give below, $\Pi_\Xi = K(\mathbb{B}_\Xi)$ is the space of all linear polynomials in 4 variables, hence is of dimension 5. On the other hand, $\mathbb{B}_\Xi$ consists of two disjoint bases, each with exactly 4 elements. Thus, (5.4) simply fails to hold here, and new ideas are needed for connecting $\dim \Pi_\Xi$ to Ξ.

Example 5.6. Let

$$\Xi := \begin{bmatrix} 0 & 0 & 3 & 0 & 0 & 2 & 3/2 & 3/2 \\ 0 & 0 & 0 & 3 & 1/2 & 1/3 & 1 & 1/2 \\ 0 & 2 & 3/2 & 3/2 & 0 & 0 & 3 & 0 \\ 1/2 & 1/3 & 1 & 1/2 & 0 & 0 & 0 & 3 \end{bmatrix} = [\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}].$$

Note that the last four columns of Ξ are obtained from the first two by interchanging the first two with the last two rows. It is easy to see that $B_1 := \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$ is a basis, i.e., that for any integer $\alpha \in \mathbb{Z}^4\backslash 0$, there exists $\xi \in B_1$ such that $\xi \cdot \alpha \in \mathbb{Z}\backslash 0$. By symmetry, $B_2 := \{\mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}\}$ is also a basis. Since these two bases are disjoint, every set in $A \in \mathbb{A}(\mathbb{B}_\Xi)$ must contain at least two elements, hence the corresponding differential operator ℓ_A annihilates all linear polynomials. This shows that Π_Ξ contains indeed all linears. We will show now that $\mathbb{B}_\Xi$ consists only of the two bases B_1 and B_2. This will prove that (5.4) fails to holds here, implying thereby that the shell condition fails to hold here; *a fortiori* the tree-condition is not valid here.

Recall, from Result 5.2 and Definition 5.3, that $X \subset \Xi$ spans if every $\alpha \in \mathbb{Z}^4\backslash 0$ is **covered** by some $\xi \in X$ in the sense that $\alpha \cdot \xi \in \mathbb{Z}\backslash 0$. In order to prove that only B_1 and B_2 are bases, we prove that any subset $X \subset \Xi$ containing neither B_1 nor B_2 cannot be spanning. Since, for any such subset X, there must be $\xi \in B_1\backslash X$ and $\eta \in B_2\backslash X$, it is sufficient to show that, for any $(\xi, \eta) \in B_1 \times B_2$, there exists $\alpha \in \mathbb{Z}^4\backslash 0$ which is only covered by ξ and η. This is evident from the following calculation:

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \Xi = \begin{bmatrix} \mathbf{6} & 16 & 39 & 51 & \mathbf{6} & 16 & 39 & 51 \\ \mathbf{6} & 16 & 39 & 15 & 0 & \mathbf{12} & 27 & 45 \\ \mathbf{6} & 16 & 21 & 33 & 3 & 2 & \mathbf{24} & 39 \\ \mathbf{6} & 16 & 39 & 33 & 3 & 14 & 33 & \mathbf{48} \\ 9 & \mathbf{6} & 0 & 9 & 0 & \mathbf{-12} & -9 & 45 \\ 0 & \mathbf{12} & 9 & 9 & 0 & 0 & \mathbf{18} & 0 \\ 0 & \mathbf{12} & 27 & 27 & 3 & 14 & 33 & \mathbf{12} \\ 3 & 2 & \mathbf{6} & 21 & 3 & 2 & \mathbf{6} & 21 \\ 3 & 2 & \mathbf{6} & 3 & 0 & 0 & 0 & \mathbf{18} \\ 3 & 14 & 33 & \mathbf{30} & 3 & 14 & 33 & \mathbf{30} \end{bmatrix} /6$$

which, for each pair $(\mathbf{i}, \mathbf{j}) \in B_1 \times B_2$ with $i + 4 \le j$, exhibits an integer vector α covered by no column of Ξ other than $\mathbf{i}$ and $\mathbf{j}$.

All the other α's needed are obtained by symmetry; for example, to find an α covered only by **2** and **5**, start with the vector $(1, 0, 1, 2)$ which, by the above, is only covered by **6** and **1** and interchange its first two with its last two entries.

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