AN IMPROVED ORDER OF APPROXIMATION FOR THIN-PLATE SPLINE INTERPOLATION IN THE UNIT DISC

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Abstract. We show that the L_p-norm of the error in thin-plate spline interpolation in the unit disc decays like $O(h^{\gamma_p+1/2})$, where $\gamma_p := \min\{2, 1+2/p\}$, under the assumptions that the function to be approximated is C^∞ and that the interpolation points contain the finite grid $\{hj : j \in \mathbb{Z}^2,\ |hj| < 1-h\}$.

1. Introduction

Let H be the set of all continuous functions $f : \mathbb{R}^2 \to \mathbb{C}$ having square integrable second order derivatives, and let $|||\cdot|||$ be the semi-norm defined on H by

$$|||f||| := \sqrt{\int_{\mathbb{R}^2} \left|\frac{\partial^2 f(x)}{\partial x_1^2}\right|^2 + 2\left|\frac{\partial^2 f(x)}{\partial x_1 \partial x_2}\right|^2 + \left|\frac{\partial^2 f(x)}{\partial x_2^2}\right|^2 dx}.$$

Let Ξ be any bounded set of non-collinear points in $\mathbb{R}^2$. Duchon [2] has shown that to each $f \in H$, there exists a unique $s \in H$ which minimizes $|||s|||$ subject to the interpolation conditions $s_{|\Xi} = f_{|\Xi}$. The function s is called the *thin-plate spline interpolant to f at Ξ* and will be denoted by $T_\Xi f$. When Ξ contains only finitely many points, Duchon further characterized $T_\Xi f$ as the unique function in $S(\phi;\Xi)$ which interpolates f at Ξ. Here $\phi : \mathbb{R}^2 \to \mathbb{R}$ is the radially symmetric function given by

$$\phi(x) := |x|^2 \log|x|, \qquad x \in \mathbb{R}^2,$$

and $S(\phi;\Xi)$ is the space of all functions g of the form

$$g = \sum_{\xi\in\Xi} \lambda_\xi \phi(\cdot - \xi) + p,$$

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where $p \in \Pi_1 := \{\text{polynomials of total degree} \le 1\}$ and the λ_ξ's satisfy

$$\sum_{\xi\in\Xi} \lambda_\xi q(\xi) = 0, \quad \forall q \in \Pi_1.$$

An important problem relating to thin-plate spline interpolation is that of determining the rate at which $T_\Xi f$ converges to f as the points Ξ become dense. Let us assume that $\Omega \subset \mathbb{R}^2$ is an open bounded domain over which the error will be measured. We assume that $\Xi \subset \overline{\Omega}$, and we define the 'density' of Ξ in Ω to be the number

$$\delta := \delta(\Xi;\Omega) := \sup_{x\in\Omega} \min_{\xi\in\Xi} |x-\xi|.$$

Thin-plate spline interpolation in Ω is said to provide *L_p-approximation of order γ* if

$$\|f - T_\Xi f\|_{L_p(\Omega)} = O(\delta^\gamma)$$

for all sufficiently smooth functions f. Duchon [3] has shown that if Ω is connected, satisfies a uniform cone condition, and has a Lipschitz boundary, then thin-plate spline interpolation in Ω provides L_p-approximation of order at least

$$\gamma_p := \min\{2, 1+2/p\}$$

for $p \in [1\,..\,\infty]$. More precisely, it was shown that

(1.1)
$$\|f - T_\Xi f\|_{L_p(\Omega)} \le \text{const}\, \delta^{\gamma_p} |||T_\Omega f - T_\Xi f|||, \quad \forall f \in H,\ p \in [1\,..\,\infty], \qquad \text{and}$$

(1.2)
$$|||T_\Omega f - T_\Xi f||| \to 0 \text{ as } \delta \to 0.$$

Powell [7] (see also [10]) has obtained similar results for the case $p = \infty$ with less restrictive assumptions on the domain Ω, and has even found the best const in (1.1) for some special cases. In the limiting case when the points Ξ are taken as the infinite grid $h\mathbb{Z}^2$ and Ω is taken as all of $\mathbb{R}^2$, it was shown by Buhmann [1] that $\|f - T_\Xi f\|_{L_\infty(\mathbb{R}^2)} = O(h^4)$ as $h \to 0$ for all sufficiently smooth f. His approach employed techniques developed in the context of approximation from shift-invariant spaces; however, this shift-invariant space approach has yet to provide any results on the approximation order of thin-plate spline interpolation in bounded domains (as defined above). Recently, Johnson [6] has shown that one should not in general expect thin-plate spline interpolation to provide L_p-approximation of order greater than $2+1/p$. Precisely, it was shown that if Ω is the unit disk $B := \{x \in \mathbb{R}^2 : |x| < 1\}$, then there exists $f \in C^\infty(\mathbb{R}^2)$ such that $\|f - T_\Xi f\|_{L_p(B)} \ne o(\delta^{2+1/p})$. Note that the difference between this upper bound on the approximation order of $2+1/p$ and Duchon's lower bound of γ_p is $\frac{1}{2} + \left|\frac{1}{p} - \frac{1}{2}\right|$.

