On Ptak's derivation of the Jordan normal form

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Some readers of [1] might appreciate the following comments that make more explicit how Ptak's beautiful insight there leads to a trivial proof of (the basics of) the Jordan normal form.

The proof of Theorem 1 of [1] can also be based on the observation that, X being finite-dimensional, the sequence $\{0\} \subseteq \ker A \subseteq \ker A^2 \subseteq \cdots$, must eventually be stationary, i.e., $\ker A^q = \ker A^{q+p}$ for some q and all $p > 0$. For such q, let X_r and X_s be the range and the kernel, respectively, of A^q, hence $\dim X = \dim X_r + \dim X_s$. Further, for any $x \in X_r \cap X_s$, $x = A^q z$ for some z, and so $z \in \ker A^{2q} = \ker A^q$, hence $x = 0$. Therefore, X is the direct sum of the two A-invariant subspaces X_s and X_r, and A is regular on X_r (since A^q is) and is nilpotent on X_s.

In the setup and notation of Theorem 2 of [1], there must be, by duality, some y_0 in Y for which $\langle x_0 A^{q-1}, y_0 \rangle \neq 0$, hence the q-order matrix $(\langle x_0 A^{j-1}, y_0 A^{*q-i} \rangle : i, j = 1, \ldots, q)$ is triangular with nonzero diagonal entries, therefore invertible, and this guarantees that X is the direct sum $X_0 + X'$, with X_0 the linear span of $(x_0 A^{j-1} : j = 1, \ldots, q)$ and X' the annihilator of $\{y_0 A^{*q-i} : i = 1, \ldots, q\}$, both of which are A-invariant. Moreover, it shows $(x_0 A^{j-1} : j = 1, \ldots, q)$ to be a basis for X_0, and the matrix representation, with respect to this basis, of A restricted to X_0 has the familiar form of a Jordan block (for the eigenvalue 0).

Now, X being finite-dimensional, there are A-invariant direct sum decompositions $X = X_1 + \cdots + X_m$ that are minimal in the sense that none of its summands is the direct sum of two nontrivial A-invariant subspaces. Take any one such. Then the matrix representation for A with respect to any basis made up from bases for the summands X_i is block diagonal, with the ith block the matrix representation of the restriction A_i of A to X_i with respect to the chosen basis for X_i.

Assuming the underlying field to be algebraically closed, the restriction A_i of A to X_i has some eigenvalue, λ_i, and, in view of the minimality of X_i, Theorem 1 ensures that $B_i := A_i - \lambda_i$ is nilpotent, while Theorem 2 then ensures that, for some $x \in X_i$ and some q, $(x B_i^{j-1} : j = 1, \ldots, q)$ is a basis for X_i, and the matrix representation of A_i with respect to that basis is a Jordan block with λ_i as its diagonal element.

Theorems 1 and 2 of [1] don't seem to assist in the proof that the Jordan normal form is unique (up to reordering of the blocks), although such uniqueness is readily established by the observation that

$$n_j := \dim \ker(A - \lambda)^j = \sum_{\lambda_i = \lambda} \min(\dim X_i, j),$$

hence $\Delta n_j := n_{j+1} - n_j$ equals the number of blocks for λ of order $> j$, giving the decomposition-independent number $-\Delta^2 n_{j-1}$ for the number of Jordan blocks for λ of order j.

[1] V. Ptak, A remark on the Jordan normal form of matrices, Linear Algebra Appl. (this issue, i.e., vol. 310, 2000, xxx–xxx)