On ascertaining inductively the dimension of the joint kernel of certain commuting linear operators *

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Abstract: Given an index set $\mathbf{X}$, a collection $\mathbb{B}$ of subsets of $\mathbf{X}$ (all of the same cardinality), and a collection $\{\ell_x\}_{x\in\mathbf{X}}$ of commuting linear maps on some linear space, the family of linear operators whose joint kernel $K = K(\mathbb{B})$ is sought consists of all $\ell_A := \prod_{a\in A}\ell_a$ with A any subset of $\mathbf{X}$ which intersects every $B \in \mathbb{B}$. The goal is to establish conditions, on $\mathbb{B}$ and ℓ, which ensure that

$$\dim K(\mathbb{B}) = \sum_{B\in\mathbb{B}} \dim K(\{B\}),$$

or, at least, one or the other of the two inequalities contained in this equality. Concrete instances of this problem arise in box spline theory, and specific conditions on ℓ were given by Dahmen and Micchelli for the case that $\mathbb{B}$ consists of the bases of a matroid.

We give a new approach to this problem, and establish the inequalities and the equality under various rather weak conditions on $\mathbb{B}$ and ℓ. These conditions involve the solvability of certain linear systems of the form $\ell_b? = \phi_b$, $b \in B$, with $B \in \mathbb{B}$, and the existence of 'placeable' elements of $\mathbf{X}$, i.e., of $x \in \mathbf{X}$ for which every $B \in \mathbb{B}$ not containing x has all but one element in common with some $B' \in \mathbb{B}$ containing x.

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1. Introduction

Given a linear space S (over some field), we attempt to determine the dimension of spaces of the form

$$K := \cap_{l \in L} \ker l,$$

with L a (finite) sequence of linear endomorphisms of S, i.e., a sequence in $L(S)$, chosen in a manner described below.

We start with a finite set $\mathbf{X}$ of **atoms**, and associate each $x \in \mathbf{X}$ with a map $\ell_x \in L(S)$. No assumption is made in advance concerning the individual ℓ_x, $x \in \mathbf{X}$, but it is assumed throughout this paper that the **atomic maps** $\{\ell_x\}_{x \in \mathbf{X}}$ commute with one another:

$$\ell_x \ell_y = \ell_y \ell_x, \quad x, y \in \mathbf{X}.$$

This means, in particular, that the product

$$\ell_A := \prod_{x \in A} \ell_x, \quad A \subseteq \mathbf{X},$$

is well-defined, without any need for ordering $\mathbf{X}$.

The joint kernel $K \subseteq S$ whose dimension we attempt to determine can be described in terms of a subset $\mathbb{B}$ of the power set $2^{\mathbf{X}}$; the latter consists of all subsets of $\mathbf{X}$. In general, the set $\mathbb{B}$ can be chosen in quite an arbitrary manner, and, in particular, there is no assumption that

$$\mathbf{X}(\mathbb{B}) := \bigcup_{B \in \mathbb{B}} B$$

covers all of $\mathbf{X}$. Once $\mathbb{B}$ is selected, the joint kernel $K = K(\mathbb{B})$ is defined as

$$K(\mathbb{B}) := \bigcap_{A \in \mathbf{A}(\mathbb{B})} \ker \ell_A,$$

where

$$2^{\mathbf{X}} \supseteq \mathbb{A} := \mathbb{A}(\mathbb{B}) := \{A \subseteq \mathbf{X} : \forall \{B \in \mathbb{B}\}\ A \cap B \neq \emptyset\}$$

is the set of all subsets of $\mathbf{X}$ which meet every $B \in \mathbb{B}$. A particularly simple situation arises when $\mathbb{B}$ consists of the *bases* of some matroid. Consistent with this, we call each *minimal* element (under inclusion) of $\mathbb{A}$ a **cocircuit** even when $\mathbb{B}$ does not have matroidal structure, and denote the set of all cocircuits by

$$\mathbb{A}_{\min}.$$

When $\mathbb{B}$ is empty, $K(\mathbb{B}) = \{0\}$. More interestingly, when $\mathbb{B}$ consists of a single set $B \subseteq \mathbf{X}$, K is simply the joint kernel of the atomic maps $\{\ell_x\}_{x \in B}$. It was the ingenious idea of Dahmen and Micchelli [DM3] to study the relation between the dimension of K and the dimensions of the "block spaces" $K(\{B\})$, $B \in \mathbb{B}$. Their work was stimulated by two nontrivial examples that occur in box spline theory (cf. [BH], [DM1], [DM2], [BeR], [DM3]), one of which we now describe.

Example 1.1. Assume that S is the space $\mathcal{D}(\mathbb{R}^s)$ of complex-valued distributions (entire functions or formal power series will do as well), and let each ℓ_x be a differential operator of the form $\ell_x = D_{v_x} - \lambda_x$, where $v_x \in \mathbb{R}^s \backslash \{0\}$, $\lambda_x \in \mathbb{C}$, and D_y, $y \in \mathbb{R}^s$, is the directional derivative in the y-direction. Define

$$\mathbb{B} := \{Y \subseteq \mathbf{X} : \ \{v_y\}_{y \in Y} \text{ is a basis for } \mathbb{R}^s\}.$$

It is easy to verify that in this case, for each $B \in \mathbb{B}$, $K(\{B\})$ is spanned by one exponential, hence, in particular, $\dim K(\{B\}) = 1$. It is much harder to prove here that

$$\dim K = \#\mathbb{B}. \tag{1.2}$$

This result was proved first for the case $\lambda = 0$ in [DM1] (see also [HS]), and for the general case in [BeR] and [DM3]. Note that (1.2) can also be written as

$$\dim K = \sum_{B \in \mathbb{B}} \dim K(\{B\}), \tag{1.3}$$

and it was Dahmen and Micchelli's observation in [DM3] that this latter formulation holds in more general settings which started research into these matters. We will get to a more systematic discussion of the literature later on in this introduction. We mention at this point that the significance of (1.3) in approximation theory lies in the fact that, whenever $\{v_x\}_{x \in \mathbf{X}} \subseteq \mathbb{Z}^s$, K determines the exponential-polynomial space in the span of the integer translates of the corresponding box spline (cf. the above-cited references for details). However, the above-mentioned connection is no more valid when $\{v_x\}_{x \in \mathbf{X}} \not\subseteq \mathbb{Z}^s$, and the analogous problem (of determining the dimension of the exponential-polynomials in that span) is hopelessly complicated. It was our desire to settle this more general problem that partly motivated the research that led to the present paper. More details can be found in §4. □

In the above example, each $B \in \mathbb{B}$, being a basis for $\mathbb{R}^s$, is of cardinality s. We retain such an assumption throughout this paper, i.e., assume that

$$\#B = s, \quad \text{all } B \in \mathbb{B}, \tag{1.4}$$

for some positive integer s, and call it the **rank** of $\mathbb{B}$. Also, because of this example (and again in consistency with matroid theory), we term the elements of $\mathbb{B}$ **bases**. Our ultimate goal is to prove (1.3) which, however, cannot be proved in general without further assumptions, as simple examples show (see Example 2.1). All methods now in the literature, as well as our approach here, separate the discussion of (1.3) into proving the inequality $\le$ (i.e., upper bounds) and the inequality $\ge$ (i.e., lower bounds). Assumptions to be made for the derivation of (1.3) fall into two essentially different categories, those involving $\mathbb{B}$ and those involving ℓ.

(i) **$\mathbb{B}$-conditions**. In addition to (1.4), we assume in the paper one or more of the following: (1) $\mathbb{B}$ is matroidal (i.e., $\mathbb{B}$ is the collection of bases for a matroid defined on a subset of $\mathbf{X}$); (2) $\mathbb{B}$ is order-closed; (3) $\mathbb{B}$ is minimum-closed; (4) $\mathbb{B}$ is fair; (5) $\mathbf{X}$ contains a replaceable element; (6) $\mathbb{B}$ satisfies the $\mathbb{E}$-condition (i.e., $\emptyset \in \mathbb{E}$); (7) $\mathbf{X}$ contains a placeable element. All these conditions will be defined in the sequel; still, as an easy reference for the reader, we record the relations observed in the paper between these various conditions in the following diagram, in which each arrow indicates a proper (i.e., nonreversible) implication, and, in addition, the absence of an otherwise possible arrow indicates that the corresponding implication does not hold in general:

$$\begin{array}{ccccccccc} & & & & & & (4) & \longrightarrow & (5) \\ & & & & & \nearrow & & & \\ (1) & \longrightarrow & (2) & \longrightarrow & (3) & & & & \\ & & & & & \searrow & & & \\ & & & & & & (6) & \longrightarrow & (7) \end{array}$$

It is probably inherent in the problem (and this is confirmed by [DDM: Theorem 6.2] and by Proposition 3.24) that, by imposing $\mathbb{B}$-conditions (of any type), one can infer only upper bounds (i.e., prove the inequality $\le$ in (1.3)), and lower bounds must incorporate knowledge on the operators involved, which is the second category of assumptions we impose:

(ii) **ℓ-conditions**, namely, assumptions on the operators $\{\ell_x\}_{x\in\mathbf{X}}$. We employ three such assumptions. One is the solvability of certain atomic systems (cf. 3.2), a second is directness of $(\mathbb{B},\ell)$ (cf. 7.1), and a third is s-additivity, from [S] (cf. 8.1).

The known methods employed for the derivation of (1.3) can be divided into **inductive** and **noninductive**. The inductive methods partition $\mathbb{B}$ into two (or more) disjoint subsets

$$\mathbb{B} = \mathbb{B}_1 \cup \mathbb{B}_2,$$

study the relations between $\dim K$ and $\dim K(\mathbb{B}_1) + \dim K(\mathbb{B}_2)$, and proceed to the consideration of each $\mathbb{B}_j$, $j = 1, 2$. This results in a binary (or higher-order) tree decomposition of $\mathbb{B}$. The only two noninductive results that we are aware of are the complex-variable proof in [BeR] that shows that in Example 1.1 one has

$$\dim K \ge \#\mathbb{B},$$

and the polynomial ideal argument in [BR] that shows that in Example 1.1 one has

$$\dim K(\mathbb{B}') \ge \#\mathbb{B}' \tag{1.5}$$

for an arbitrary subset $\mathbb{B}' \subseteq \mathbb{B}$ (as matter of fact, the argument in [BeR] also implies (1.5), but no formal statement to that extent is made there). The latter result (1.5) is of particular interest because it proves lower bounds while the matching upper bounds might be invalid; moreover, these lower bounds require no $\mathbb{B}$-conditions. We are unaware of noninductive methods for the derivation of *upper bounds*. (The proof in [BR] that shows equality to hold in (1.5) in case $\mathbb{B}$ is order-closed is only seemingly noninductive, since it invokes a result from [DR] which is proved there by an inductive method.) As for inductive arguments, all those that we are aware of (including, thus, those of the present paper) require some $\mathbb{B}$-conditions and, moreover, the $\mathbb{B}$-conditions which are known to suffice for lower bounds imply matching upper bounds as a by-product.

The two basic operations in matroid theory are **deletion** and **restriction**, and these operations play a major role in our more general context as well. Precisely, for a given $y \in \mathbf{X}$, we *delete* y from $\mathbf{X}$ to obtain

$$\mathbb{B}_{\backslash y} := \{B \in \mathbb{B} : y \notin B\},$$

and *restrict* $\mathbb{B}$ to y to obtain

$$\mathbb{B}_{|y} := \{B \in \mathbb{B} : y \in B\},$$

and in this way to form a partition of $\mathbb{B}$ into two sets. Note that

$$\mathbb{B} \subseteq \mathbb{B}' \quad\Longrightarrow\quad K(\mathbb{B}) \subseteq K(\mathbb{B}'), \tag{1.6}$$

and hence both spaces $K(\mathbb{B}_{\backslash y})$ and $K(\mathbb{B}_{|y})$ are subspaces of $K = K(\mathbb{B})$.

In principle, we study the exactness of sequences of the form

$$0 \to ? \hookrightarrow K \xrightarrow{j} ? \to 0,$$

where the unknown terms should be related to the space of deletion and the space of restriction. Thanks to (1.6), we have (at least) two options to consider:

$$0 \to K(\mathbb{B}_{\backslash y}) \hookrightarrow K \xrightarrow{j} ? \to 0 \tag{1.7}$$

and

$$0 \to K(\mathbb{B}_{|y}) \hookrightarrow K \stackrel{i}{\longrightarrow} ? \to 0. \tag{1.8}$$

The next step to be made then is the selection of the appropriate map j or i and of the corresponding space now missing in the above sequences. Before we discuss such completions of the above sequences, we require some further notations and definitions.

Guided by Example 1.1, we refer to the elements of $\{Y \subseteq \mathbf{X} : \exists\{B \in \mathbb{B}\} B \subseteq Y\}$ as the **spanning** subsets of $\mathbf{X}$, and to the collection

$$\mathbb{H} = \mathbb{H}(\mathbb{B})$$

of all maximally nonspanning subsets as **hyperplanes**. We also need the family

$$\mathbb{I} = \mathbb{I}(\mathbb{B}) := \bigcup_{B \in \mathbb{B}} 2^B$$

of all **independent** subsets of $\mathbf{X}$. We say that $\mathbb{B}$ is **matroidal** whenever $\mathbb{I}(\mathbb{B})$ defines a matroid on $\mathbf{X}(\mathbb{B})$, which means (cf. [W]) that $\mathbb{I}(\mathbb{B}) \neq \emptyset$ and, for any $I_1, I_2 \in \mathbb{I}(\mathbb{B})$ with $\#I_1 = \#I_2 + 1$, there is $y \in I_1$ for which $I_2 \cup \{y\}$ is still independent. Finally, for $Y \subseteq \mathbf{X}$, we set

$$\mathbb{B}_Y := \{B \in \mathbb{B} : \; B \subseteq Y\}. \tag{1.9}$$

This is consistent with the notation $\mathbb{B}_{\backslash y}$ introduced earlier if $\backslash y$ is interpreted as $\mathbf{X}\backslash\{y\}$.

The DM map. Assuming that $\mathbb{B}$ is matroidal, Dahmen and Micchelli offer in [DM3] the following choice for the missing map and space in (1.7). They choose the map j as

$$j : K \to \mathop{\times}_{A \in \mathbf{A}_{\min}(\mathbb{B}_{\backslash y})} K(\mathbb{B}_{\mathbf{X}\backslash A}) \; : f \mapsto \big(\ell_A f\big)_{A \in \mathbf{A}_{\min}(\mathbb{B}_{\backslash y})}, \tag{1.10}$$

and employ it, in [DM3: Theorem 3.1], to prove the upper bound

$$\dim K \leq \sum_{B \in \mathbb{B}} \dim K(\{B\}) \tag{1.11}$$

for a matroidal $\mathbb{B}$. They also assert (cf. [DM3: Theorem 3.3]) equality in (1.11) under additional ℓ-conditions, and one of the by-products of the present paper (cf. §5) is the bridging of an apparent gap in the proof of the supporting Lemma 3.2 of [DM3]. Shen in [S] introduces a condition, called 's-additivity' (cf. §8), on an abelian semi-group G of linear maps on S and, using the DM map, shows his condition to be necessary and sufficient for equality in (1.11) to hold for all maps ℓ from $\mathbf{X}$ into G and all matroidal $\mathbb{B}$ with rank s. Jia, Riemenschneider and Shen [JRS1] refine and extend the results of [S], from a matroidal $\mathbb{B}$ to an "order-closed" $\mathbb{B}$, a notion introduced in [BR]. Further, [JRS1] prove that if G is a semi-group of differential (resp. difference) operators, generated by polynomials in s indeterminates over some algebraically closed field of characteristic 0, and the linear space S is a space of formal power series (resp., sequences) in s indeterminates over the same field, then G is s-additive (cf. Corollary 3.5 and Theorem 4.4 in [JRS1]). More recently, Dahmen, Dress and Micchelli [DDM], using homological algebra and a replaceability condition (cf. §2), derived (1.3), i.e., equality in (1.11), for the matroidal and order-closed structure under certain ℓ-conditions (cf. [DDM: Theorems 6.2, 6.5]). Those conditions are stronger than the basic solvability condition 3.2 assumed in the present paper.

It is the DM map that naturally gives rise to the notion of "replaceability". More about this map and the exactness of the corresponding short sequence is given in §2 and §8.