The purpose of the present paper is to build on Duchon's work to obtain L_p-approximation of order $\gamma_p + 1/2$ in a special case. Our point of attack is the factor $|||T_\Omega f - T_\Xi f|||$ on the right hand side of (1.1) which, according to (1.2), decays to 0 as $\delta \to 0$. It seems plausible that if f is sufficiently smooth, then this factor might decay to zero as some power of δ. We can see immediately, that one should not in general expect this factor to decay faster than $O(\sqrt{\delta})$:

Theorem 1.3. *If $\Omega = B$, then there exists $f \in C^\infty(\mathbb{R}^2)$ such that*

$$|||T_\Omega f - T_\Xi f||| \neq o(\sqrt{\delta}) \text{ as } \delta \to 0.$$

Proof. According to [6], there exists $f \in C^\infty(\mathbb{R}^2)$ such that $\|f - T_\Xi f\|_{L_2(\Omega)} \neq o(\delta^{5/2})$. If $|||T_\Omega f - T_\Xi f||| = o(\sqrt{\delta})$, then it follows from (1.1) that $\|f - T_\Xi f\|_{L_2(\Omega)} = o(\delta^{5/2})$ which is a contradiction. $\square$

In order to investigate the decay of $|||T_\Omega f - T_\Xi f|||$ in the most favorable of circumstances, we make the following simplifying assumptions:
First, we assume that the function to be approximated, f, belongs to $C^\infty(\mathbb{R}^2)$. Second, we assume that our domain Ω is the open unit disc B. And last, we assume that our centres Ξ satisfy

$$\Xi_h \subset \Xi \subset \overline{B}, \qquad \text{where } \Xi_h := h\mathbb{Z}^2 \cap (1-h)B. \tag{1.4}$$

Note that $S(\phi; \Xi_h) \subset S(\phi; \Xi)$ and $\delta(\Xi; B) \leq \delta(\Xi_h; B) = O(h)$ as $h \to 0$.

Under these assumptions, we show that the factor $|||T_B f - T_\Xi f|||$ decays to 0 as $O(\sqrt{h})$. Precisely, we show the following

Theorem 1.5. *If Ξ satisfies (1.4) and $f \in C^\infty(\mathbb{R}^2)$, then $|||T_B f - T_\Xi f||| = O(\sqrt{h})$ as $h \to 0$, and consequently*

$$\|f - T_\Xi f\|_{L_p(B)} = O(h^{\gamma_p + 1/2}) \text{ as } h \to 0,$$

where $\gamma_p := \min\{2, 1 + 2/p\}$, $p \in [1\,..\,\infty]$.

Note that, for $p = 2$, we obtain L_2-approximation of order $5/2$ (modulo assumption (1.4)) which matches Johnson's upper bound on the L_2-approximation order.

In the sequel we use standard multi-index notation: $D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}$. The Laplacian operator is denoted $\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$. For multi-indices $\alpha \in \{0, 1, 2, \dots\}^2$, we define $|\alpha| := \alpha_1 + \alpha_2$, while for $x \in \mathbb{R}^2$, we define $|x| := \sqrt{x_1^2 + x_2^2}$. For multi-indices α, we employ the notation $()^\alpha$ to represent the monomial $x \mapsto x^\alpha$, $x \in \mathbb{R}^2$. The space of bivariate polynomials of total degree $\leq k$ can then be expressed as $\Pi_k := \operatorname{span}\{()^\alpha : |\alpha| \leq k\}$. For $x \in \mathbb{R}^2$, we define the complex exponential e_x by $e_x(t) := e^{ix \cdot t}$, $t \in \mathbb{R}^2$. The Fourier transform of a function f can then be expressed as $\widehat{f}(w) := \int_{\mathbb{R}^2} e_{-w}(x) f(x)\, dx$. The space of compactly supported C^∞ functions is denoted $C_c^\infty(\mathbb{R}^2)$. If μ is a distribution and g is a test function, then the application of μ to g is denoted $\langle g, \mu \rangle$. Familiarity with tempered distributions is assumed throughout the sequel. Two important facts in this regard are first that the Fourier transform of ϕ can be identified on $\mathbb{R}^2 \backslash 0$ with $8\pi\, |\cdot|^{-4}$ (cf. [4]), and second that

$$|||f||| = (2\pi)^{-1} \left\| |\cdot|^2 \widehat{f} \right\|_{L_2(\mathbb{R}^2)} = \|\Delta f\|_{L_2(\mathbb{R}^2)}$$

which is an application of the Plancherel Theorem [8; page 172].

2. A Preliminary Result

Definition 2.1. Let $\mathcal{F}$ be the collection of all functions of the form $\phi*\mu+p$, where $p\in\Pi_1$ and μ is a distribution of order at most 1 satisfying

$$\begin{aligned} &(i) \qquad \operatorname{supp}\mu\subset\overline{B}, \qquad \text{and} \\ &(ii) \qquad |\widehat{\mu}(w)|\le \mathrm{const}\frac{|w|^2}{1+|w|^{3/2}}, \quad w\in\mathbb{R}^2. \end{aligned}$$

We point out that since ϕ and all its first order derivatives are continuous and exhibit only polynomial growth at ∞, and since μ is a compactly supported distribution of order at most 1, it follows that the function $\phi*\mu$, defined by

$$\phi*\mu(x):=\langle\phi(x-\cdot),\mu\rangle,$$

is continuous, has only polynomial growth at ∞, and satisfies $(\phi*\mu)\widehat{\ }=\widehat{\phi}\widehat{\mu}$. The purpose of this section is to prove the following

Theorem 2.2. *Let Ξ satisfy (1.4). If $f\in\mathcal{F}$, then the following hold:*

$$\begin{aligned} &(i) \qquad f\in H, \\ &(ii) \qquad |||f-T_\Xi f|||=O(\sqrt{h}) \text{ as } h\to 0, \\ &(iii) \qquad T_B f=f, \\ &(iv) \qquad \|f-T_\Xi f\|_{L_p(B)}=O(h^{\gamma_p+1/2}) \text{ as } h\to 0, \end{aligned}$$

where $\gamma_p:=\min\{2,1+2/p\}$, $p\in[1\,..\,\infty]$.

The following lemma is crucial to proving (i).

Lemma 2.3. *If $g\in C_c(\mathbb{R}^2)$ satisfies $|g(w)|\le\mathrm{const}\,|w|^3$, then*

$$\langle g,\widehat{\phi}\rangle=8\pi\int_{\mathbb{R}^2}|w|^{-4}\,g(w)\,dw.$$

Proof. Let $\sigma\in C_c^\infty(\mathbb{R}^2)$ satisfy $\sigma_{|_B}=1$, and define the tempered distribution $\widehat{\nu}$ according to

$$\langle g,\widehat{\nu}\rangle:=8\pi\int_{\mathbb{R}^2}|w|^{-4}\left(g(w)-\sum_{|\alpha|\le 2}\frac{D^\alpha g(0)}{\alpha!}\sigma(w)w^\alpha\right)dw, \quad g\in C_c^\infty(\mathbb{R}^2).$$

(Note that $\left|g(w)-\sum_{|\alpha|\le 2}\frac{D^\alpha g(0)}{\alpha!}\sigma(w)w^\alpha\right|\le\mathrm{const}(g)\,|w|^3$ and hence the above integrand is absolutely integrable.) Since $\widehat{\nu}=\widehat{\phi}$ on $\mathbb{R}^2\backslash 0$, it follows that $\phi=\nu+p$ for some polynomial p. In order to show that $p\in\Pi_2$, we will estimate the growth of $|\nu(x)|$ for large $|x|$. Assume $|x| \geq 1$, and put $k_x(w) := e_x(w) - \sum_{|\alpha|\leq 2} \frac{D^\alpha e_x(0)}{\alpha!}\sigma(w)w^\alpha$, $w \in \mathbb{R}^2$. Since $\widehat{\nu}$ can be identified with an integrable function on $\mathbb{R}^2 \backslash B$, it follows that

$$\nu(x) = (2\pi)^{-2}\langle e_x, \widehat{\nu}\rangle = (2\pi)^{-2} 8\pi \int_{\mathbb{R}^2} |w|^{-4}\, k_x(w)\, dw.$$

Since $|e_x| = 1$ and $\max_{|\alpha|\leq 2} \|D^\alpha e_x(0)\| \leq \text{const}\, |x|^2$, we have the crude estimate $|k_x(w)| \leq \text{const}\ |x|^2$, $\forall w \in \mathbb{R}^2$. Noting that $D^\alpha k_x(0) = 0$, $\forall\, |\alpha| \leq 2$, it follows from Taylor's theorem that for $w \in B$,

$$|k_x(w)| \leq \text{const}\, |w|^3 \max_{|\alpha|=3} \|D^\alpha k_x\|_{L_\infty(B)} \leq \text{const}\, |w|^3\, |x|^3 .$$