The atomic map. It is quite surprising that this simple idea was not used before. Here we choose i in (1.8) to be the restriction of ℓ_y to K, and thus obtain, for any $y \in \mathbf{X}$, the following short sequence

$$0 \to K(\mathbb{B}_{|y}) \hookrightarrow K \xrightarrow{i} K(\mathbb{B}_{\backslash y}) \to 0. \tag{1.12}$$

We will readily observe in §2 that $K(\mathbb{B}_{|y}) \subseteq \ker i$ and $\ell_y K(\mathbb{B}) \subseteq K(\mathbb{B}_{\backslash y})$, hence (1.12) is a short sequence in the homological sense. However, we can infer neither upper bounds nor lower bounds from this sequence, since, in general, the sequence is inexact in two different locations: first, we do not expect in general to have $K(\mathbb{B}_{|y}) = \ker i$, and further, we do not expect in general i to be onto. The derivation of upper bounds relies on the first exactness, the derivation of lower bounds relies on the second exactness. It is the desire to prove that $K(\mathbb{B}_{|y}) = \ker i$ that leads to the notion of placeability (§2) and the further desire to prove the ontoness of i that leads to the $\mathbb{E}$-condition (cf. §3).

The rest of this paper is organized as follows. Section 2 is devoted to the derivation of upper bounds using either of the above two approaches. Its main result is Theorem 2.16. The equality (1.3) is obtained (via the atomic map) in §3, which contains the main result of this paper (Theorem 3.19). An example relevant to box spline theory is studied in §4, and the application of Theorem 3.19 to matroidal and minimum-closed structures (together with some improvements) are discussed in §5 and §6, respectively (see, in particular, Theorems 5.2 and 6.4), with an application of the results on matroids and minimum-closed sets presented in §7. The DM map is revisited in §8, which is the counterpart of §3.

Our joint venture that led eventually to the present paper was initiated by the reading of [DDM]. We take this opportunity to thank the authors of [DDM] for making that preprint of their paper available to us.

2. Replaceability and placeability

We describe in this section $\mathbb{B}$-conditions which allow us to obtain upper bounds on $\dim K$ in terms of $\dim K(\mathbb{B}_{\backslash y})$ and $\dim K(\mathbb{B}_{|y})$ for a suitably chosen $y \in \mathbf{X}$. We emphasize that no ℓ-conditions are imposed here, hence these bounds are valid for an arbitrary S and arbitrary $\ell : X \to L(S)$. One might wonder whether it may be possible to establish realistic upper bounds on $\dim K$ without any $\mathbb{B}$-conditions, especially since (1.5) shows that this might be the case for *lower bounds*. The following example hints at the difficulties in obtaining such upper bounds without $\mathbb{B}$-conditions.

Example 2.1. Let $\mathbf{X}$, $\mathbb{B}$ and $\{\ell_x\}_{x\in\mathbf{X}}$ be as in Example 1.1. We assume that $\{v_x\}_{x\in\mathbf{X}}$ are held fixed, select an arbitrary $\mathbb{B}' \subseteq \mathbb{B}$, and consider the possible influence of the choice of the constants $\lambda := \{\lambda_x\}_x$ on $\dim K(\mathbb{B}')$ (such considerations are intimately related to the notions of "algebraic multiplicity" and "geometric multiplicity" of a zero of an analytic ideal, cf., e.g., [AGV]). Ideally, we would like $\dim K(\mathbb{B}')$ to be independent of the choice of λ, as is the case for certain $\mathbb{B}'$. [BR] shows that for an arbitrary $\mathbb{B}' \subseteq \mathbb{B}$ and for a generic choice of λ, $K(\mathbb{B}')$ is spanned by $\#\mathbb{B}'$ exponentials, hence its dimension is $\#\mathbb{B}'$. On the other hand, if we choose $\mathbb{B}'$ to consist of pairwise disjoint bases, then $\mathbb{A}_{\min}$ consists of all sets containing exactly one element from each $B \in \mathbb{B}'$, hence, with the choice $\lambda_x = 0$, all x, $K(\mathbb{B}')$ necessarily equals the space of all s-variate polynomials of degree $< k := \#\mathbb{B}'$ (since it trivially contains the latter polynomial set, yet it can contain no nontrivial homogeneous polynomial of degree k), and hence $\dim K(\mathbb{B}') = \binom{k+s-1}{s} > k = \#\mathbb{B}'$ (unless $s = 1$).

Now, suppose that we choose $\mathbb{B}'$ as above and want to derive lower bounds and upper bounds on $\dim K(\mathbb{B}')$ without specifying the choice of λ. In view of the above discussion, the best possible lower bound is (1.5), and this is a realistic bound since it generically coincides with the correct dimension. In contrast, we cannot provide an upper bound better than $\dim K(\mathbb{B}') \le \binom{k+s-1}{s}$, which, generically, is a gross overestimate of the correct dimension, and deviates from the desired estimate (1.3). □

The example shows, in particular, that the computation of $\dim K$ for a general $\mathbb{B}$ might require detailed knowledge of the interplay between the atomic maps involved. In contrast, we compute $\dim K$ in this paper under mild general assumptions on the atomic maps. It is therefore understandable that we must employ in our course suitable $\mathbb{B}$-conditions.

Since the DM map and the atomic map require different $\mathbb{B}$-conditions, we separate the discussion accordingly. In these discussions, we use intensively the following simple fact which follows from the observation that, for any $Y \subseteq \mathbf{X}$ and any $A \in \mathbb{A}(\mathbb{B}_{\backslash Y})$, $Y \cup A \in \mathbb{A}$ (where, as mentioned before, $\backslash Y := \mathbf{X} \backslash Y$).

Proposition 2.2. *For any $\mathbf{X}$, $\mathbb{B}$ and ℓ, and any $Y \subseteq \mathbf{X}$, ℓ_Y maps K into $K(\mathbb{B}_{\backslash Y})$.*

2.1. The DM map, Jia's intersection condition, and replaceability

We consider the DM map j defined in (1.10) and the corresponding short sequence (1.7). Because of Proposition 2.2, j is well-defined, and further, one observes that $\ker j = K(\mathbb{B}_{\backslash y})$.

We find it useful in this section to index the target of j by $H \in \mathbb{H}$ rather than by $A \in \mathbb{A}_{\min}(\mathbb{B}_{\backslash y})$. This is possible, because

$$\mathbb{A}_{\min}(\mathbb{B}_{\backslash y}) = \{\mathbf{X}\backslash(y \cup H) : y \notin H \in \mathbb{H}\} \cup \{\mathbf{X}\backslash H : y \in H \in \mathbb{H}\} = \{\mathbf{X}\backslash(y \cup H) : H \in \mathbb{H}\}.$$

Further, since $\mathbb{B}_{y \cup H} = \emptyset$ in case $y \in H \in \mathbb{H}$, the only nontrivial components of the elements in the target

$$\mathop{\times}_{H \in \mathbb{H}} K(\mathbb{B}_{y \cup H})$$

of the DM map are those belonging to

$$\mathbb{H}_{\backslash y} := \{H \in \mathbb{H} : y \notin H\}.$$

Therefore we infer from (1.7), (1.10) the following inequality:

$$\dim K \le \dim K(\mathbb{B}_{\backslash y}) + \sum_{H \in \mathbb{H}_{\backslash y}} \dim K(\mathbb{B}_{y \cup H}). \tag{2.3}$$

The arguments so far are valid for a general $\mathbb{B}$, and hence (2.3) holds in general. It corresponds to writing $\mathbb{B}$ as the union

$$\mathbb{B} = \mathbb{B}_{\backslash y} \cup \bigcup_{H \in \mathbb{H}_{\backslash y}} \mathbb{B}_{y \cup H}, \tag{2.4}$$

but, offhand, there is no reason to believe that this is a partition of $\mathbb{B}$, since we might find the same basis B in two different $\mathbb{B}_{y \cup H}$ (this is the case, e.g., for the $\mathbb{B}'$ in Example 2.1). In case the union in (2.4) is not disjoint, (2.3) will not lead to the desired upper bound (1.11) on $\dim K$.

This means that we are led to require the **intersection condition**,

$$\forall \{H, H' \in \mathbb{H}_{\backslash y}\} \quad \mathbb{B}_{y \cup H} \cap \mathbb{B}_{y \cup H'} = \emptyset, \text{ unless } H = H', \tag{2.5}$$

first suggested by Jia, in [J].

Lemma 2.6. *The intersection condition (2.5) is satisfied (for y) if and only if, for every $B \in \mathbb{B}_{|y}$, there is at most one $H \in \mathbb{H}_{\backslash y}$ containing $B\backslash y$.*

Proof: We observe that any $B \in \mathbb{B}_{y\cup H} \cap \mathbb{B}_{y\cup H'}$ must contain y, i.e., is in $\mathbb{B}_{|y}$. Thus the condition $B \in \mathbb{B}_{y\cup H} \cap \mathbb{B}_{y\cup H'}$, $H \neq H'$, is equivalent to the condition that $(B\backslash y)$ is contained in the two different hyperplanes H and H'. □

The intersection condition (2.5), as we will prove in a moment, is equivalent to having y "replaceable" in $\mathbb{B}$, in the sense of the following definition.

Definition 2.7. *y is* **replaceable** *in $\mathbb{B}$ (or,* $\mathbb{B}$**-replaceable***) if for every $B \in \mathbb{B}_{|y}$ and every $B' \in \mathbb{B}$ there is some $x \in B'$ so that $(B\backslash y) \cup x \in \mathbb{B}$.*

For example, if $\#\mathbf{X}(\mathbb{B}) \leq s+1$, then every $x \in \mathbf{X}(\mathbb{B})$ is replaceable. Thus, the simplest $\mathbb{B}$ without a replaceable element is $\{12, 34\}$.

Proposition 2.8. *y is $\mathbb{B}$-replaceable if and only if (2.5) holds (for y).*

Proof: '$\Longleftarrow$': Let $B \in \mathbb{B}_{|y}$ and $B' \in \mathbb{B}$. Since $B\backslash y$ is not spanning, there exists $H \in \mathbb{H}_{\backslash y}$ containing $B\backslash y$. We claim that, necessarily,

$$H = H' := \{x \in \mathbf{X} : (B\backslash y) \cup x \notin \mathbb{B}\}.$$

For, $H \subseteq H'$ since H contains $B\backslash y$ but contains no basis (in particular no basis of the form $(B\backslash y) \cup x$). On the other hand, if there were $x \in H'\backslash H$, then $(B\backslash y) \cup x$ would be not spanning, hence would be contained in some hyperplane H'', and this hyperplane could not be H, since H does not contain x. This would give us two distinct hyperplanes both containing $B\backslash y$, hence neither one containing y, and this would contradict (2.5), by Lemma 2.6. But now, knowing that H' is a hyperplane, we know that it cannot contain B', hence there is some $x \in B'\backslash H'$ and, by the very definition of H', $(B\backslash y) \cup x \in \mathbb{B}$ for each such x.

'$\Longrightarrow$': If $\mathbb{B}$ fails to satisfy (2.5) (for y), then there exist two distinct hyperplanes H, H' not containing y for which there is some $B \in \mathbb{B}_{y\cup H} \cap \mathbb{B}_{y\cup H'}$. B is necessarily of the form $(B\backslash y) \cup y$ with $B\backslash y \subseteq H \cap H'$. Since $H \neq H'$, the union $H \cup H'$ properly contains H and H', hence spans, i.e., contains a basis B'. For $x \in B'$, $(B\backslash y) \cup x$ is a subset of either H or H' (since $(B\backslash y) \in H \cap H'$ and $x \in H \cup H'$), hence cannot span. This means that $y \in B$ is not replaceable by any $x \in B'$. □

We note that this proposition is close to [DDM: Lemma 6.4]. We also note, for later use, the following characterization of $\mathbb{B}$ being matroidal (this is a standard result; cf., e.g., [W: Theorem 1.2.1]).

Proposition 2.9. *The collection $\mathbb{B} \neq \emptyset$ is matroidal if and only if every $y \in \mathbf{X}(\mathbb{B})$ is $\mathbb{B}$-replaceable.*

Proof: '$\Longrightarrow$': Let $y \in B \in \mathbb{B}$ and $B' \in \mathbb{B}$. Since $\#(B\backslash y) < \#B'$ and both sets are independent, the assumption that $\mathbb{B}$ is matroidal implies that there must be $x \in B'$ so that $(B\backslash y) \cup x$ is independent, hence a basis.

'$\Longleftarrow$': Let $P, Q \in \mathbb{I}$ with $\#P < \#Q$. Then there are P', Q' with $P \cup P'$, $Q \cup Q'$ in $\mathbb{B}$. We order P' in any manner and replace sequentially each $p' \in P' \subseteq (P \cup P') \in \mathbb{B}$ by an element from the basis $Q \cup Q'$. At the end, we obtain a basis of the form $P \cup P''$, with $P'' \subseteq Q \cup Q'$. Since $\#P'' = \#P' > \#Q'$, we must have $P'' \cap Q \neq \emptyset$, and any set of the form $P \cup q$ for some $q \in P'' \cap Q$ is independent. □

Corollary 2.10. *For any independent $(s-1)$-set C, $\mathbb{B}_{|C}$ is matroidal.*

Proof: Every $x \in \mathbf{X}(\mathbb{B}_{|C})$ is either in C or else completes C to a basis, hence, either way, is replaceable. □

To summarize: if y is $\mathbb{B}$-replaceable, then the union (2.4) is disjoint and therefore the estimate (2.3) provides a first inductive step toward the final desired upper bound (1.11). However, our primary aim in this paper is the application of the atomic map to which we now turn our attention.

2.2. The atomic map and placeability

Considering (1.12), we observe that, by Proposition 2.2, the map i is well-defined (i.e., maps into $K(\mathbb{B}_{\backslash y})$). Further, we always have that $K(\mathbb{B}_{|y}) \subseteq \ker i = \ker \ell_y \cap K$: the inclusion $K(\mathbb{B}_{|y}) \subseteq K$ is due to $\mathbb{B}_{|y} \subseteq \mathbb{B}$, while the inclusion $K(\mathbb{B}_{|y}) \subseteq \ker \ell_y$ follows from the fact that $\{y\}$ is a cocircuit in $\mathbb{B}_{|y}$.

The atomic map provides the necessary inductive step towards an upper bound if, in addition to the above, we also know that $K(\mathbb{B}_{|y}) \supseteq \ker i$. For this, we introduce the following notion of placeability:

Definition 2.11. *We say that Y is* **placeable into** *B if $Y \cup C \in \mathbb{B}$ for some $C \subseteq B$. If Y is placeable into every $B \in \mathbb{B}$, then we say that Y is placeable (in $\mathbb{B}$), or,* **$\mathbb{B}$-placeable***.*

For example, if $\#\mathbf{X} \leq s+1$, then every $x \in \mathbf{X}(\mathbb{B})$ is placeable. Thus, the simplest $\mathbb{B}$ without a placeable element is $\{12, 34\}$. Further, y is replaceable in $\mathbb{B}$ iff for each $B \in \mathbb{B}$, $B \backslash y$ is $\mathbb{B}$-placeable. On the other hand, there may be some replaceable atom even though none of the atoms are placeable, as is the case for $\mathbb{B} = \{123, 126, 129, 345, 678\}$, in which 9 is trivially replaceable, while none of the atoms in 345 can be placed in 678 and vice versa, and 1, 2, or 9 cannot be placed in either.

Further, if $\mathbb{B}$ is matroidal, then every independent element, i.e., every $y \in \mathbf{X}(\mathbb{B})$, is placeable (by Proposition 2.13 below), but the converse does not hold, as the following example shows. Let $\mathbf{X} = 12345 := \{1,2,3,4,5\}$ and take $\mathbb{B} := \binom{\mathbf{X}}{3} \backslash \{123, 124\}$ (with $\binom{\mathbf{X}}{d}$ the collection of all d-subsets of $\mathbf{X}$). Since every $x \in \mathbf{X}$ is placeable into any $B \in \binom{\mathbf{X}}{3}$ in three different ways, the removal of two sets cannot destroy placeability of any $x \in \mathbf{X}$. On the other hand, 5 in 125 cannot be replaced by anything in 234, hence $\mathbb{B}$ fails to be matroidal, by Proposition 2.9.

The following, extended, example shows that the placeability of every $x \in \mathbf{X}$ fails to imply various other conditions.

Example 2.12. Let $\mathbf{X} = 12345678 := \{1, ..., 8\}$ and let $\mathbb{B} := \binom{\mathbf{X}}{3} \backslash \{123, 124, 567, 568\}$. Then, every $x \in \mathbf{X}$ is $\mathbb{B}$-placeable, but no $x \in \mathbf{X}$ is $\mathbb{B}$-replaceable: given $1 \leq x \leq 4$, x in $B = 56x$ cannot be replaced by any element from 578, and a similar argument applies to $x > 4$. In particular, $\mathbb{B}$ is neither matroidal (by Proposition 2.9), nor is it minimum-closed (by Proposition 6.6, and for whatever ordering we choose to impose on $\mathbf{X}$), hence cannot be order-closed.