Employing these two estimates on $\mathbb{R}^2 \backslash\, |x|^{-1/3}\, B$ and $|x|^{-1/3}\, B$, respectively, we obtain

$$\begin{aligned} |\nu(x)| &\leq \text{const} \int_{|w|\geq x^{-1/3}} |w|^{-4}\, |k_x(w)|\ dw + \text{const} \int_{|w|< x^{-1/3}} |w|^{-4}\, |k_x(w)|\ dw \\ &\leq \text{const}\ |x|^2 \int_{|w|\geq x^{-1/3}} |w|^{-4}\ dw + \text{const}\, |x|^3 \int_{|w|< x^{-1/3}} |w|^{-4}\, |w|^3\ dw \\ &\leq \text{const}\ |x|^2 \int_{|x|^{-1/3}}^{\infty} r^{-4} r\, dr + \text{const}\ |x|^3 \int_0^{|x|^{-1/3}} r^{-1} r\, dr = \text{const}\ |x|^{8/3} . \end{aligned}$$

Since $|\phi(x)|$ is also bounded by const $|x|^{8/3}$ for $|x| \geq 1$, it follows that $|p(x)| = |\phi(x) - \nu(x)| \leq$ const $|x|^{8/3}$ for $|x| \geq 1$. Hence, $p \in \Pi_2$. Therefore, there exists constants a_α, $|\alpha| \leq 2$ such that

$$\langle g, \widehat{\phi}\rangle = \langle g, \nu\rangle + \sum_{|\alpha|\leq 2} a_\alpha D^\alpha g(0), \quad \forall g \in C_c^\infty(\mathbb{R}^2). \tag{2.3}$$

Now if $g \in C_c^\infty(\mathbb{R}^2)$ satisfies $|g(w)| \leq \text{const}\, |w|^3$, then $D^\alpha g(0) = 0$, $\forall\, |\alpha| \leq 2$ and consequently (2.3) reduces to $\langle g, \widehat{\phi}\rangle = 8\pi \int_{\mathbb{R}^2} |w|^{-4}\, g(w)\, dw$. $\square$

Proof of Theorem 2.2 (i). Let $f \in \mathcal{F}$, and let μ and p be as in Definition 2.1. In order to show that $f \in H$, we must show that $D^\alpha f \in L_2\ \forall\, |\alpha| = 2$. Assume $|\alpha| = 2$. Then $(D^\alpha f)\widehat{\ } = -()^\alpha \widehat{\phi}\widehat{\mu}$. Now if $g \in C_c^\infty(\mathbb{R}^2)$, then $g_1 := -()^\alpha \widehat{\mu} g \in C_c^\infty(\mathbb{R}^2)$ and $|g_1(w)| \leq \text{const}\, |w|^4$, and so it follows by Lemma 2.3 that

$$\begin{aligned} &\langle g, (D^\alpha f)\widehat{\ }\rangle = \langle g_1, \widehat{\phi}\rangle = 8\pi \int_{\mathbb{R}^2} |w|^{-4}\, g_1(w)\, dw \\ &= -8\pi \int_{\mathbb{R}^2} |w|^{-4}\, w^\alpha \widehat{\mu}(w) g(w)\, dw. \end{aligned}$$

It follows from condition (ii) of Definition 2.1 that

$$\left|-8\pi\left|w\right|^{-4}w^{\alpha}\widehat{\mu}(w)\right| \le \mathrm{const}\,(1+|w|)^{-3/2}.$$

Hence, $-8\pi\left|\cdot\right|^{-4}()^{\alpha}\widehat{\mu}\in L_2(\mathbb{R}^2)$, and it now follows from the Plancherel Theorem [8; page 172] that $D^{\alpha}f\in L_2(\mathbb{R}^2)$. $\square$

In order to get a handle on the quantity $|||f-T_{\Xi}f|||$, we make use of the fact [2] that

$$|||g-T_{\Xi}g||| = \min\{|||g-s|||: s\in S(\phi;\Xi)\}, \quad \forall g\in H.$$