The lack of a replaceable atom makes this example inappropriate for an application of the DM map. On the other hand, we will verify (cf. Example 6.8) that $\mathbb{B}$ here satisfies the $\mathbb{E}$-condition, and this guarantees a successful binary decomposition of $\mathbb{B}$ via the atomic map. □

We have just seen that total placeability (i.e., having every $y \in \mathbf{X}(\mathbb{B})$ placeable) falls short of implying that $\mathbb{B}$ is matroidal. In this regard, it is useful to note the following two propositions.

Proposition 2.13.
(i) If $\mathbb{B}$ is matroidal, then every independent set is placeable.
(ii) If every independent $(s-1)$-set is placeable, then $\mathbb{B}$ is matroidal.

Proof: (i): This is a standard matroid argument. Let $C \in \mathbb{I}$ and $B \in \mathbb{B}$. We are to prove that $B \cup C$ contains some $B' \in \mathbb{B}_{|C}$. This is certainly so in case $\#C = s$. In the contrary case, B contains some $(\#C+1)$-set C', and, $\mathbb{B}$ being matroidal, this implies that, for some $y \in C'$, $C \cup y \in \mathbb{I}$. Downward induction on $\#C$ then completes the proof.

(ii): Since every independent $(s-1)$-set is placeable if and only if every element of $\mathbf{X}(\mathbb{B})$ is replaceable, Proposition 2.9 supplies the proof. $\square$

Proposition 2.14. *If, for every $Y \subseteq \mathbf{X}$, every $y \in \mathbf{X}(\mathbb{B}_{|Y})$ is $\mathbb{B}_{|Y}$-placeable, then $\mathbb{B}$ is matroidal.*

Proof: In view of Proposition 2.9, it suffices to show that every $y \in \mathbf{X}(\mathbb{B})$ is replaceable. Let $B, B' \in \mathbb{B}$ and let $y \in B$. We need to find $x \in B'$ such that $(B\backslash y) \cup x \in \mathbb{B}$, and may assume without loss that $y \notin B'$ (otherwise, choose x to be y). We prove the existence of such an atom x by (downward) induction on $\#Y$, with $Y := B \cap B'$, there being nothing to prove when $\#Y = s$. Also, when $\#Y = s-1$, we choose x as the single element of $B'\backslash B$. So, assume $\#Y < s-1$. Then $B\backslash(y \cup Y)$ is not empty. Let b be one of its elements. By assumption, b is $\mathbb{B}_{|Y}$-placeable, hence we can place b in B', i.e., there is some $B'' \in \mathbb{B}_{|Y}$ for which $B''\backslash B' = \{b\}$. This implies that $\#(B \cap B'') > \#Y$. Thus, by induction, there exists $x \in B''$ for which $(B\backslash y) \cup x \in \mathbb{B}$. This x differs from b (since b is in $B\backslash y$, hence cannot complete this latter set to a basis), and thus x is in B'. $\square$

The proposition makes clear that total placeability, while being preserved under deletion (unless, of course, we delete the atom in question), cannot be preserved under restriction. Indeed, we see that, in Example 2.12, 2 fails to be $\mathbb{B}_{|1}$-placeable into 134.

The next lemma prepares for the main result of this section.

Lemma 2.15. *Let $Y \subseteq \mathbf{X}$, and set $\mathbb{Y} := \mathbb{A}(\{Y\})$. Then, $\mathbb{A}(\mathbb{B}_{|Y}) \supseteq \mathbb{A}(\mathbb{B}) \cup \mathbb{Y}$, with equality if and only if Y is $\mathbb{B}$-placeable. In the latter case,*

$$K(\mathbb{B}_{|Y}) = K \cap \bigcap_{y \in Y} \ker \ell_y.$$

Proof: The containment $\mathbb{A}(\mathbb{B}_{|Y}) \supseteq \mathbb{A}(\mathbb{B}) \cup \mathbb{Y}$ is straightforward.

Assume that Y is not $\mathbb{B}$-placeable. Then there exists $B \in \mathbb{B}$ for which $B \cup Y$ fails to contain an element of $\mathbb{B}_{|Y}$, hence $\mathbf{X}\backslash(B \cup Y) \in \mathbb{A}(\mathbb{B}_{|Y})$, yet $\mathbf{X}\backslash(B \cup Y)$ is neither in $\mathbb{A}(\mathbb{B})$ (since its complement contains B) nor in $\mathbb{Y}$ (since it is disjoint from Y).

Assume that Y is $\mathbb{B}$-placeable, and let $A \notin \mathbb{A}(\mathbb{B}) \cup \mathbb{Y}$. Since $A \notin \mathbb{A}(\mathbb{B})$, $\mathbf{X}\backslash A$ contains some $B \in \mathbb{B}$, and since $A \notin \mathbb{Y}$, $\mathbf{X}\backslash A$ must contain Y. Since Y is $\mathbb{B}$-placeable, there is $\mathbb{B}_{|Y} \ni B' \subseteq B \cup Y \subseteq \mathbf{X}\backslash A$, hence $A \notin \mathbb{A}(\mathbb{B}_{|Y})$. $\square$

Theorem 2.16. *If y is $\mathbb{B}$-placeable, then*

$$\dim K \le \dim K(\mathbb{B}_{|y}) + \dim K(\mathbb{B}_{\backslash y}), \tag{2.17}$$

with equality if and only if ℓ_y maps K onto $K(\mathbb{B}_{\backslash y})$.

Proof: Since y is $\mathbb{B}$-placeable, Lemma 2.15 implies that $K(\mathbb{B}_{|y}) = K \cap \ker \ell_y$, hence that $\ker i$ in (1.12) coincides with $K(\mathbb{B}_{|y})$. Thus, (1.12) is exact at K and (2.17) follows. Equality in (2.17) holds if and only if (1.12) is also exact at $K(\mathbb{B}_{\backslash y})$, i.e., if and only if ℓ_y maps K onto $K(\mathbb{B}_{\backslash y})$. $\square$

Use of the atomic map and the placeability notion seems to be more applicable and powerful than the alternative idea of the DM map and the notion of replaceability. For example, using the former approach we obtain in the next two sections the equality (1.3) under ℓ-conditions which are weaker than those employed in [DM3] and [DDM], and weaker than the s-additivity used in [S] and [JRS1]. Further, the replaceability of $y \in \mathbf{X}$ is a *necessary* (albeit not sufficient) condition for the exactness of the short sequence employed in the DM map, while, in contrast, the short sequence (1.12) can be exact even for nonplaceable y's. Indeed, if we choose λ in Example 2.1 in such a way that K is spanned by (pure) exponentials (which is the generic choice), then, for an arbitrary $\mathbb{B}' \subseteq \mathbb{B}$ and an arbitrary $y \in \mathbf{X}$, the sequence (1.12) (with $\mathbb{B}'$ replacing $\mathbb{B}$) can be easily shown to be exact (since then $K(\mathbb{B}')$ is spanned by eigenvectors of ℓ_y).

3. $\mathbb{E}$-condition and special solvability

In this section we discuss conditions on the map $\ell : X \to L(S)$ and on $\mathbb{B}$ under which there is equality in the inequality (2.17).

We expect equality in (2.17) in case ℓ_y maps K *onto* $K(\mathbb{B}_{\backslash y})$, i.e., in case the equation

$$\ell_y? = f$$

has solutions in K for any $f \in K(\mathbb{B}_{\backslash y})$. For this reason, our ℓ-conditions are connected to the solvability of systems of the form

$$(C, \varphi) : \quad \ell_c? = \varphi_c, \quad c \in C, \tag{3.1}$$

with $C \subseteq \mathbf{X}$ and φ a map into S and defined (at least) on C.

Definition. *We call the system* (C, φ)
(i) **special**, *or, more explicitly,* $\mathbb{B}$**-special** *if* $\varphi_c \in K(\mathbb{B}_{\backslash c})$*, all* $c \in C$*;*
(ii) **compatible** *if* $\ell_c \varphi_b = \ell_b \varphi_c$ *for all* $c, b \in C$*;*
(iii) **independent** *(resp.* **basic***) if* $C \in \mathbb{I}$ *(resp.* $C \in \mathbb{B}$*).*

It goes without saying that such compatibility is a *necessary* condition for the solvability of such a system.

Solvability condition 3.2. *Any special compatible basic system is solvable.*

As an example, in Example 1.1 one can easily verify that *any* compatible basic system is solvable. However, our solvability condition requires only the solvability of "special" systems, hence might hold even when some more general compatible basic systems fail to admit solutions. For example, we can allow S to be finite-dimensional, e.g., to be K itself, which is not possible with other approaches in the literature.

For our subsequent purposes, it will be important to know that the solution of the special compatible basic system is *in* K, but this fact is free:

Lemma 3.3. *Any solution of a special basic system* (B, φ) *lies in* K*.*

Proof: Let f be a solution, and let $A \in \mathbb{A}(\mathbb{B})$. Then A contains some element b of our B, therefore $\ell_A f = \ell_{A\backslash b} \ell_b f = \ell_{A\backslash b} \varphi_b$. Since $A \in \mathbb{A}(\mathbb{B})$, it follows that $(A\backslash b) \in \mathbb{A}(\mathbb{B}_{\backslash b})$, and thus, because we assume that $\varphi_b \in K(\mathbb{B}_{\backslash b})$, we obtain $\ell_A f = 0$. □

3.1. The set $\mathbb{E}$

The solvability condition is all we need for the derivation of (1.3) in case the "effective" rank is 1 (by Theorem 5.2, since $\mathbb{B}_{|C}$ is matroidal for any independent $(s-1)$-set C, by Corollary 2.10). For "effective" rank higher than 1, we use Lemma 3.3 to show that ℓ_y maps K *onto* $K(\mathbb{B}_{\backslash y})$, by extending the equation $\ell_y? = \varphi_y$ with $\varphi_y \in K(\mathbb{B}_{\backslash y})$ to a special compatible basic system (B, φ), but the existence of such an extension is not trivial. Our proof that this is possible is by induction, and requires that $\mathbb{B}$ **satisfies the $\mathbb{E}$-condition**, by which we mean that

$$\emptyset \in \mathbb{E},$$

with $\mathbb{E} = \mathbb{E}(\mathbb{B})$ the following peculiar subset of $\mathbb{I}$.

Definition 3.4. *Let $\mathbb{E} = \mathbb{E}(\mathbb{B})$ be the collection of all those $C \in \mathbb{I}$ which either are in $\mathbb{B}$, or else there is some $b \in \mathbf{X}\backslash C$, called a* **$\mathbb{B}$-extender for** *$C$, which satisfies the following two conditions:*
(i) $C \cup b \in \mathbb{E}$;
(ii) if $\mathbb{B}_{\backslash b} \neq \emptyset$, then $C \in \mathbb{E}(\mathbb{B}_{\backslash b})$.

The recursion required in this definition does terminate after finitely many steps. For, the first branch leads to a set of higher cardinality, hence this branch terminates after exactly $s - \#C$ steps. The second branch keeps the cardinality of C the same but decreases the number of bases, thus it is guaranteed to terminate since $\#\mathbb{B}$ is finite.

Note also that $\mathbb{E}(\mathbb{B})$ is not (in general) monotone in $\mathbb{B}$. For, while $\mathbb{B} \subseteq \mathbb{B}'$ implies that $\mathbb{I}(\mathbb{B}) \subseteq \mathbb{I}(\mathbb{B}')$, this resulting increase in independent sets could lead to a nontrivial $\mathbb{B}'_{\backslash b}$ where before we had $\mathbb{B}_{\backslash b} = \emptyset$ (hence $C \in \mathbb{E}$ merely because $C \cup b \in \mathbb{E}$), without guaranteeing that C lies in $\mathbb{E}(\mathbb{B}'_{\backslash b})$ (since, before, we did not need to know whether or not $C \in \mathbb{E}(\mathbb{B}_{\backslash b})$). On the other hand, if we make an appropriate assumption, such as that, for all Y, $\mathbb{B}_Y = \emptyset \implies \mathbb{B}'_Y = \emptyset$, in order to avoid this objection, then we get the trivial conclusion that $\mathbb{B} = \mathbb{B}'$.

As an example, an independent $(s-1)$-set C is in $\mathbb{E}$ if and only if $\{b \in \mathbf{X} : C \cup b \in \mathbb{B}\} \in \mathbb{A}$, as the proof of the following connection between $\mathbb{E}$ and the $\mathbb{B}$-placeable subsets of $\mathbf{X}$ makes clear.

Proposition 3.5. *Every $C \in \mathbb{E}$ is $\mathbb{B}$-placeable, and the converse is true if $\#C = s - 1$. In particular, if $s = 2$, then $y \in \mathbb{E}$ if and only if y is $\mathbb{B}$-placeable.*

Proof: We prove the first claim by (downward) induction on $\#C$ and induction on $\#\mathbb{B}$, it being trivial if $\#C = s$ or $\#\mathbb{B} = 1$.

Assume that $\#C < s$, and let $B \in \mathbb{B}$. Since $C \in \mathbb{E}$, it has an extender, b say. In particular, $C \cup b \in \mathbb{E}$. If $b \in B$, we apply our induction hypothesis to $C \cup b$ to conclude that $C \cup b$ is placeable in B, *a fortiori* C is placeable there. If $b \notin B$, then $B \in \mathbb{B}_{\backslash b}$. Since $\mathbb{B}_{\backslash b}$ is a (nonempty) proper subset of $\mathbb{B}$, and since we know that we still have $C \in \mathbb{E}(\mathbb{B}_{\backslash b})$, we can conclude by induction that C is placeable in B.

It remains to show that a $\mathbb{B}$-placeable C of cardinality $s - 1$ is in $\mathbb{E}$: If C is placeable, then every $B \in \mathbb{B}$ must meet the set $C' := \{b \in \mathbf{X} : C \cup b \in \mathbb{B}\}$. This means that $\mathbb{B}_{\backslash C'} = \emptyset$, hence $C \in \mathbb{E}$ follows by induction on $\#C'$, it being trivially true when $\#C' = 1$. □

In general, placeability does not imply membership in $\mathbb{E}$. For example, not every $\mathbb{E}$ contains the empty set (cf. Proposition 3.11 below). As a concrete example, if $\mathbf{X} = 12345 := \{1, 2, 3, 4, 5\}$ and $\mathbb{B} = \{123, 234, 245, 135\}$, then 5 can be placed into any basis, but does not make it into $\mathbb{E}$ since no 2-set containing 5 is placeable and therefore no such set makes it into $\mathbb{E}$ (in view of the last proposition). As it turns out, the additional condition needed for a placeable C to be in $\mathbb{E}$ is that it be already in $\mathbb{E}(\mathbb{B}_{|C})$. This is a consequence of the following lemma.

Lemma 3.6. *If $Y \subseteq \mathbf{X}$ contains some placeable C, then $Y \in \mathbb{E}$ if and only if $Y \in \mathbb{E}(\mathbb{B}_{|C})$.*

Proof: We begin with the observation that, for any $Z \subseteq \mathbf{X}\backslash C$,

$$\mathbb{B}_{\backslash Z} = \emptyset \iff \mathbb{B}_{|C\backslash Z} = \emptyset. \tag{3.7}$$

Indeed, the necessity is trivial. As for the sufficiency, if $B \in \mathbb{B}_{\backslash Z}$, then C, being placeable, can be placed into B, and this provides an element of $\mathbb{B}_{|C\backslash Z}$.

'$\Longrightarrow$': The proof is by (downward) induction on $\#Y$ and induction on $\#\mathbb{B}$, it being trivially true when $\#\mathbb{B} \leq 1$ or if $\#Y = s$. Assume that $\#\mathbb{B} > 1$, and that $\#Y < s$. Since we assume $Y \in \mathbb{E}$, there exists a $\mathbb{B}$-extender, say b, for Y. We now verify that b is also a $\mathbb{B}_{|C}$-extender for Y: (i) Since $Y \cup b$ is in $\mathbb{E}$ and is larger than Y, induction on $\#Y$ ensures that $Y \cup b \in \mathbb{E}(\mathbb{B}_{|C})$. (ii) If $\mathbb{B}_{|C\backslash b} \neq \emptyset$, then, by (3.7), $\mathbb{B}_{\backslash b} \neq \emptyset$, hence $Y \in \mathbb{E}(\mathbb{B}_{\backslash b})$. Therefore, by induction on $\#\mathbb{B}$, $Y \in \mathbb{E}(\mathbb{B}_{|C\backslash b})$.

'$\Longleftarrow$': Since (3.7) is an equivalence, the argument just given also works with $\mathbb{B}$ and $\mathbb{B}_{|C}$ interchanged. □

Note that, offhand, Lemma 3.6 implies nothing about the relationship between $\mathbb{E}$ and $\mathbb{E}(\mathbb{B}_{|C})$. This reflects the fact that, in general, $\mathbb{E}(\mathbb{B})$ is not a monotone function of $\mathbb{B}$.