The upshot is that rather than being forced to estimate $|||f-T_{\Xi}f|||$ directly, we may instead estimate $|||f-s_h|||$ where $s_h\in S(\phi;\Xi_h)\subset S(\phi;\Xi)$ can be chosen at our convenience. Given the form of $f\in\mathcal{F}$, namely $f=\phi*\mu+p$, a natural way to construct s_h would be to first convolve μ with some function $\psi(\cdot/h)$ and then put $s_h=\sum_{j\in\mathbb{Z}^2}(\psi(\cdot/h)*\mu)(hj)\phi(\cdot-hj)$. The only problem with this attempt is that the coefficients $(\psi(\cdot/h)*\mu)(hj)$ will not in general vanish when hj is outside $(1-h)B$, and hence we cannot expect s_h to belong to $S(\phi;\Xi_h)$. This problem can be overcome by convolving $\psi(\cdot/h)$ not with μ, but rather with $\mu((1+r_0h)\cdot)$. With ψ and r_0 chosen appropriately, it will follow that $(\psi(\cdot/h)*\mu)(hj)=0$ whenever hj is outside $(1-h)B$.

Let $\psi:=\eta*\sigma$, where $\eta\in C_c(\mathbb{R}^2)$ and $\sigma\in C_c^{\infty}(\mathbb{R}^2)$ are some functions satisfying

$$\sup_{j\in\mathbb{Z}^2}|\delta_{0,j}-\widehat{\eta}(w-2\pi j)| \le \mathrm{const}\,|w|^2, \quad w\in\mathbb{R}^2 \tag{2.4}$$

$$|1-\widehat{\sigma}(w)| \le \mathrm{const}\,|w|^2, \quad w\in\mathbb{R}^2. \tag{2.5}$$

For example, one could choose $\eta(x)=\chi_{[-1..1]^2}(x)(1-|x_1|)(1-|x_2|)$ and $\sigma(x)=c^{-1}\chi_B(x)\exp(-1/(1-|x|^2))$ with $c=\int_B\exp(-1/(1-|x|^2))\,dx$. Let $r_0>2$ be such that $\operatorname{supp}\psi\subset(\frac{r_0}{2}-1)B$.

Lemma 2.6. *Let $f\in\mathcal{F}$, and let μ be as in Definition 2.1. If μ_h and s_h are given by*

$$\mu_h := ((1+r_0h)^2\psi(\cdot/h))*(\mu((1+r_0h)\cdot)),$$
$$s_h := \sum_{j\in\mathbb{Z}^2}\mu_h(hj)\phi(\cdot-hj),$$

then $s_h\in S(\phi;\Xi_h)$ whenever $0<h<r_0^{-1}$.

Proof. It is a straightforward exercise to verify that the choice of r_0 ensures that $\operatorname{supp}\mu_h\subset(1-h)B$. Hence it remains only to show that $\sum_{\xi\in\Xi_h}\mu_h(\xi)q(\xi/h)=0$ for all $q\in\Pi_1$. For that note that $\sum_{\xi\in\Xi_h}\mu_h(\xi)q(\xi/h)=\sum_{j\in\mathbb{Z}^2}\mu_h(hj)q(j)$. If we put $g(x):=\mu_h(hx)q(x)$, then we obtain from Poisson's summation formula (cf. [9], Chapter 7) that $\sum_{j\in\mathbb{Z}^2} g(j) = \sum_{j\in\mathbb{Z}^2} \widehat{g}(2\pi j)$. Now $\widehat{\mu}_h = h^2\widehat{\psi}(h\cdot)\widehat{\mu}(\cdot/(1+r_0h))$; hence, if $q = \sum_{|\alpha|\leq 1} i^{-|\alpha|}a_\alpha()^\alpha$, then

$$\widehat{g} = \sum_{|\alpha|\leq 1} a_\alpha D^\alpha(h^{-2}\widehat{\mu}_h(\cdot/h)) = \sum_{|\alpha|\leq 1} a_\alpha D^\alpha(\widehat{\eta}\widehat{\sigma}\widehat{\mu}(\cdot/(h+r_0h^2))).$$

Condition (2.4) ensures that $D^\alpha(\widehat{\eta}\widehat{\sigma}\widehat{\mu}(\cdot/(h+r_0h^2))) = 0$ at $2\pi j$ whenever $j\in\mathbb{Z}^2\backslash 0$ and $|\alpha|\leq 1$. On the other hand, Definition 2.1 (ii) ensures that $D^\alpha(\widehat{\eta}\widehat{\sigma}\widehat{\mu}(\cdot/(h+r_0h^2))) = 0$ at 0 for all $|\alpha|\leq 1$. Hence,

$$\sum_{\xi\in\Xi_h} \mu_h(\xi)q(\xi/h) = \sum_{j\in\mathbb{Z}^2} \widehat{g}(2\pi j) = 0.$$

□

The effect of convolving $\widehat{\psi}(\cdot/h)$ with the $(1+r_0h)$-dilate of μ rather than with μ itself is that s_h is best compared not to f, but rather to the $(1+r_0)$-dilate of f. For this, we define

$$f_h := (1+r_0h)^{-2}f((1+r_0h)\cdot), \tag{2.7}$$

and use the triangle inequality to write

$$|||f-s_h||| \leq |||f-f_h||| + |||f_h-s_h|||.$$

We consider each of these terms separately in the following two lemmata.

Lemma 2.8. *Let $f\in\mathcal{F}$. If f_h is as defined in (2.7), then*

$$|||f-f_h||| = O(\sqrt{h}) \text{ as } h\to 0.$$

proof. Let μ be as in Definition 2.1, and note that

$$\begin{aligned} |||f-f_h||| &= (2\pi)^{-1}\left\| |\cdot|^2\left(\widehat{f}-\widehat{f_h}\right)\right\|_{L_2(\mathbb{R}^2\backslash 0)} \\ &= (2\pi)^{-1}\left\| |\cdot|^2\left[\widehat{\phi}\widehat{\mu} - (1+r_0h)^{-4}\widehat{\phi}(\cdot/(1+r_0h))\widehat{\mu}(\cdot/(1+r_0h))\right]\right\|_{L_2(\mathbb{R}^2\backslash 0)} \\ &= 4\left\| |\cdot|^{-2}\left(\widehat{\mu}-\widehat{\mu}(\cdot/(1+r_0h))\right)\right\|_{L_2}. \end{aligned}$$

Now,

$$\left\||\cdot|^{-2}(\widehat{\mu}-\widehat{\mu}(\cdot/(1+r_0h))\right\|_{L_2}^2 = \int_{\mathbb{R}^2} |w|^{-4}\left|\widehat{\mu}(w)-\widehat{\mu}(w/(1+r_0h))\right|^2\,dw.$$

We estimate this integral by breaking $\mathbb{R}^2$ into the three pieces B, $h^{-1}B\backslash B$, and $\mathbb{R}^2\backslash h^{-1}B$. For the first piece, we note that since $\widehat{\mu}$ is entire it can be written as a power series

$$\widehat{\mu}(w) = \sum_\alpha c_\alpha w^\alpha,$$

and it follows from Definition 2.1 (ii) that $c_{(0,0)} = c_{(1,0)} = c_{(0,1)} = 0$. Thus

$$\widehat{\mu}(w)-\widehat{\mu}(w/(1+r_0h)) = \sum_{|\alpha|\ge 2} c_\alpha(w^\alpha-(w/(1+r_0h))^\alpha) = \sum_{|\alpha|\ge 2} c_\alpha w^\alpha(1-(1+r_0h)^{-|\alpha|}).$$

It now follows that

$$\begin{aligned}
&\left\| |\cdot|^{-2}(\widehat{\mu} - \widehat{\mu}(\cdot/(1+r_0h))\right\|_{L_2(B)} = \left\| |\cdot|^{-2} \sum_{|\alpha|\ge 2} c_\alpha()^\alpha(1-(1+r_0h)^{-|\alpha|})\right\|_{L_2(B)} \\
&\le \sum_{|\alpha|\ge 2} |c_\alpha|(1-(1+r_0h)^{-|\alpha|}) \left\| |\cdot|^{-2}()^\alpha\right\|_{L_2(B)} \\
&\le r_0\sqrt{\pi}h \sum_{|\alpha|\ge 2} |c_\alpha||\alpha| \le \operatorname{const} h,
\end{aligned}$$

where we have used the estimate $1-(1+r_0h)^{-|\alpha|} \le r_0h|\alpha|$ and the fact that $\left\| |\cdot|^{-2}()^\alpha\right\|_{L_2(B)} \le \sqrt{\pi}$.

For the second piece, we note that $\widehat{\mu}(w)-\widehat{\mu}(w/(1+r_0h)) = (1-(1+r_0h)^{-1})w\cdot(\nabla\widehat{\mu}(\xi))$ for some ξ between $w/(1+r_0h)$ and w. Since μ is compactly supported, it follows from Definition 2.1 (ii) that $|\nabla\widehat{\mu}(w)| \le \operatorname{const}(1+\sqrt{|w|})$. Consequently, $|\widehat{\mu}(w)-\widehat{\mu}(w/(1+r_0h))| \le \operatorname{const} h(1+|w|^{3/2})$. Hence,

$$\begin{aligned}
&\int_{h^{-1}B\setminus B} |w|^{-4}\left|\widehat{\mu}(w)-\widehat{\mu}(w/(1+r_0h))\right|^2\,dw \le \operatorname{const}\int_{h^{-1}B\setminus B} |w|^{-4}h^2|w|^3\,dw \\
&= \operatorname{const} h^2\int_1^{h^{-1}} r^{-4}r^3r\,dr = \operatorname{const} h^2(h^{-1}-1) \le \operatorname{const} h.
\end{aligned}$$