Corollary 3.8. *$C \in \mathbb{E}$ if and only if the following two conditions hold:*
(a) C is $\mathbb{B}$-placeable;
(b) $C \in \mathbb{E}(\mathbb{B}_{|C})$.

Proof: Because of Proposition 3.5, it is sufficient to prove that, for a $\mathbb{B}$-placeable C, $C \in \mathbb{E}$ if and only if $C \in \mathbb{E}(\mathbb{B}_{|C})$. But this is just the special case $Y = C$ in the lemma.

□

Note that we could not use (a) and (b) of this corollary to *define* $\mathbb{E}$ because the equivalence proved in this corollary is a tautology whenever $\mathbb{B} = \mathbb{B}_{|C}$.

Lemma 3.9. *Assume that $\mathbb{B} = \mathbb{B}_{|C}$.*
(a) If $Y \cup C \in \mathbb{E}$, then $J \in \mathbb{E}$ for any $Y \backslash C \subseteq J \subseteq Y \cup C$.
(b) If $Y \in \mathbb{E}$, then $J \in \mathbb{E}$ for any $Y \subseteq J \subseteq Y \cup C$.

Proof: (a): The proof is by (downward) induction on $\#J$, there being nothing to prove when $\#J = \#(Y \cup C)$. If now $\#J < \#(Y \cup C)$, then there exists $x \in (Y \cup C)\backslash J$. We verify that any such x is an extender for J: (i) $J \cup x \in \mathbb{E}$ by induction hypothesis; (ii) any such x is necessarily in C (since $Y\backslash C \subseteq J$), hence $\mathbb{B}_{\backslash x} = \emptyset$.

(b): The proof is by induction on $\#\mathbb{B}$ and (downward) induction on $\#Y$, it being trivially true when $\#\mathbb{B} = 1$ or $\#Y = s$. So, assume that $\#\mathbb{B} > 1$ and $\#Y < s$. We are to verify that $J \in \mathbb{E}$. Since $Y \in \mathbb{E}$, it has an extender, b say. Since $Y \cup b$ is in $\mathbb{E}$ and larger than Y, induction on $\#Y$ implies that $J \cup b \in \mathbb{E}$, hence we are done in case $b \in J$. Otherwise, to verify that b is an extender for J, assume that $\mathbb{B}_{\backslash b} \neq \emptyset$. Then $Y \in \mathbb{E}(\mathbb{B}_{\backslash b})$, hence $J \in \mathbb{E}(\mathbb{B}_{\backslash b})$, by induction on $\#\mathbb{B}$ (applicable since $J \cup b \in \mathbb{E}$ implies that $\#\mathbb{B}_{\backslash b} < \#\mathbb{B}$, and since $\mathbb{B}_{\backslash b} = (\mathbb{B}_{\backslash b})_{|C}$). □

Corollary 3.10. *Assume that $\mathbb{B} = \mathbb{B}_{|C}$ and $Y \subseteq \mathbf{X}$. Then $Y \in \mathbb{E}$ if and only if $Y \cup C \in \mathbb{E}$ if and only if $Y\backslash C \in \mathbb{E}$. In particular, $\emptyset \in \mathbb{E}(\mathbb{B}_{|C})$ if and only if $C \in \mathbb{E}(\mathbb{B}_{|C})$.*

Proof: Both implications '$\Longrightarrow$' are special cases of the lemma, as is the first '$\Longleftarrow$', while the second '$\Longleftarrow$' follows from (b) of the lemma since $Y \cup C = (Y\backslash C) \cup C$. □

The final results in this subsection aim at providing efficient methods for an inductive verification of the $\mathbb{E}$-condition, i.e., the condition $\emptyset \in \mathbb{E}$.

Proposition 3.11. *Assume $\#\mathbb{B} > 1$. If $\emptyset \in \mathbb{E}$, then, for some $b \in \mathbb{E}$, $\emptyset \in \mathbb{E}(\mathbb{B}_{|b})$ and $\emptyset \in \mathbb{E}(\mathbb{B}_{\backslash b})$. Conversely, if $\emptyset \in \mathbb{E}(\mathbb{B}_{\backslash b})$ for some $b \in \mathbb{E}$, then $\emptyset \in \mathbb{E}$.*

Proof: For the sake of both claims here, we note that

$$b \in \mathbb{E} \quad \Longrightarrow \quad b \in \mathbb{E}(\mathbb{B}_{|b}) \quad \Longrightarrow \quad \emptyset \in \mathbb{E}(\mathbb{B}_{|b}). \tag{3.12}$$

Indeed, the first implication corresponds to the choice $C = b$ in Corollary 3.8, and the second implication corresponds to the choice $C = b$ in Corollary 3.10.

Proof of first claim: Let $C \subseteq \mathbf{X}$ be a maximal set for which $\mathbb{B} = \mathbb{B}_{|C}$ and note that $\#C < s$ (since $\#\mathbb{B} > 1$). Since $\emptyset \in \mathbb{E}$, $C \in \mathbb{E}$ by Corollary 3.10, hence has a $\mathbb{B}$-extender, b. We now verify that any such b does the job: $\mathbb{B}_{\backslash b} \neq \emptyset$ (for, if $\mathbb{B}_{\backslash b}$ were empty, then $\mathbb{B} = \mathbb{B}_{|C\cup b}$, and this would contradict the maximality of C), and therefore, because b extends C, we have $C \in \mathbb{E}(\mathbb{B}_{\backslash b})$, or equivalently (by Corollary 3.10, since $\mathbb{B}_{\backslash b} = (\mathbb{B}_{\backslash b})_{|C})$, $\emptyset \in \mathbb{E}(\mathbb{B}_{\backslash b})$. Further, since $C \cup b \in \mathbb{E}$, the choice $Y = b$ in Corollary 3.10 provides the conclusion that $b \in \mathbb{E}$, and hence, by (3.12), $\emptyset \in \mathbb{E}(\mathbb{B}_{|b})$.

Proof of second claim: Since $b \in \mathbb{E}$ and $\emptyset \in \mathbb{E}(\mathbb{B}_{\backslash b})$, b is a $\mathbb{B}$-extender for $\emptyset$. □

A repeated application of the last proposition leads to the following partial unraveling of the condition $\emptyset \in \mathbb{E}$:

Corollary 3.13. *$\emptyset \in \mathbb{E}$ if and only if $\mathbf{X}$ contains a sequence $b_1, \ldots, b_r$ for which $b_j \in \mathbb{E}(\mathbb{B}_{\backslash b_1,\ldots,b_{j-1}})$, all j, while $\#\mathbb{B}_{\backslash b_1,\ldots,b_r} = 1$.*

We note that, by Proposition 5.1, $\emptyset \in \mathbb{E}$ in case $\mathbb{B}$ is matroidal. This provides the following strengthening of the above corollary.

Corollary 3.14. *$\emptyset \in \mathbb{E}$ if and only if $\mathbf{X}$ contains a sequence $b_1, \ldots, b_r$ for which $b_j \in \mathbb{E}(\mathbb{B}_{\backslash b_1,\ldots,b_{j-1}})$, all j, while $\mathbb{B}_{\backslash b_1,\ldots,b_r}$ is matroidal.*

Remark 3.15. A complete unraveling of the condition in Proposition 3.11 produces a binary tree whose nodes are of the form $\mathbb{B}_{|Y\backslash Z}$ for certain $Y, Z \subseteq \mathbf{X}$ with $Y \cap Z = \emptyset$. Further, each such node is either a leaf, in which case it contains exactly one $B \in \mathbb{B}$, or else it is the disjoint union of its two children, $\mathbb{B}_{|(b\cup Y)\backslash Z}$ and $\mathbb{B}_{|Y\backslash(Z\cup b)}$, with $b \in \mathbf{X}\backslash(Y \cup Z)$ $\mathbb{B}_{|Y\backslash Z}$-placeable. Finally, $\mathbb{B}$ is the root of this tree. Since such a tree is obtainable whenever $\mathbb{B}$ satisfies the $\mathbb{E}$-condition, we call it an **$\mathbb{E}$-tree for** $\mathbb{B}$. □

Lemma 3.16. *Every node of an $\mathbb{E}$-tree for $\mathbb{B}$ satisfies the $\mathbb{E}$-condition.*

Proof: We proceed by induction on $\#\mathbb{B}$, it being trivially true if $\#\mathbb{B} = 1$. If $\#\mathbb{B} > 1$, and b is the placeable element used to split the root node, $\mathbb{B}$, then, by Corollary 3.8, $b \in \mathbb{E}$ if and only if $b \in \mathbb{E}(\mathbb{B}_{|b})$, and, by Corollary 3.10, this latter condition is equivalent to having $\emptyset \in \mathbb{E}(\mathbb{B}_{|b})$, and this condition holds by induction hypothesis. Thus $b \in \mathbb{E}$, and, by Proposition 3.11, this implies that $\emptyset \in \mathbb{E}$ since $\emptyset \in \mathbb{E}(\mathbb{B}_{\backslash b})$ by induction hypothesis. □

We have proved the following characterization of the $\mathbb{E}$-condition.

Theorem 3.17. *$\emptyset \in \mathbb{E}$ if and only if there is an $\mathbb{E}$-tree for $\mathbb{B}$.*

Without the requirement that the b used to split $\mathbb{B}_{|Y\backslash Z}$ be $\mathbb{B}_{|Y\backslash Z}$-placeable, every $\mathbb{B}$ would have such a tree.

With or without this placeability requirement, the leaves of such a tree constitute the partition of $\mathbb{B}$ into its elements.

3.2. Dimension estimates

We now turn our attention to the main topic of this section, namely the connection between the content of $\mathbb{E}$ and the validity of (1.3). The central ingredient for our argument is the following proposition for whose proof the set $\mathbb{E}$ was tailor-made.

Proposition 3.18. *If the solvability condition 3.2 holds, then, any special compatible system (C, φ) with $C \in \mathbb{E}$ can be extended to a special compatible basic system, hence has solutions in K.*

Proof: The proof is by (downward) induction on $\#C$ and induction on $\#\mathbb{B}$. The statement is trivial if $\mathbb{B}$ is empty or if C is a basis.

Let $\mathbb{B}$ and $C \in \mathbb{E}\backslash\mathbb{B}$ be given, and assume that we already know the claim for larger $C \in \mathbb{E}$ as well as for any set $C' \in \mathbb{E}(\mathbb{B}')$ with $\#\mathbb{B}' < \#\mathbb{B}$.

Let b be an extender for C. Then $C \cup b$ is in $\mathbb{E}$ and larger than C. We claim that we can correspondingly find some $\varphi_b \in K(\mathbb{B}_{\backslash b})$ so that the extended special system $(C \cup b, \varphi)$ is still compatible. For this, it is necessary and sufficient that φ_b solve the system $(C, \ell_b\varphi)$. There are two cases:

(i) if $\mathbb{B}_{\backslash b} = \emptyset$, then $\mathbb{B} = \mathbb{B}_{|b}$, hence $\{b\} \in \mathbb{A}$, therefore $\ker \ell_b \supseteq K \supseteq K(\mathbb{B}_{\backslash c})$ for all $c \in C$ and, in particular, $\ell_b\varphi_c = 0$ for all $c \in C$, thus the trivial choice $\varphi_b = 0$ solves $(C, \ell_b\varphi)$.

(ii) if $\mathbb{B}_{\backslash b} \neq \emptyset$, then we know that $C \in \mathbb{E}(\mathbb{B}_{\backslash b})$, and the system $(C, \ell_b\varphi)$ is compatible and $\mathbb{B}_{\backslash b}$–special (since (C, φ) is compatible and $\mathbb{B}$-special, and because of Proposition 2.2). Also, since $C \cup b \in \mathbb{E}$, it is contained in some $B \in \mathbb{B}$ and this B is necessarily not in $\mathbb{B}_{\backslash b}$. This implies that $\mathbb{B}_{\backslash b}$ is a *proper* subset of $\mathbb{B}$. It follows, by induction hypothesis, that $(C, \ell_b\varphi)$ has a solution in $K(\mathbb{B}_{\backslash b})$, and any such is suitable as φ_b.

Since $C \cup b$ is in $\mathbb{E}$ and larger than C, induction now allows the conclusion that our present (extended) special and compatible system is part of a special compatible basic system. □

We are now ready to state and prove the main result of this paper:

Theorem 3.19.

(a) Assume that the solvability condition 3.2 holds. Then, for any $y \in \mathbb{E}$, ℓ_y maps K onto $K(\mathbb{B}_{\backslash y})$, and

$$\dim K = \dim K(\mathbb{B}_{|y}) + \dim K(\mathbb{B}_{\backslash y}). \tag{3.20}$$

(b) Assume that $\emptyset \in \mathbb{E}$. Then

$$\dim K \leq \sum_{B \in \mathbb{B}} \dim K(\{B\}), \tag{3.21}$$

with equality in case the solvability condition 3.2 holds.

Proof: (a): Since $y \in \mathbb{E}$, Proposition 3.18 implies that the linear equation $\ell_y? = \varphi_y$ with $\varphi_y \in K(\mathbb{B}_{\backslash y})$ can be extended to a special compatible basic system (B, φ), and, by assumption, this is solvable, while, by Lemma 3.3, any solution of such a system is in K. This proves that ℓ_y maps K *onto* $K(\mathbb{B}_{\backslash y})$. On the other hand, since $y \in \mathbb{E}$, it is $\mathbb{B}$-placeable (by Proposition 3.5). Now apply Theorem 2.16.

(b): We prove this part by induction on $\#\mathbb{B}$, it being trivially true when $\#\mathbb{B} = 1$. Assume that $\#\mathbb{B} > 1$. Then, by Proposition 3.11, there exists $b \in \mathbb{E}$ for which $\emptyset$ is contained in both $\mathbb{E}(\mathbb{B}_{|b})$ and $\mathbb{E}(\mathbb{B}_{\backslash b})$. In particular, neither $\mathbb{B}_{|b}$ nor $\mathbb{B}_{\backslash b}$ is empty, hence both are of cardinality $< \#\mathbb{B}$, and induction therefore provides the inequalities

$$\dim K(\mathbb{B}_{|b}) \leq \sum_{B \in \mathbb{B}_{|b}} \dim K(\{B\}), \qquad \dim K(\mathbb{B}_{\backslash b}) \leq \sum_{B \in \mathbb{B}_{\backslash b}} \dim K(\{B\}). \tag{3.22}$$

On the other hand, since b is in $\mathbb{E}$, hence placeable, Theorem 2.16 implies that

$$\dim K \leq \dim K(\mathbb{B}_{|b}) + \dim K(\mathbb{B}_{\backslash b}). \tag{3.23}$$

Combining (3.22) and (3.23), we obtain (3.21).

For the equality assertion, note that, as soon as the solvability condition 3.2 is assumed with respect to $\mathbb{B}$, it automatically holds with respect to any subset $\mathbb{B}' \subseteq \mathbb{B}$ (since $K(\mathbb{B}') \subseteq K$). Therefore, if the solvability condition 3.2 holds, then, by (a) and by induction, we have equality in (3.22) and (3.23), hence obtain equality in (3.21). □

The final claim in this section provides a partial converse of (b) in the above theorem.

Proposition 3.24. *Let B be a basis in $\mathbb{B}$ that satisfies, for some ordering $B = (b_1, ..., b_s)$,*

$$\mathbb{B}_{\backslash a} \neq \emptyset \quad \Longrightarrow \quad I_a := \{b \in B : b < a\} \in \mathbb{E}(\mathbb{B}_{\backslash a}), \quad a \in B.$$

If

$$\dim K = \sum_{B' \in \mathbb{B}} \dim K(\{B'\}),$$

then any $\mathbb{B}$-special compatible system (B, φ) is solvable.