For the third piece we use the bound $|\widehat{\mu}(w)-\widehat{\mu}(w/(1+r_0h))| \le \operatorname{const}(1+\sqrt{|w|})$ (a consequence of Definition 2.1 (ii)) to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2\setminus h^{-1}B} |w|^{-4}\left|\widehat{\mu}(w)-\widehat{\mu}(w/(1+r_0h))\right|^2\,dw \le \operatorname{const}\int_{\mathbb{R}^2\setminus h^{-1}B} |w|^{-4}|w|\,dw \\
&= \operatorname{const}\int_{h^{-1}}^{\infty} r^{-3}r\,dr = \operatorname{const} h.
\end{aligned}$$

□

Lemma 2.9. *Let $f\in\mathcal{F}$. If f_h is as defined in (2.7) and μ_h, s_h are as defined in Lemma 2.6, then*

$$|||f_h - s_h||| = O(\sqrt{h}) \text{ as } h\to 0.$$

Proof. Assume $0<h\le r_0^{-1}$. In order to simplify the notation, we introduce

$$\widehat{\nu}_h := \widehat{\mu}(\cdot/(1+r_0h)),$$

and note that $\widehat{f_h} = \widehat{\phi}\widehat{\nu}_h$ on $\mathbb{R}^2\backslash 0$. On the other hand, for $w \in \mathbb{R}^2\backslash 0$, we have $\widehat{s}_h(w) = \sum_{j\in\mathbb{Z}^2} \widehat{\phi}(w)\mu_h(hj)e^{-ihj\cdot w}$. If we define $g(x) := \mu_h(hx)e^{-ihx\cdot w}$, $x \in \mathbb{R}^2$, then we obtain from Poisson's summation formula (cf. [9], Chapter 7) that $\sum_{j\in\mathbb{Z}^2} g(j) = \sum_{j\in\mathbb{Z}^2} \widehat{g}(2\pi j)$. Hence,

$$\begin{aligned}\widehat{s}_h(w) &= \widehat{\phi}(w)\sum_{j\in\mathbb{Z}^2} g(j) = \widehat{\phi}(w)\sum_{j\in\mathbb{Z}^2}\widehat{g}(2\pi j)\\ &= \widehat{\phi}(w)\sum_{j\in\mathbb{Z}^2} h^{-2}\widehat{\mu}_h(w+2\pi j/h)\\ &= \widehat{\phi}(w)\sum_{j\in\mathbb{Z}^2}\widehat{\psi}(hw+2\pi j)\widehat{\nu}_h(w+2\pi j/h).\end{aligned}$$

Hence,

$$\begin{aligned}|||f_h - s_h||| &= (2\pi)^{-1}\left\| |\cdot|^2\left(\widehat{f}_h - \widehat{s}_h\right)\right\|_{L_2(\mathbb{R}^2\backslash 0)}\\ &= 4\left\| |\cdot|^{-2}\left[\widehat{\nu}_h - \sum_{j\in\mathbb{Z}^2}\widehat{\psi}(h\cdot+2\pi j)\widehat{\nu}_h(\cdot+2\pi j/h)\right]\right\|_{L_2}\end{aligned}$$

(2.10)

$$\le 4\left\| |\cdot|^{-2}\,\widehat{\nu}_h(1-\widehat{\psi}(h\cdot))\right\|_{L_2} + 4\left\| |\cdot|^{-2}\sum_{j\in\mathbb{Z}^2\backslash 0}\widehat{\psi}(h\cdot+2\pi j)\widehat{\nu}_h(\cdot+2\pi j/h))\right\|_{L_2}.$$

We consider first the term $\left\| |\cdot|^{-2}\,\widehat{\nu}_h(1-\widehat{\psi}(h\cdot))\right\|_{L_2}$. Since $1 \le 1 + r_0 h \le 2$, it follows from Definition 2.1 (ii) that

$$|\widehat{\nu}_h(w)|^2 \le \text{const}\frac{|w|^4}{1+|w|^3}, \quad w \in \mathbb{R}^2. \tag{2.11}$$

From (2.4) and (2.5) we obtain

$$\left|1-\widehat{\psi}(w)\right|^2 \le \text{const}\frac{|w|^4}{1+|w|^4}, \quad w \in \mathbb{R}^2.$$