Proof: We are to prove that, for any such $B \in \mathbb{B}$, any $\mathbb{B}$–special compatible system (B, φ) is solvable. Since, by Lemma 3.3, any solution of such a system is necessarily in K, this is equivalent to proving that the map

$$P : S \to S^s : f \mapsto (\ell_b f)_{b \in B}$$

carries K *onto* the space $\Phi := \Phi_{b_s}$, where, for $a, b, c \in B$, we define

$$\Phi_a := \{(\varphi_b) \in \mathop{\times}_{b \leq a} K(\mathbb{B}_{\backslash b}) : \forall \{b, c \leq a\}\ \ell_c \varphi_b = \ell_b \varphi_c\}.$$

Since P maps K into Φ and $K \cap \ker P = K(\{B\})$, while

$$\dim K = \sum_{B' \in \mathbb{B}} \dim K(\{B'\})$$

by assumption, it is therefore sufficient to prove that

$$\dim \Phi \leq \sum_{B' \in \mathbb{B} \backslash B} \dim K\{(B'\}). \tag{3.25}$$

For this, we claim, and prove inductively, that

$$\dim \Phi_a \leq \sum_{b \leq a} \dim \left(K(\mathbb{B}_{\backslash b}) \cap \bigcap_{c < b} \ker \ell_c \right). \tag{3.26}$$

The case $a = b_1$ is trivial, since $\Phi_{b_1} = K(\mathbb{B}_{\backslash b_1})$, and (3.26) asserts that $\dim \Phi_{b_1} \leq \dim K(\mathbb{B}_{\backslash b_1})$. Assume, thus, that (3.26) holds for $a \leq c := b_{k-1}$, and consider the case $a = b_k$. We note that every element $\varphi \in \Phi_a$ is of the form (ψ, φ_a), with $\psi \in \Phi_c$, and φ_a satisfying suitable compatibility conditions. Conversely, if $\psi \in \Phi_c$ is extendible to $\varphi = (\psi, \varphi_a) \in \Phi_a$, then it is easily checked that any other such extension (ψ, φ'_a) must satisfy

$$\varphi_a - \varphi'_a \in K(\mathbb{B}_{\backslash a}) \cap \bigcap_{b < a} \ker \ell_b.$$

Therefore, it readily follows that

$$\dim \Phi_a \leq \dim \Phi_c + \dim \left(K(\mathbb{B}_{\backslash a}) \cap \bigcap_{b < a} \ker \ell_b \right),$$

and (3.26) follows.

Further, if $\mathbb{B}_{\backslash a} \neq \emptyset$, then the set $I_a = \{b \in B : b < a\}$ is in $\mathbb{E}(\mathbb{B}_{\backslash a})$ by assumption, hence is $\mathbb{B}_{\backslash a}$-placeable by Proposition 3.5. Thus it follows from Lemma 2.15 that

$$K(\mathbb{B}_{\backslash a}) \cap \bigcap_{b<a} \ker \ell_b = K((\mathbb{B}_{\backslash a})_{|I_a}),$$

and this equality holds trivially when $\mathbb{B}_{\backslash a} = \emptyset$. Therefore, since $\Phi = \Phi_{b_s}$, we obtain from (3.26) and the last equation that

$$\dim \Phi \leq \sum_{a \in B} \dim K((\mathbb{B}_{\backslash a})_{|I_a}) \leq \sum_{a \in B} \sum_{B' \in (\mathbb{B}_{\backslash a})_{|I_a}} \dim K(\{B'\}).$$

Here, the second inequality is trivial for all a with $\mathbb{B}_{\backslash a} = \emptyset$ and follows for all other $a \in B$ from Theorem 3.19(b), since $I_a \in \mathbb{E}(\mathbb{B}_{\backslash a})$ by assumption, hence in $\mathbb{E}((\mathbb{B}_{\backslash a})_{|I_a})$ (by Corollary 3.8), therefore $\emptyset \in \mathbb{E}((\mathbb{B}_{\backslash a})_{|I_a})$ (by Corollary 3.10 applied to $C = I_a$). Further, this double sum equals $\sum_{B' \in \mathbb{B} \backslash B} \dim K(\{B'\})$ since $\mathbb{B} \backslash B$ is the disjoint union of the sets $(\mathbb{B}_{\backslash a})_{|I_a}$, $a \in B$. □

Note that all inequalities established during the proof must actually be equalities. Further, in terms of the $\mathbb{E}$-tree for $\mathbb{B}$ (see Remark 3.15), the only bases which offhand satisfy the conditions of the proposition are those belonging to the largest matroidal node (of the tree) of the form $\mathbb{B}_{|Y}$.

4. An example

In this section, we apply our results from the previous one to an example whose solution is important in box spline theory. Since Approximation Theory and in particular box spline theory is not an issue in this paper, we discuss neither the connections nor the applications of this example to box splines. However, the example has intrinsic importance for the discussion in this paper since it provides a naturally arising instance when the condition $\emptyset \in \mathbb{E}$ holds (and hence (1.3) holds), while the seemingly more verifiable, but stronger, conditions (minimum-closed, order-closed, matroidal) are invalid.

The example goes as follows: M is an s-dimensional linear subspace of $\mathbb{R}^d$ which is spanned by integer vectors. We associate each $x \in \mathbf{X}$ with a vector $v_x \in \mathbb{Q}^d \backslash 0$ and define the **coverage of** x as the set

$$Z_x(M) := \{\alpha \in M \cap \mathbb{Z}^d : \ v_x \cdot \alpha \in \mathbb{Z} \backslash 0\}. \tag{4.1}$$

This set does not change if we replace v_x by its orthogonal projection onto M, and this orthogonal projection is again in $\mathbb{Q}^d$ since M is spanned by integer vectors. We may, and do, therefore assume without loss of generality that $v_x \in M$ for all $x \in \mathbf{X}$. More generally, the **coverage** of $Y \subseteq \mathbf{X}$ is, by definition, the union

$$Z_Y(M) := \cup_{y \in Y} Z_y(M).$$

Further, we say that $x \in \mathbf{X}$ **weakly** covers $Z \subseteq \mathbb{Z}^d \cap M$ in case $\{0\} \neq v_x \cdot Z \subseteq \mathbb{Z}$, and call x an M**-integer** if it weakly covers all of $M \cap \mathbb{Z}^d$. We note that x is an $\mathbb{R}^d$-integer exactly when v_x is a nontrivial integer vector. Finally, for ease of notation, we use the orthogonal complement of $Y \subseteq \mathbf{X}$ to mean the orthogonal complement of the corresponding set of vectors in $\mathbb{Q}$:

$$Y^\perp := \cap_{y \in Y} {v_y}^\perp.$$

Based on these notions, we define the set $\mathbb{B}$ as follows. An illustration for this definition can be found in Example 6.9.

Definition 4.2. *The collection* $\mathbb{B} = \mathbb{B}(M)$ *consists of those sets* $B \subseteq \mathbf{X}$ *of cardinality* s *for which* $M \cap B^\perp = \{0\}$*, and, in some ordering* $B = \{b_1, ..., b_s\}$*, and for each* $j = 1, ..., s$*,* $B_j := \{b_1, ..., b_j\}$ *covers* $(M \cap \mathbb{Z}^d) \backslash {B_j}^\perp$*.*

Since $\dim M = s = \#B$, the condition that $M \cap B^\perp = \{0\}$ implies that, for each $B \in \mathbb{B}$, $(v_b)_{b \in B}$ is a basis for M (recall our assumption that, for each x, $v_x \in M$). Each $B \in \mathbb{B}$ is not just a set, but an ordered set, but we do not count two such B as different elements of $\mathbb{B}$ if they only differ in the ordering of the elements.

Further, if $B = (b_1, \ldots, b_s)$ and, for some $j > 1$, b_j is an M-integer, then we can exchange it with its neighbor to the left, i.e., then also $B' := (B_{j-2}, b_j, b_{j-1}, b_{j+1}, \ldots, b_s)$ satisfies the condition of the definition: Since $(b_1, \ldots, b_j)$ covers $(M \cap \mathbb{Z}^d) \backslash (b_1, \ldots, b_j)^\perp$ regardless of the order in which we write the terms in the sequence $(b_1, \ldots, b_j)$, the only issue is whether (B_{j-2}, b_j) covers $(M \cap \mathbb{Z}^d) \backslash (B_{j-2}, b_j)^\perp$. However, $M \cap \mathbb{Z}^d$ is the union of $(M \cap \mathbb{Z}^d) \backslash {B_{j-2}}^\perp$ with $(M \cap \mathbb{Z}^d) \cap {B_{j-2}}^\perp$, and the first set is covered by B_{j-2}, while b_j, being an M-integer, covers everything in $M \cap \mathbb{Z}^d$ except those elements in ${b_j}^\perp$. Thus, (B_{j-2}, b_j) covers everything in $M \cap \mathbb{Z}^d$ except for $(M \cap \mathbb{Z}^d) \cap {B_{j-2}}^\perp \cap {b_j}^\perp$, i.e., everything in $(M \cap \mathbb{Z}^d) \backslash (B_{j-2}, b_j)^\perp$.

A slightly different definition of $\mathbb{B}$ could have been to merely require the s-set $B \in \mathbb{B}$ to cover all nontrivial integers in M. If $s \leq 3$, this latter variant can be proved to be equivalent to the one we had chosen above. However, it is possible to give examples for $s = 4$ of s-sets of rational vectors which cover all the (nontrivial) integers in a space of dimension 4, which nevertheless do not satisfy the terms of the definition for any ordering of its elements.

If $M = \mathbb{R}^d$, and each x is an M-integer (which is equivalent in this case to having $v_x \in \mathbb{Z}^d$), then $\mathbb{B}$ consists exactly of all subsets B whose corresponding $\{v_x\}_{x \in B}$ form a basis for $\mathbb{R}^d$. In other words, the present setting generalizes the box spline setup described in Example 1.1. As a matter of fact, it is the present setup that one needs to study when the 'directions' of a box spline are permitted to be rational vectors.

Our technical goal is to prove the following two results:

Lemma 4.3. *Every* M*-integer is in* $\mathbb{E}$*.*

Lemma 4.4. $\emptyset \in \mathbb{E}$.

This latter lemma, when combined with Theorem 3.19, provides us with the following.

Theorem 4.5.

$$\dim K \leq \sum_{B \in \mathbb{B}} \dim K(\{B\}), \tag{4.6}$$

and equality holds whenever the solvability condition 3.2 holds.

The problem germane to box spline theory needs only the case $M = \mathbb{R}^d$, and ℓ and S as in Example 1.1. For this case, we need only the upper bound result (4.6), with the matching lower bound already being provided by (1.5) (recall that, for this choice of the operators, $\dim K(\{B\}) = 1$). The fact that we have chosen to define the problem on linear subspaces $M \subseteq \mathbb{R}^d$ is technical: we need it for the inductive proof. We did not restrict attention to the specific ℓ of Example 1.1 simply because the results of §3 allow us to prove the above theorem without prescribing ℓ.

We prove Lemma 4.4 simultaneously with Lemma 4.3:

Proof of Lemma 4.4 and Lemma 4.3. We use induction on $s = \dim M$, the proof being obvious if $\dim M = 1$, since for any rank-1 $\mathbb{B}$, we always have $\emptyset \in \mathbb{E}$, and further $\{y\}$ is then in $\mathbb{B}$ iff y is an M-integer. Assume thus that $s = \dim M > 1$.

First note that the first element in any basis must be an M-integer: Indeed, with b_1 the first element in question, it must cover $(M \cap \mathbb{Z}^d) \backslash b_1{}^\perp$, hence $\mathbb{Z} \supseteq v_{b_1} \cdot (M \cap \mathbb{Z}^d) \neq \{0\}$, the inequality due to the fact that v_{b_1} is a rational vector and part of a basis for M. In particular, there are M-integers in $\mathbf{X}$ unless $\mathbb{B}$ is empty.

Further, we claim that any M-integer y can be placed into any $B \in \mathbb{B}$: Let $B := \{b_1, ..., b_s\}$ and set $v_j := v_{b_j}$, all j. Since $v_y \in M \backslash 0$, and $(v_1, \ldots, v_s)$ is a basis for M, there is a smallest $r \geq 1$ for which

$$\operatorname{span}\{v_1, ..., v_r\} = \operatorname{span}\{v_1, ..., v_{r-1}, v_y\}. \tag{4.7}$$

We contend that y can replace b_r in B. Since the role of b_r in B is to cover $(M \cap \mathbb{Z}^d \cap B_{r-1}^\perp) \backslash B_r^\perp$, we need to prove that y covers this set as well. Since y weakly covers all of $M \cap \mathbb{Z}^d$, we need only verify that it does not vanish on $B_{r-1}^\perp \backslash B_r^\perp$, i.e., that $v_y \cdot \alpha \neq 0$, all $\alpha \in B_{r-1}^\perp \backslash B_r^\perp$. For this, observe that, by choice of r, $v_y \in cv_r + \operatorname{span}(v_j)_{j<r}$ for some nonzero c, hence, for any $\alpha \in B_{r-1}^\perp \backslash B_r^\perp$, $v_y \cdot \alpha = cv_r \cdot \alpha \neq 0$. Thus, indeed, $B' := (B \backslash b_r) \cup y \in \mathbb{B}$, and y is thus placeable. After placing y into B, the discussion following the definition of $\mathbb{B}$ implies that we can place y as the first element of B' (without changing the order of the rest).

Since y is placeable, it is, by Corollary 3.8, in $\mathbb{E}$ if and only if it is in $\mathbb{E}(\mathbb{B}_{|y})$, and, by Corollary 3.10 (applied with $C = \{y\}$ and $Y = \emptyset$), this latter condition is equivalent to having $\emptyset \in \mathbb{E}(\mathbb{B}_{|y})$. Thus, to prove that $y \in \mathbb{E}$, it remains to show that $\emptyset \in \mathbb{E}(\mathbb{B}_{|y})$. For this we apply induction on s: we first define M' to be the subspace of M which is perpendicular to v_y. Since y covers $(M \backslash M') \cap \mathbb{Z}^d$, but certainly does not cover any $\alpha \in (M' \cap \mathbb{Z}^d)$, we conclude that $B \in \mathbb{B}(M')$ if and only if $(y, B) \in \mathbb{B}$. In particular, $\emptyset \in \mathbb{E}(\mathbb{B}_{|y})$ if and only if $\emptyset \in \mathbb{E}(\mathbb{B}(M'))$, and the latter condition holds by induction hypothesis since $\dim M' = s - 1$.

To complete the inductive step (and thereby the proof of the two lemmata), it remains to show that $\emptyset \in \mathbb{E}$, which we prove by induction on the number of M-integers in $\mathbf{X}$. Assume that there is an M-integer y. Since y is in $\mathbb{E}$, it can serve as an extender for $\emptyset$, provided $\emptyset \in \mathbb{E}(\mathbb{B}_{\backslash y})$ in case $\mathbb{B}_{\backslash y} \neq \emptyset$. But the latter proviso holds by our induction hypothesis (on the number of M-integers, of which $\mathbf{X} \backslash y$ is guaranteed to contain at least one since $\mathbb{B}_{\backslash y} \neq \emptyset$, but fewer than does $\mathbf{X}$). □

5. Matroid structure and special solvability

In this section, we prove the dimension formula (1.3) under the assumption that $\mathbb{B}$ is matroidal.

Recall from Proposition 2.9 that $\mathbb{B}$ is matroidal if and only if each independent x is replaceable in $\mathbb{B}$, and, from Proposition 2.13, that, if $\mathbb{B}$ is matroidal, then every independent element x is $\mathbb{B}$-placeable, with the converse not true in general.

Proposition 5.1. *$\mathbb{B}$ is matroidal if and only if $\mathbb{E} = \mathbb{I}$.*

Proof: '$\Longrightarrow$': It is sufficient to prove that, for any $C \subset B \in \mathbb{B}$, and any $b \in B \backslash C$, $C \in \mathbb{I}(\mathbb{B}_{\backslash b})$ in case $\mathbb{B}_{\backslash b} \neq \emptyset$. For this, if $B' \in \mathbb{B}_{\backslash b}$, then, since $\mathbb{B}$ is matroidal, C is placeable in B', i.e., extendible to a basis using only elements of B' and, since $b \notin B'$, this implies that $C \in \mathbb{I}(\mathbb{B}_{\backslash b})$.

'$\Longleftarrow$': If $\mathbb{E} = \mathbb{I}$, then, by Proposition 3.5, every independent set is placeable, hence $\mathbb{B}$ is matroidal by Proposition 2.13. □

If $\mathbb{B}$ is matroidal, then, for any $x \in \mathbf{X}$, also $\mathbb{B}_{|x}$ and $\mathbb{B}_{\backslash x}$ are matroidal.

For matroidal $\mathbb{B}$, we have the following theorem.

Theorem 5.2. *If $\mathbb{B}$ is matroidal and $\dim K < \infty$, then, the following are equivalent:*

(i) The solvability condition 3.2 holds.

(ii) For all $Y \subseteq \mathbf{X}$ and $y \in Y$, ℓ_y maps $K(\mathbb{B}_Y)$ onto $K(\mathbb{B}_{Y\backslash y})$.

(iii)

$$\dim K = \sum_{B \in \mathbb{B}} \dim K(\{B\}).$$

Proof: (i)$\Longrightarrow$(ii): Let $y \in Y \subseteq \mathbf{X}$. Since $\mathbb{B}$ is matroidal, so is $\mathbb{B}_Y$, hence $y \in \mathbb{E}(\mathbb{B}_Y)$ by Proposition 5.1. Since, for every $b \in Y$, $K(\mathbb{B}_{Y\backslash b}) \subset K(\mathbb{B}_{\backslash b})$, the solvability condition 3.2 (which is assumed to hold with respect to $\mathbb{B}$) holds with respect to $\mathbb{B}_Y$, too. Therefore, Theorem 3.19 (a), with $\mathbb{B}$ there replaced by $\mathbb{B}_Y$, implies that ℓ_y maps $K(\mathbb{B}_Y)$ onto $K(\mathbb{B}_{Y\backslash y})$.

(ii)$\Longrightarrow$(iii): The proof is by induction on $\#\mathbb{B}$, it being trivially true when $\#\mathbb{B} \leq 1$. Assume $\#\mathbb{B} > 1$, and choose $y \in \mathbf{X}(\mathbb{B})$ so that $\mathbb{B}_{\backslash y} \neq \emptyset$. Since $y \in \mathbb{E}$ (by Proposition 5.1), we conclude from Theorem 3.19 that

$$\dim K = \dim K(\mathbb{B}_{|y}) + \dim K(\mathbb{B}_{\backslash y}),$$

and this equals $\sum_{B \in \mathbb{B}_{|y}} \dim K(\{B\}) + \sum_{B \in \mathbb{B}_{\backslash y}} \dim K(\{B\})$ by induction (applicable since both $\mathbb{B}_{|y}$ and $\mathbb{B}_{\backslash y}$ have smaller cardinality than $\mathbb{B}$), and this equals

$$\sum_{B \in \mathbb{B}} \dim K(\{B\})$$

since $\mathbb{B}$ is the disjoint union of $\mathbb{B}_{|y}$ and $\mathbb{B}_{\backslash y}$.

(iii)$\Longrightarrow$(i): This implication is a special case of Proposition 3.24, since $\mathbb{B}_{\backslash b}$ is matroidal for any b, hence, by Proposition 5.1, every $B \in \mathbb{B}$ satisfies the conditions imposed on B in Proposition 3.24. □

Corollary 5.3. *If $\mathbb{B}$ is matroidal, $\dim K < \infty$, and $s > 2$, then any of the conditions (i)-(iii) in Theorem 5.2 is equivalent to the following condition:*

(iv) For every (some) $r \in (1\,\mathrm{d}s)$, and every $Y \subseteq \mathbf{X}$, every $\mathbb{B}_Y$–special compatible independent system (C, φ) with $\#C = r$ has solutions in $K(\mathbb{B}_Y)$.

Proof: (i) $\Longrightarrow$ (iv): Given a $\mathbb{B}_Y$-special compatible independent system (C, φ), since $\mathbb{B}_Y$ is matroidal, we have $C \in \mathbb{E}(\mathbb{B}_Y)$ by Proposition 5.1. Hence, by Proposition 3.18, (C, φ) can be extended to a $\mathbb{B}_Y$-special compatible basic system (B, φ). Since every $\mathbb{B}_Y$-special system is also $\mathbb{B}$-special, assumption (i) implies the solvability of (B, φ), and any of its solutions is necessarily in $K(\mathbb{B}_Y)$, by Lemma 3.3. Hence, (C, φ) has solutions in $K(\mathbb{B}_Y)$.

(iv) $\Longrightarrow$ (ii): Given an independent $y \in Y$ and $\varphi \in K(\mathbb{B}_{Y\backslash y})$, we know, from Proposition 5.1 and the fact that $\mathbb{B}_Y$ is matroidal, that $y \in \mathbb{E}$. Therefore, the proof of Proposition 3.18 shows that the equation (y, φ) can be extended, step by step, to a $\mathbb{B}_Y$-special compatible basic system. Instead of performing all the $s-1$ steps of this extension process, we can stop after $r-1$ steps to obtain a $\mathbb{B}_Y$-special compatible independent system of r equations, which admits a solution in $K(\mathbb{B}_Y)$, since (iv) is assumed. □

Remark. We note that, in [DM3], (iii) is obtained under the (explicit) assumption that every compatible independent system (C, φ) is solvable and the (implicit) assumption that any $\mathbb{B}$–special compatible independent system (C, φ) actually has solutions in K. Since we are able to derive (iii) from (i), we are in effect avoiding the possibly hard task of verifying that certain systems not only are solvable, but have solutions in some subspace (like K). In fact, since we show that (i), (ii), and (iv) are all equivalent, we are incidentally closing the gap in the argument for (i)$\Longrightarrow$(iii) in [DM3: Lemma 3.2] by showing that any $\mathbb{B}$-special compatible independent system (C, φ) has solutions in K if any $\mathbb{B}$-special compatible basic system is solvable.

6. Order-closed and minimum-closed

We now consider a weakening of the assumption that $\mathbb{B}$ is matroidal. Any total order on $\mathbf{X}$ induces a corresponding partial order on $\mathbb{B}$ by the prescription

$$B := (b_1 < \cdots < b_s) \ \le \ B' := (b'_1 < \cdots < b'_s) \quad \Longleftrightarrow \quad b_j \le b'_j, \quad j = 1, \ldots, s.$$

We recall from [BR] the following weakening of being matroidal.

Definition. *We call* $\mathbb{B}$ **order-closed in** $\mathbb{B}'$ *if*
(i) $\mathbb{B} \subseteq \mathbb{B}'$*;*
(ii) $\mathbb{B}'$ *is matroidal;*
(iii) $B' \le B$ *for some* $B' \in \mathbb{B}'$ *and* $B \in \mathbb{B}$ *implies that* $B' \in \mathbb{B}$.

We note that, for any $Y \subseteq \mathbf{X}$, $\mathbb{B}_Y$ is order-closed in $\mathbb{B}'_Y$ if $\mathbb{B}$ is order-closed in $\mathbb{B}'$.

Lemma 6.1. *Any order-closed* $\mathbb{B}$ *has a unique minimal element, namely the unique minimal element of the associated matroidal* $\mathbb{B}'$.

Proof: Let $B =: (b_1 < \cdots < b_s)$ be a minimal element of $\mathbb{B}$. If not $B \le B'$ for all $B' \in \mathbb{B}'$, then there would exist $B' = (b'_1, \ldots . b'_s) \in \mathbb{B}'$ so that $b'_j < b_j$ for some j. Assume without loss that j is the smallest such index. Since $\mathbb{B}'$ is matroidal and both $(b_1, \ldots, b_{j-1})$ and $(b'_1, \ldots, b'_j)$ are in $\mathbb{I}(\mathbb{B}')$, there would exist some $k \le j$ so that $(b_1, \ldots, b_{j-1}, b'_k) \in \mathbb{I}(\mathbb{B}')$ and, further, $(b_1, \ldots, b_{j-1}, b'_k)$ could be completed to an element $B'' = (b_1, \ldots, b_{j-1}, b'_k, \ldots)$ of $\mathbb{B}'$ using elements from B. Since $b'_k \le b'_j < b_j$, it would follow that $B'' < B$, hence $B'' \in \mathbb{B}$ (since $\mathbb{B}$ is order-closed), and this would contradict the minimality of B. $\square$

In particular, given an ordering on $\mathbf{X}$, any matroidal $\mathbb{B}$ has a unique minimal element, which we will denote by

$$\min \mathbb{B}.$$

Since we often need only this consequence of order-closedness, we give it a special name.

Definition 6.2. *We call* $\mathbb{B}$ **minimum-closed in** $\mathbb{B}'$ *if*
(i) $\mathbb{B} \subseteq \mathbb{B}'$*;*
(ii) $\mathbb{B}'$ *is matroidal;*
(iii) for all $Y \subseteq \mathbf{X}$, $\min \mathbb{B}'_Y \in \mathbb{B}_Y$.

Again, if $\mathbb{B}$ is minimum-closed in $\mathbb{B}'$, so is $\mathbb{B}_Y$ in $\mathbb{B}'_Y$ for any $Y \subseteq \mathbf{X}$.

Proposition 6.3. *If* $(b_1 < \cdots < b_s) := \min \mathbb{B}$, *with* $\mathbb{B}$ *minimum-closed in* $\mathbb{B}'$, *then, for any* $k \geq 0$, $I_k := (b_1, \ldots, b_k)$ *is in* $\mathbb{E}$, *as well as in* $\mathbb{E}(\mathbb{B}_{\backslash b_{k+1}})$ *if* $\mathbb{B}_{\backslash b_{k+1}}$ *is not empty. In particular,* $\emptyset \in \mathbb{E}$.

Proof: The proof is by induction on $\#\mathbb{B}$ and (downward) induction on $k \leq s$, it being trivially true when $\#\mathbb{B} = 1$ or $k = s$.

So assume that $\#\mathbb{B} > 1$ and $k < s$. Then, by induction hypothesis, $I_k \cup b_{k+1} \in \mathbb{E}$. If $\mathbb{B}_{\backslash b_{k+1}} \neq \emptyset$, let B' be its minimal element. Then $I_k \subseteq B'$ since, otherwise, we could complete I_k to an element B'' of the matroidal $\mathbb{B}'_{\backslash b_{k+1}}$ by elements from B', and this would imply that $B'' < B'$, contradicting the minimality of B'. Therefore, I_k is the initial segment of the minimal element for $\mathbb{B}_{\backslash b_{k+1}}$, hence in $\mathbb{E}(\mathbb{B}_{\backslash b_{k+1}})$, by induction hypothesis (on $\#\mathbb{B}$). This verifies that $I_k \in \mathbb{E}$. $\square$

For a minimum-closed $\mathbb{B}$, we have the following dimension formula.

Theorem 6.4. *If* $\mathbb{B}$ *is minimum-closed (in particular, if* $\mathbb{B}$ *is order-closed) in* $\mathbb{B}'$, *then*

$$\dim K \leq \sum_{B \in \mathbb{B}} K(\{B\}), \tag{6.5}$$

with equality in case the solvability condition 3.2 holds. Further, if equality holds, then any $\mathbb{B}$*-special compatible system* $(\min \mathbb{B}, \varphi)$ *is solvable.*

Proof: By Proposition 6.3, $\emptyset \in \mathbb{E}$, hence the claim here follows from part (b) of Theorem 3.19, with the final statement true by the same Proposition 6.3 and Proposition 3.24. $\square$

In the rest of this section we make several observations relevant to minimum-closedness.

Proposition 6.6. *If* $\mathbb{B}$ *is minimum-closed in* $\mathbb{B}'$ *and* $\#\mathbb{B} > 1$, *then* $y := \max\{x \in \mathbf{X}(\mathbb{B}) : \mathbb{B}_{\backslash x} \neq \emptyset\}$ *is well-defined and replaceable.*

Proof: Let $B, B' \in \mathbb{B}$ and assume $y \in B$. We need to find $b \in B'$ that replaces y. If $y \in B'$, take $b = y$. Otherwise, since $\mathbb{B}'$ is matroidal and $B, B' \in \mathbb{B} \subseteq \mathbb{B}'$, we can find $b \in B'$ for which $B'' := (B\backslash y) \cup b \in \mathbb{B}'$. Since $Y := B'' \cup y$ contains a basis (namely B) from $\mathbb{B}$ and $\mathbb{B}$ is minimum-closed, $\mathbb{B}$ must contain $\min \mathbb{B}'_Y$. However, $\min \mathbb{B}'_Y = B''$ because $y > b$ (by the maximality of y and the fact that $B \in \mathbb{B}_{\backslash b}$, hence $\mathbb{B}_{\backslash b} \neq \emptyset$). $\square$

Proposition 6.7. *Let* $\mathbb{B} \subseteq \mathbb{B}'$ *for some matroidal* $\mathbb{B}'$. *Then,* $\mathbb{B}$ *is minimum-closed in* $\mathbb{B}'$ *if and only if, for each* $Y \subseteq \mathbf{X}$ *of cardinality* $s + 1$,

$$\mathbb{B}_Y \neq \emptyset \quad \Longrightarrow \quad \min \mathbb{B}'_Y \in \mathbb{B}.$$

Proof: The implication '$\Longrightarrow$' is trivial.

'$\Longleftarrow$': Let $Y \subseteq \mathbf{X}$, and assume that $B \in \mathbb{B}_Y$. Let $B' := \min \mathbb{B}'_Y$. We need to show that $B' \in \mathbb{B}$, and for this we can assume without loss that $Y = B \cup B'$ (since otherwise we can replace Y by its subset $B \cup B'$). We prove the desired result by induction on $\#Y$. If $\#Y \leq s + 1$, then $B' = \min \mathbb{B}'_Y \in \mathbb{B}$, by assumption. Assume now that $\#Y > s + 1$. Let y be the maximal element in $B\backslash B'$ and C be the set of all elements in B which are larger than y. Then, $C \subseteq B \cap B'$ by the choice of y.

First we observe that we need only to prove that $\mathbb{B}_{Y\backslash y} \neq \emptyset$. Indeed, we clearly have $B' \subseteq (Y\backslash y)$, and therefore $B' = \min \mathbb{B}'_{Y\backslash y}$, and hence, by the induction hypothesis (applicable since $Y\backslash y$ has one less element and our proof goes by induction on $\#Y$), $B' \in \mathbb{B}$.

Since $\mathbb{B}'$ is matroidal, we can replace $y \in B$ by an element $x \in B'$.

We claim that $x < y$ for any such x: if not, every element in the subset $C \cup x$ of B' is larger than y. Thus, B' contains at least $\#C + 1$ elements which are larger than y, while B contains only $\#C$ elements larger than y, and this is impossible, since $B' < B$.

We now let $B'' := (B \backslash y) \cup x$. We claim that it suffices to prove that $B'' = \min \mathbb{B}'_{B \cup x}$. Indeed, $B \cup x$ consists of $s + 1$ atoms, and contains a basis from $\mathbb{B}$ (viz. B), therefore, $B'' \in \mathbb{B}$ by the hypothesis of the proposition. Since $B'' \subseteq Y \backslash y$, this proves that $\mathbb{B}_{Y \backslash y} \neq \emptyset$, and, by the above observation, completes the proof of the proposition.

Thus, it remains to show that $B'' = B''' := \min \mathbb{B}'_{B \cup x}$. If not, then $B''' < B''$. Since B'' misses only y (from $B \cup x$), B''' must miss then a larger element, and because we already proved that $x < y$, this missed atom must belong to C. But then B''' contains only $\#C - 1$ atoms larger than y, while B' contains at least $\#C$ atoms larger than y, hence B' cannot be smaller than B'''. This contradicts the minimality of B', thereby completing the proof. □

We now give examples to show that, in general, the implications

$$\text{order-closed} \implies \text{minimum-closed} \implies \emptyset \in \mathbb{E}$$

proved and used in this section cannot be reversed even if we permit complete freedom in the choice of the matroidal $\mathbb{B}'$ in which $\mathbb{B}$ is to be order-, resp., minimum-closed, and also permit complete freedom in the ordering.

The first example shows that a $\mathbb{B}$ satisfying the $\mathbb{E}$-condition need not be minimum-closed in any matroidal $\mathbb{B}'$ and in any ordering.

Example 6.8. Let $\mathbf{X}$ and $\mathbb{B}$ be as in Example 2.12. Since $\mathbf{X}$ contains no $\mathbb{B}$-replaceable atom, Proposition 6.6 shows that $\mathbb{B}$ is never minimum-closed regardless of the ordering we choose for $\mathbf{X}$. On the other hand, we claim that $\emptyset \in \mathbb{E}(\mathbb{B})$. One binary tree that proves this claim goes as follows: we choose 3 (which was verified to be placeable). In $\mathbb{B}_{|3}$ every atom is replaceable (since only one 3-set that contains 3 is not a basis), hence is a matroid, by Proposition 2.9. As for $\mathbb{B}_{\backslash 3}$, here 4 is placeable (since it was so in the beginning). Again, $\mathbb{B}_{\backslash 3|4}$ is matroidal, and we need to look only at $\mathbb{B}_{\backslash 3,4}$, which is an order-closed subset for the ordering $\{1, 2, 7, 8, 5, 6\}$ and with $\mathbb{B}' := \binom{\mathbf{X} \backslash \{3,4\}}{3}$.

□

The next example is a strengthening of the preceding one, in that it shows that the results on minimum-closedness are not general enough to solve the problem of §4; i.e., while Lemma 4.4 asserts that $\emptyset \in \mathbb{E}$, the stronger assertion "$\mathbb{B}$ is minimum-closed" is, in general, invalid for $\mathbb{B}$ considered in §4.

Example 6.9. Let $\mathbf{X} = 123456 := \{1, \ldots, 6\}$, and $\mathbb{B} = \{12, 23, 13, 14, 25, 36\}$. This is the $\mathbb{B}$ obtained in §4 for the choice $M = \mathbb{R}^d$, $d = 2$, and

$$v_1 = (1, 0),\ v_2 = (0, 1),\ v_3 = (1, 1),\ v_4 = (1/2, 1),\ v_5 = (1, 1/2),\ v_6 = (1/2, -1/2).$$

We now assume that $\mathbb{B}$ is minimum-closed in some matroidal $\mathbb{B}'$ and with respect to some ordering $<$ on $\mathbf{X}$, and derive from this a contradiction.

First, due to the symmetries in $\mathbb{B}$, we can assume without loss of generality that $4 < 5 < 6$. Then, we consider the following three subsets of $\mathbf{X}$:
(a): $Y = 356$. The only basis in $\mathbb{B}_Y$ is 36. If $35 \in \mathbb{B}'$, then, as $5 < 6$ implies $35 < 36$, $\mathbb{B}_Y$ would not be minimum-closed in $\mathbb{B}'_Y$. Therefore, we must have $35 \notin \mathbb{B}'$.

(b): $Y = 346$. Repeating the argument in (a), with 4 replacing 5, we conclude that $34 \notin \mathbb{B}'$. Consequently, since $34, 35 \notin \mathbb{B}'$, and $\mathbb{B}'$ is matroidal, also $45 \notin \mathbb{B}'$ (since, otherwise, 3 could not be placed into the basis 45).
(c): $Y = 245$. Since $25 \in \mathbb{B} \subset \mathbb{B}'$, and $45 \notin \mathbb{B}'$, we must have $24 \in \mathbb{B}'$ (otherwise, 4 cannot be placed into 25). Thus $\mathbb{B}_Y = \{25\}$ while $\mathbb{B}'_Y = \{24, 25\}$, and since $24 < 25$, $\mathbb{B}_Y$ is not minimum-closed in $\mathbb{B}'_Y$. □

The final example shows that the results concerning minimum-closed $\mathbb{B}$ are a true generalization of their order-closed counterparts, by exhibiting a minimum-closed $\mathbb{B}$ which fails to be order-closed in any matroidal $\mathbb{B}'$ containing it and any ordering of $\mathbf{X}$.

Example 6.10. Let $\mathbf{X} = 12345 := \{1, ..., 5\}$, and take $\mathbb{B} := \binom{\mathbf{X}}{3} \backslash \{125, \ 135, \ 245, \ 345\}$. The resulting $\mathbb{B}$ is trivially minimum-closed in $\binom{\mathbf{X}}{3}$ with respect to the natural ordering, since each of the four sets omitted contains the largest atom, 5, hence is the minimum basis in some $\mathbb{B}^0_Y$ only in the trivial case when it equals Y.

Assume now that $\mathbb{B}$ is order-closed in some matroidal $\mathbb{B}'$ and with respect to some ordering $<$ on $\mathbf{X}$. We show that this assumption is untenable.

First, we claim that, with this assumption, necessarily $125 \in \mathbb{B}'$ and prove this by contradiction. Indeed, if $125 \notin \mathbb{B}'$, then necessarily $135 \in \mathbb{B}'$ since otherwise no element from $123 \in \mathbb{B} \subseteq \mathbb{B}'$ could be used to replace 4 in $145 \in \mathbb{B} \subseteq \mathbb{B}'$. With that, comparison of $135 \in \mathbb{B}' \backslash \mathbb{B}$ with $145 \in \mathbb{B}$ implies that $4 < 3$. On the other hand, the same argument shows that (still under the assumption $125 \notin \mathbb{B}'$) also 245 is necessarily in $\mathbb{B}'$, and, now, comparison of $245 \in \mathbb{B}' \backslash \mathbb{B}$ with $235 \in \mathbb{B}$ shows that $3 < 4$, a contradiction.

Since 2 and 3 enter the definition of $\mathbb{B}$ symmetrically, as do 1 and 4, it follows that necessarily all the four sets excluded from $\mathbb{B}$ must be in $\mathbb{B}'$. In particular, both 135 and 245 must be in $\mathbb{B}'$, yet, as we just saw, this leads to the contradictory conclusions that $4 < 3$ and $3 < 4$. We have reached a contradiction. □

Note that all the 3-sets actively involved in this example are in $\mathbb{B}_{|5}$. We can therefore think of this example as being of rank 2, with the atom 5 added only in order to make $\mathbb{B}$ minimum-closed. With this, $\mathbb{B}$ itself reduces to that simplest of pathological examples in the context of this paper, namely the set

$$12, \ 34$$

which must fail to be order-closed since the dimension theorem fails for it in general.

7. Dimension equalities without $\mathbb{B}$-conditions

In this section, we consider a nice application of the material detailed in the last two sections. This application is based on the following ℓ-condition, which we show later on to imply our solvability condition 3.2 under the additional assumption that (7.4) holds.

Definition 7.1. *We call the pair* $(\mathbb{B}, \ell)$ **direct** *if, for every $B \in \mathbb{B}$ and every $x \in \mathbf{X} \backslash B$, ℓ_x defines a (linear) automorphism on $K(\{B\})$.*

For example, the pair $(\mathbb{B}, \ell)$ of Example 1.1 is direct for a generic choice of the constants $\{\lambda_x\}_x$. Note that ℓ_x is a (linear) automorphism on $K(\{B\})$ exactly when it is 1-1 on $K(\{B\})$ since, in any case, for any $f \in K(\{B\})$ and any $b \in B$, $\ell_b(\ell_x f) = \ell_x(\ell_b f) = 0$, hence ℓ_x maps $K(\{B\})$ into itself.

We chose the term "direct" since, for a direct $(\mathbb{B}, \ell)$, the sum $\sum_{B \in \mathbb{B}} K(\{B\})$ is direct. This implies at once that, for a direct $(\mathbb{B}, \ell)$, (7.2) $$\dim K \geq \sum_{B\in \mathbb{B}} \dim K(\{B\}),$$

since, by (1.6), we always have the inclusion

(7.3) $$K \supseteq \sum_{B\in \mathbb{B}} K(\{B\}).$$

It is also clear that, for a matroidal $\mathbb{B}$, equality holds in (7.2), since the converse inequality

(7.4) $$\dim K \leq \sum_{B\in \mathbb{B}} \dim K(\{B\})$$

holds for such $\mathbb{B}$, by virtue of Theorem 3.19(b) and the fact that every matroidal $\mathbb{B}$ satisfies the $\mathbb{E}$-condition. However, the following theorem seems to be less obvious:

Theorem 7.5. *Assume that the pair $(\mathbb{B}_0, \ell)$ is direct and $\mathbb{B}_0$ is matroidal. Then the equality*

$$\dim K = \sum_{B\in \mathbb{B}} \dim K(\{B\})$$

holds for an arbitrary $\mathbb{B} \subseteq \mathbb{B}_0$.

In view of (7.2), we need only to prove (7.4). We present two different arguments for (7.4), each of which proves (7.4) in a more general setup than required here. The first approach relaxes the requirement that $\mathbb{B}_0$ be matroidal, and the second approach relaxes the ℓ-condition of directness.

Our first generalized version of Theorem 7.5 reads as follows:

Theorem 7.6. *Theorem 7.5 holds even if we assume that $\mathbb{B}_0$, in lieu of being matroidal, merely satisfies*

$$\dim K(\mathbb{B}_0) \leq \sum_{B\in \mathbb{B}_0} \dim K(\{B\}).$$

Thus, this stronger version of Theorem 7.5 applies to any $\mathbb{B}_0$ satisfying the $\mathbb{E}$-condition (in particular, to order-closed or minimum-closed $\mathbb{B}_0$), as well as to any fair $\mathbb{B}_0$ (cf. the next section).

Proof: Consider the map

$$P : K(\mathbb{B}_0) \to \underset{B\in \mathbb{B}_0\backslash \mathbb{B}}{\times} K(\{B\}) : f \mapsto (\ell_{X\backslash B} f)_{B\in \mathbb{B}_0\backslash \mathbb{B}}.$$

P is well-defined (i.e., into) by Proposition 2.2, and

$$\ker P = K(\mathbb{B}_0) \cap \bigcap_{B\in \mathbb{B}_0\backslash \mathbb{B}} \ker \ell_{X\backslash B} \supseteq K.$$

Further, P is *onto* since, for any $B, B' \in \mathbb{B}_0$,

$$\ell_{\mathbf{X}\backslash B'} K(\{B\}) = \begin{cases} 0, & B' \neq B; \\ K(\{B'\}), & B' = B, \end{cases}$$

and $K(\{B\}) \subseteq K(\mathbb{B}_0)$ for all $B \in \mathbb{B}_0$. It follows that

$$\dim \ker P + \sum_{B\in \mathbb{B}_0\backslash \mathbb{B}} \dim K(\{B\}) = \dim K(\mathbb{B}_0) \leq \sum_{B\in \mathbb{B}_0} \dim K(\{B\}).$$

Hence,

$$\dim K \leq \dim \ker P \leq \sum_{B\in \mathbb{B}} \dim K(\{B\}),$$

and the desired equality now follows from (7.2). □

The other approach for the proof of Theorem 7.5 goes as follows. We introduce a new atom x and define ℓ_x to be the zero map. Further, using $\mathbf{X} \cup x$ as the atom set, we attempt to find a new set $\mathbb{B}'$ of bases of rank $s+1$ that satisfies the following three conditions:

(i) $(B,x) \in \mathbb{B}' \iff B \in \mathbb{B}$.

(ii) $K(\{B'\}) = 0$, for every $B' \in \mathbb{B}'_{\backslash x}$.

(iii) $\dim K(\mathbb{B}') \le \sum_{B\in\mathbb{B}'} \dim K(\{B\})$.

Proposition 7.7. *If $\mathbb{B}$ has a rank-$(s+1)$ "extension" $\mathbb{B}'$ that satisfies the conditions (i-iii) specified above, then (7.4) holds.*

Proof: We observe that, since $\ell_x = 0$, $K(\{(B,x)\}) = K(\{B\})$, while $K(\{B'\}) = 0$, for any other $B' \in \mathbb{B}'$, because of assumption (ii). Thus, (iii) leads to

$$\dim K(\mathbb{B}') \le \sum_{B\in\mathbb{B}} \dim K(\{B\}).$$

The claim then follows from the fact that $K \subseteq K(\mathbb{B}')$ which can be observed in the following way. Given $A \in \mathbb{A}(\mathbb{B}')$, we have two possibilities to consider: (1) $x \in A$. In such a case $\ell_A = 0$ and therefore it annihilates K. (2) $x \notin A$. Then, since A intersects every (B,x), $B \in \mathbb{B}$, it must intersect every $B \in \mathbb{B}$, hence lies in $\mathbb{A}(\mathbb{B})$. Thus, indeed, $K \subseteq K(\mathbb{B}')$. □

Consequently, the inequality (7.4) required for the proof of Theorem 7.5 is established, as soon as we demonstrate the existence of a $\mathbb{B}'$ which satisfies (i-iii), as we do in the next proposition.

Proposition 7.8. *Assume that $\mathbb{B}_0$ is matroidal and the pair $(\mathbb{B}_0, \ell)$ is direct. For an arbitrary $\mathbb{B} \subseteq \mathbb{B}_0$, and a new atom $x \notin \mathbf{X}$, define*

$$\mathbb{B}' := \{(B,x):\ B \in \mathbb{B}\} \cup \{(B,y):\ B \in \mathbb{B}_0,\ y \in \mathbf{X}\backslash B\}.$$

Then $\mathbb{B}'$ satisfies conditions (i-iii) above, and hence (7.4) holds (by Proposition 7.7).

Proof: The fact that (ii) is satisfied follows from the directness of $(\mathbb{B}_0, \ell)$. Condition (i) trivially follows from the definition of $\mathbb{B}'$. The last condition, (iii), will follow from Theorem 6.4 as soon as we show that $\mathbb{B}'$ is minimum-closed in

$$\mathbb{B}'_0 := \{(B,y):\ B \in \mathbb{B}_0,\ y \in (\mathbf{X} \cup x)\backslash B\},$$

in any ordering that makes x the maximal atom.

For that, we first want to show that $\mathbb{B}'_0$ is matroidal. Here, we consider two bases (B,y), (B',z) in $\mathbb{B}'_0$ (namely, $B, B' \in \mathbb{B}_0$), choose $a \in (B,y)$ and search for a replacement for a in (B',z). If $a = y$, we can replace it by any atom in $(B' \cup z)\backslash B$. Otherwise, $a \in B$, and in this case we consider two different possibilities. (a): $(B\backslash a) \cup y \in \mathbb{B}_0$. Then we can write $(B,y) = (B'',a)$, with $B'' \in \mathbb{B}_0$, and proceed as in the previous case. (b): $(B\backslash a) \cup y \notin \mathbb{B}_0$. Since $B, B' \in \mathbb{B}_0$, and $\mathbb{B}_0$ is matroidal, there exists $b \in B'$, for which $(B\backslash a) \cup b \in \mathbb{B}_0$. Since we assume that $(B\backslash a) \cup y \notin \mathbb{B}_0$, we must have $b \neq y$, and hence this b is an appropriate replacement for a.

To prove that $\mathbb{B}'$ is minimum-closed in $\mathbb{B}'_0$, we first observe that all the bases in $\mathbb{B}'_0\backslash\mathbb{B}'$ contain x. Now, let $Y \subseteq \mathbf{X} \cup x$ be of cardinality $> s+1 = \operatorname{rank} \mathbb{B}'_0$. If Y contains a basis $(B,x) \in \mathbb{B}'_0\backslash\mathbb{B}'$, then $B \in \mathbb{B}_0$, and choosing any $y \in Y\backslash(B \cup x)$, we obtain a basis $(B,y) \in \mathbb{B}'$. Since x is maximal in our ordering, $(B,y) < (B,x)$, and hence (B,x) is not the minimal basis of $\mathbb{B}'_0$ on Y. Therefore, $\mathbb{B}'$ is minimum-closed in $\mathbb{B}'_0$, as claimed. □

We want to unravel a little bit the three conditions (i-iii) required of the "extension" $\mathbb{B}'$. Condition (i) determines an initial set of bases in $\mathbb{B}'$, and the subsequent problem is to determine $\mathbb{B}'_{\backslash x}$. Condition (ii) is an ℓ-condition, and asserts that for every basis $B' \in \mathbb{B}'_{\backslash x}$ and every $b \in B'$, ℓ_b is 1-1 on $K(\{B\backslash b\})$. This is a weakening of the assumption that $(\mathbb{B}, \ell)$ be direct, since we may try to construct $\mathbb{B}'$ in such a way that $\mathbb{B}'_{\backslash x}$ is small. In contrast, Condition (iii) (which should be regarded as a $\mathbb{B}$-condition on $\mathbb{B}'$, because upper bound assertions do not require ℓ-conditions) pulls the situation in the opposite direction, since the $\mathbb{B}$-conditions which are known to imply upper bounds usually assert a certain "richness" property of the underlying set of bases.

The following example illustrates further the conditions (i-iii).

Example 7.9. Let $\mathbb{B}_0$ be the collection of all s-sets in $\mathbf{X}$, hence $\mathbb{B}_0$ is matroidal. Assume that, in some ordering $<$ on $\mathbf{X}$, the following condition is satisfied: for every $B \in \mathbb{B}_0$ and every $y \in \mathbf{X}$ with $y > b$ for all $b \in B$, ℓ_y is 1-1 on $K(\{B\})$. We claim that then the inequality (7.4) holds for an arbitrary $\mathbb{B} \subseteq \mathbb{B}_0$ (i.e., an arbitrary collection of s-sets).

To verify this, we show that, given $\mathbb{B} \subseteq \mathbb{B}_0$, we can construct $\mathbb{B}'$ that satisfies (i-iii) (and then invoke Proposition 7.8). We define $\mathbb{B}'_0$ to be the collection of all $(s+1)$-sets in $X \cup x$ (with x a new atom and $\ell_x = 0$), and define

$$\mathbb{B}' := \{(B, x) : \ B \in \mathbb{B}\} \cup \{B' : \ B' \subseteq \mathbf{X}, \ \#B' = s+1\}.$$

Here, condition (i) trivially holds, and condition (ii) holds, since, for the largest atom b in every $(s+1)$-set $B' \in \mathbb{B}'_{\backslash x}$, ℓ_b is assumed to be 1-1 on $K(\{B\backslash b\})$. As for condition (iii), one verifies, as in the proof for Proposition 7.8, that, with x chosen to be the largest atom in $X \cup x$, $\mathbb{B}'$ is minimum-closed in $\mathbb{B}'_0$. □

We close this section with a proof that, in the presence of the upper bound (7.4), directness implies the solvability condition 3.2.

Proposition 7.10. *If $(\mathbb{B}, \ell)$ is direct and satisfies (7.4), then every $\mathbb{B}$-special basic compatible system is solvable.*

Proof: Let $\mathbb{B}' \subseteq \mathbb{B}$. By Theorem 7.6, the assumptions imply that

$$\dim K(\mathbb{B}') = \sum_{B \in \mathbb{B}'} \dim K(\{B\}),$$

while, by directness, $\sum_{B \in \mathbb{B}'} K(\{B\})$ is direct and in $K(\mathbb{B}')$. Hence, altogether,

$$K(\mathbb{B}') = \bigoplus_{B \in \mathbb{B}'} K(\{B\}).$$

For each $B \in \mathbb{B}$, let P_B be the projector on K onto $K(\{B\})$ corresponding to this direct sum decomposition of K. Since each ℓ_y maps each of these summands $K(\{B\})$ into itself, ℓ_y commutes with each such P_B. Hence, if the basic system (B', φ) is special and compatible, then, for each $B \in \mathbb{B}$, the system

$$\ell_b ? = P_B \varphi_b, \quad b \in B', \tag{7.11}$$

is compatible and, further, $P_B \varphi_b = 0$ in case $b \in B$, since by assumption, the original system is special, hence

$$\varphi_b \in K(\mathbb{B}_{\backslash b}) = \bigoplus_{B \in \mathbb{B}_{\backslash b}} K(\{B\}).$$

In particular, $P_{B'}\varphi_b = 0$ for all $b \in B'$, hence $f_{B'} := 0$ solves (7.11) in case $B = B'$. In the contrary case, at least one of the ℓ_b involved is invertible on $K(\{B\})$, hence the system has a solution in $K(\{B\})$, namely

$$f_B := (\ell_{b|K(\{B\})})^{-1} P_B \varphi_b.$$

It follows that $f \in K$ given by the identity

$$P_B f = f_B, \quad B \in \mathbb{B}$$

solves the original system. □

8. A replaceability condition and s-additivity

In the last five sections we analysed the dimension of K with the aid of the atomic map, hence are now in a position to enlarge on the remarks at the end of §2 concerning the relative merits of the two approaches, via the DM map and via the atomic map, to the bounding of $\dim K$. In view of the examples discussed in this paper, the $\mathbb{B}$-conditions required for the application of the atomic map (e.g., placeability) are more likely to hold than their DM counterparts (replaceability). Secondly (and more importantly), the ℓ-condition we use in the atomic approach (i.e., the solvability condition 3.2) is weaker than the one we need for the implementation of the DM map (the s-additivity, see below). This means that as long as we have in hand $\mathbb{B}$-conditions which allow us to decompose $\mathbb{B}$ through the atomic map (for example if $\emptyset \in \mathbb{E}$), we can get no better results by using the DM map. This observation applies, in particular, to matroidal, order-closed, and minimum-closed structures. The notion of replaceability plays an important role in the discussion in [DDM: §6], and hence various results obtained there are related to those of this section. We mention, however, that the method and the ℓ-condition that we employ here differ from the ones used in [DDM].

In this section, we consider as an ℓ-condition the notion of s-additivity, which was introduced in [S] and was successfully applied in [S] and [JRS1] for a matroidal and order-closed $\mathbb{B}$ respectively. While we already derived, in §5 and §6, results stronger than their counterparts from [S] and [JRS1], the approach of [JRS1] can be extended to yield new dimension results which are not obtained in the previous sections. This is due to the fact that the existence of a replaceable atom (needed here) does not imply the existence of a placeable element. It thus requires a complementary discussion of estimates for K via the DM map and the notion of replaceability.

For this discussion, let

$$G$$

denote the abelian semi-group generated by (the elements of) $\ell(\mathbf{X})$. Since this discussion involves the joint kernel of an arbitrary sequence L in G, we also use the letter K for such a joint kernel, i.e., write

$$K(L) := \bigcap_{l \in L} \ker l,$$

and trust that the reader will have no difficulty distinguishing between $K(L)$ for a sequence L in G and $K(\mathbb{B})$ for a collection $\mathbb{B}$ of subsets of $\mathbf{X}$.

Definition 8.1. *We say that G is s-**additive** in case*

$$\dim K(L, gh) = \dim K(L, g) + \dim K(L, h)$$

for arbitrary $(s-1)$-sequences L and arbitrary elements g, h (in G).

Before making use of this condition, it is perhaps useful to compare it to the solvability condition 3.2 placed on ℓ in §3, as is done in the following proposition which also fully answers the question raised in [RJS].

Proposition 8.2. *G is s-additive if and only if, for every matroidal $\mathbb{B}$ and every $\ell : \mathbf{X} \to G$ with $\dim K < \infty$, every special compatible basic system is solvable.*

Proof: It follows from [S: Theorem (2.4)] that G is s-additive if and only if (iii) of Theorem 5.2 holds for an arbitrary matroidal $\mathbb{B}$ (of rank s) and $\ell : \mathbf{X} \to G$. But, for each fixed matroidal $\mathbb{B}$ and $\ell : \mathbf{X} \to G$, (iii) is equivalent to (i) of Theorem 5.2 which says that any special compatible basic system is solvable. $\square$

We note that a comparison of s-additivity and the solvability condition 3.2 has also been made in [JRS2]. In particular, [JRS2: Theorem (2.11)] can be derived from Proposition 8.2 and Corollary 5.3.

The following lemma will play an important role in the proof of the main induction step in the next theorem. It is a variant of [JRS1: Theorem (2.1)], and employs the notation

$$K_\ell(\mathbb{B}) := \bigcap_{A \in \mathbf{A}_{\min}(\mathbb{B})} \ker \ell_A$$

whenever the dependence on the particular map ℓ needs stressing.

Lemma 8.3. *Assume that $y \in \mathbf{X}$, $H \in \mathbb{H}$, and $\ell : \mathbf{X} \to G$ satisfy the following conditions:*
(i) $\dim K < \infty$;
(ii) For arbitrary $\ell' : y \cup H \to G$,

$$\dim K_{\ell'}(\mathbb{B}_{y\cup H}) = \sum_{B \in \mathbb{B}(y\cup H)} \dim K_{\ell'}(\{B\});$$

(iii) y is $\mathbb{B}$-replaceable.
Then, for each $\varphi \in K(\mathbb{B}_{y\cup H})$, the system

$$\text{(8.4)} \qquad \begin{aligned} &\ell_{\mathbf{X}\backslash(y\cup H)} ? = \varphi \\ &\ell_{\mathbf{X}\backslash(y\cup H')} ? = 0 \qquad \forall H' \in \mathbb{H}\backslash H \end{aligned}$$

has solutions in K.

Proof: We apply [JRS1: Theorem (2.1)] to prove this lemma. For this, note that $\mathbb{B} \subseteq \mathbb{B}'$, where $\mathbb{B}' := \{B \subseteq \mathbf{X}(\mathbb{B}) : \#B = s\}$ is matroidal. Then the conditions (i) and (ii) are exactly the same as the conditions (i) and (ii) of that theorem.

As to condition (iii), since y is $\mathbb{B}$-replaceable,

$$H = \{x \in \mathbf{X} : (B\backslash y) \cup x \notin \mathbb{B}\}$$

for each $B \in \mathbb{B}_{y\cup H}$, as proved at the beginning of the proof of Proposition 2.8. This implies that, for all $B \in \mathbb{B}_{y\cup H}$ and for all $x \in \mathbf{X}\backslash H$, $(B\backslash y) \cup x \in \mathbb{B}$ which is the condition (iii) of [JRS1: Theorem (2.1)].

The solvability of (8.4) therefore follows from that theorem. We now prove that each such solution f is necessarily in K. This means that we need to show that $\ell_{\mathbf{X}\backslash H'} f = 0$, for all $H' \in \mathbb{H}$. If $H' \neq H$, then already $\ell_{\mathbf{X}\backslash(y\cup H')} f = 0$. Otherwise, $H' = H$, and there are two possibilities to consider: (a) $y \in H$. In this case $\mathbb{B}_{y\cup H} = \emptyset$, and hence $\varphi = 0$, and thus $\ell_{\mathbf{X}\backslash H} f = \ell_{\mathbf{X}\backslash(y\cup H)} f = \varphi = 0$. (b) $y \notin H$. Here, we compute $\ell_{\mathbf{X}\backslash H} f = \ell_y \ell_{\mathbf{X}\backslash(y\cup H)} f = \ell_y \varphi = 0$, with the last equality since $\varphi \in K(\mathbb{B}_{y\cup H})$, and y is a cocircuit in $\mathbb{B}_{y\cup H}$. $\square$

We note that the proof, in Proposition 3.18, of the solvability of system (3.1) does not rely on any dimension identity involving subspaces of K. This, as we saw in §3, gives a new approach to the study of the joint kernels and solves problems which cannot be easily solved by the other way. On the other hand, the proof of the solvability of the system (8.4) relies on condition (ii) which is a dimension identity for the subspace $K(\mathbb{B}_{y\cup H})$ of K. This motivates the following definition.

Definition 8.5. *We say that* $\mathbb{B}$ *is* **fair** *if, for all* $Y \subseteq \mathbf{X}$ *with* $\#\mathbb{B}_Y > 1$*, there exists a* $\mathbb{B}_Y$*-replaceable* y *for which* $\mathbb{B}_{Y\backslash y} \neq \emptyset$.

Note that $\mathbb{B}$ is fair in case it satisfies some property which (i) is inherited by subsets (i.e., holds with respect to any $\mathbb{B}_Y$, $Y \subseteq \mathbf{X}$), and which (ii) implies, in case $\#\mathbb{B} > 1$, the existence of a replaceable $y \in \mathbf{X}$ whose corresponding $\mathbb{B}_{\backslash y}$ and $\mathbb{B}_{|y}$ are not empty. An instance of such a property is minimum-closedness (which is obviously inherited by subsets, and which satisfies (ii) by Proposition 6.6), and, hence, we have we have the following.

Corollary 8.6. *Every minimum-closed* $\mathbb{B}$ *is fair.*

This last corollary is not extremely useful, since results on minimum-closed $\mathbb{B}$ were already established in §6 by other means, and the results below on fair structures will not improve upon those from §6. It is more significant to note that a fair $\mathbb{B}$ need not be minimum-closed, since otherwise our main result here (Theorem 8.10) would become a weaker version of (the first part of) Theorem 6.4. The next example serves this purpose.

Example 8.7. Let $M := \mathbb{R}^d$, $d = 2$, and let $\mathbb{B}$ be chosen as in §4, with respect to the present M. It can be checked then that $\mathbb{B}$ is fair. Precisely, given $Y \subseteq \mathbf{X}$, if Y contains only integer vectors, then $\mathbb{B}_Y$ is matroidal (as observed in the discussion prior to Lemma 4.3) and hence every $y \in \mathbf{X}(\mathbb{B}_Y)$ is $\mathbb{B}_Y$-replaceable. Otherwise, every noninteger $y \in \mathbf{X}(\mathbb{B}_Y)$ is $\mathbb{B}_Y$-replaceable: since such y appears second, hence last, in every basis B that contains it, its only contribution is to cover the nonzero integers on the line which is not covered by the other atom in that basis, or equivalently, to cover a nonzero integer α, α' on this line which is closest to the origin. Given another basis B', there must be $b \in B'$ that covers α and this b can replace y in B. □

On the other hand, Example 6.9 exhibits a special case of the above setup which is not minimum-closed, hence fair cannot imply minimum-closedness. As a matter of fact, since that example satisfies the $\mathbb{E}$-condition (as does every $\mathbb{B}$ of §4), we see that even the $\mathbb{E}$-condition combined with the assumption that $\mathbb{B}$ is fair does not imply minimum-closedness. Finally, the following example shows that $\mathbb{B}$ can be fair without satisfying the $\mathbb{E}$-condition. This means that the results in this section concerning $\dim K$ could not have been derived directly from their counterparts in §3.

Example 8.8. Let $\mathbb{B} = \{123, 124, 125, 246, 147, 367, 467, 567\}$. Then only 4 is placeable, and, in $\mathbb{B}_{\backslash 4} = \{123, 125, 367, 567\}$, only 3 and 5 are placeable, and, with $x = 3$ or 5, $\mathbb{B}_{\backslash 4|x} = \{12x, x67\}$ cannot be split any further by a placeable element. Thus, by Theorem 3.17, $\mathbb{B}$ does not satisfy the $\mathbb{E}$-condition. On the other hand, $\mathbb{B}$ is fair, as one verifies directly. □

Proposition 8.9. *If* $\mathbb{B}$ *is fair, then*

$$\dim K \leq \sum_{B\in\mathbb{B}} \dim K(\{B\}).$$

Proof: We use induction on $\#\mathbb{B}$, it being trivially true in case $\#\mathbb{B} \leq 1$. So, assume that $\#\mathbb{B} > 1$. Then, there is, by assumption, a replaceable $y \in \mathbf{X}(\mathbb{B})$, and, by Proposition 2.8, this implies (2.5) which, in turn, implies that $\mathbb{B}$ is the *disjoint* union of the collections $\mathbb{B}_{y\cup H}$, $H \in \mathbb{H}$, and $\mathbb{B}_{\backslash y}$. Further, since $y \in \mathbf{X}(\mathbb{B})$, $\#\mathbb{B}_{\backslash y} < \#\mathbb{B}$, and since $\mathbb{B}_{\backslash y} \neq \emptyset$, by assumption, also $\#\mathbb{B}_{y\cup H} < \#\mathbb{B}$, $H \in \mathbb{H}$. Therefore, induction together with (2.3) finishes the proof. □

We have been reminded that this proposition was already proved in [J], with the ‘intersection conditions’ used there enforcing, by Proposition 2.8, what we have called here ‘fair’.

Theorem 8.10. *Suppose that* $\mathbb{B}$ *(of rank* s*) is fair and* G *is* s*-additive. Then, for arbitrary* $\ell : \mathbf{X} \to G$ *with* $\dim K < \infty$,

$$\dim K = \sum_{B \in \mathbb{B}} \dim K(\{B\}).$$

Proof: The proof is by induction on $\#\mathbb{B}$, it being trivially true when $\#\mathbb{B} \le 1$. Let $y \in \mathbf{X}(\mathbb{B})$ be $\mathbb{B}$-replaceable, with $\mathbb{B}_{\backslash y} \neq \emptyset$. The major induction step is to prove that the short sequence

$$0 \to K(\mathbb{B}_{\backslash y}) \hookrightarrow K \xrightarrow{j} \mathop{\times}_{H \in \mathbb{H}} K(\mathbb{B}_{y \cup H}) \to 0 \tag{8.11}$$

is exact, with j defined by (1.10), but using $H \in \mathbb{H}$ rather than $A \in \mathbb{A}_{\min}(\mathbb{B}_{\backslash y})$ to index the components of j's target, as discussed at the beginning of §2.1.

Since $\ker j = K(\mathbb{B}_{\backslash y})$, the sequence is exact at K. To prove that the sequence (8.11) is exact, it remains to show that j is *onto*. This follows by applying Lemma 8.3 to each $H \in \mathbb{H}$. Lemma 8.3 can be applied to each $H \in \mathbb{H}$, since, for each $H \in \mathbb{H}$, (i) holds by assumption and (iii) holds by the choice of y, while, finally, since $y \in \mathbf{X}(\mathbb{B})$, $\#\mathbb{B}_{\backslash y} < \#\mathbb{B}$ and since $\mathbb{B}_{\backslash y} \neq \emptyset$, $\#\mathbb{B}_{y \cup H} < \#\mathbb{B}$ for each H, hence (ii) holds by induction hypothesis.

It follows from the exactness of the sequence (8.11) that

$$\dim K = \dim K(\mathbb{B}_{\backslash y}) + \sum_{H \in \mathbb{H}} \dim K(\mathbb{B}_{y \cup H}).$$

Since y is $\mathbb{B}$-replaceable, $\bigcup_{H \in \mathbb{H}} \mathbb{B}_{y \cup H}$ is a disjoint union, by Proposition 2.8. Therefore, $\mathbb{B}$ is the disjoint union of $\mathbb{B}_{\backslash y}$ and $\{\mathbb{B}_{y \cup H},\, H \in \mathbb{H}\}$. Applying the induction hypothesis to $K(\mathbb{B}_{\backslash y})$ and $\{K(\mathbb{B}_{y \cup H}) : H \in \mathbb{H}\}$, we have

$$\dim K = \sum_{B \in \mathbb{B}} \dim K(\{B\}).$$

□

We remark that the exactness of (8.11) gives the exactness of the “Hom” of the sequence (6.31) of [DDM], and Theorem 8.10 holds if $\mathbb{B}$ is ‘strongly coherent’ as defined in [DDM]. Interested readers should consult [DDM] for details.

As noted before, being fair is implied by minimum-closedness, thereby is also implied by order-closedness, and these implications are proper. Our result, thus, improves [JRS1: Theorem (2.3)], since the latter concerns order-closed sets. The argument we use, however, is essentially the one in [JRS1].

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