Consequently,

$$\begin{aligned}\left\| |\cdot|^{-2}\,\widehat{\nu}_h(1-\widehat{\psi}(h\cdot))\right\|_{L_2}^2 &\le \text{const}\int_{\mathbb{R}^2}|w|^{-4}\,\frac{|w|^4}{1+|w|^3}\frac{|hw|^4}{1+|hw|^4}\,dw\\ &= \text{const}\int_0^\infty \frac{h^4r^4}{(1+r^3)(1+h^4r^4)}r\,dr\\ &\le \text{const}\left(\int_0^1 h^4\,dr + \int_1^{1/h}\frac{h^4r^5}{r^3}\,dr + \int_{1/h}^\infty \frac{r}{1+r^3}\,dr\right)\end{aligned}$$

(2.12)

$$\le \text{const}\left(h^4 + h^4\int_1^{1/h} r^2\,dr + \int_{1/h}^\infty r^{-2}\,dr\right) \le \text{const}\,h.$$

Let $C := [-\frac{1}{2} \,.\,.\, \frac{1}{2})^2$. Employing the partition of $\mathbb{R}^2$, $\mathbb{R}^2 = \cup_{k\in\mathbb{Z}^2} 2\pi h^{-1}(k+C)$, we expand the square of the second term at the bottom of (2.10) as

$$\begin{aligned}
&\left\| |\cdot|^{-2} \sum_{j\in\mathbb{Z}^2\backslash 0} \widehat{\nu}_h(\cdot + 2\pi j/h)\widehat{\psi}(h\cdot + 2\pi j) \right\|_{L_2}^2 \\
&= \sum_{k\in\mathbb{Z}^2} \left\| |\cdot|^{-2} \sum_{j\in\mathbb{Z}^2\backslash 0} \widehat{\nu}_h(\cdot + 2\pi j/h)\widehat{\psi}(h\cdot + 2\pi j) \right\|_{L_2(2\pi h^{-1}(k+C))}^2 .
\end{aligned} \tag{2.13}$$

For $j \in \mathbb{Z}^2\backslash 0$ and $k \in \mathbb{Z}^2\backslash\{-j\}$, we have

$$\begin{aligned}
&\left\| |\cdot|^{-2}\, \widehat{\nu}_h(\cdot + 2\pi j/h)\widehat{\psi}(h\cdot + 2\pi j) \right\|_{L_2(2\pi h^{-1}(k+C))}^2 \\
&= \int_{2\pi h^{-1}(k+C)} |w|^{-4} \left|\widehat{\nu}_h(w + 2\pi j/h)\right|^2 \left|\widehat{\psi}(hw + 2\pi j)\right|^2 \, dw \\
&= \int_{2\pi h^{-1}C} \left|2\pi h^{-1}k + w\right|^{-4} \left|\widehat{\nu}_h(w + 2\pi (k+j)/h)\right|^2 \left|\widehat{\psi}(hw + 2\pi (k+j))\right|^2 \, dw \\
&\le \text{const} \int_{2\pi h^{-1}C} \left|2\pi h^{-1}k + w\right|^{-4} \left|w + 2\pi (k+j)/h\right| \left|\widehat{\psi}(hw + 2\pi (k+j))\right|^2 \, dw \\
&\le \text{const} \int_{2\pi h^{-1}C} \left|2\pi h^{-1}k + w\right|^{-4} \left|w + 2\pi (k+j)/h\right| \|\widehat{\sigma}\|_{L_\infty(2\pi(j+k+C))}^2 |hw|^4 \, dw, \qquad \text{by (2.4)}, \\
&\le \text{const}\, h^4 \left|h^{-1}(k+j)\right| \|\widehat{\sigma}\|_{L_\infty(2\pi(j+k+C))}^2 \int_{2\pi h^{-1}C} \frac{|w|^4}{\left|2\pi h^{-1}k + w\right|^4} \, dw \\
&\le \text{const}\, h (1+|k|)^{-4} \left|k+j\right| \|\widehat{\sigma}\|_{L_\infty(2\pi(j+k+C))}^2 .
\end{aligned}$$

For $j \in \mathbb{Z}^2\backslash 0$ and $k = -j$, we have

$$\begin{aligned}
&\left\| |\cdot|^{-2}\, \widehat{\nu}_h(\cdot + 2\pi j/h)\widehat{\psi}(h\cdot + 2\pi j) \right\|_{L_2(2\pi h^{-1}(k+C))}^2 \\
&= \int_{2\pi h^{-1}(k+C)} |w|^{-4} \left|\widehat{\nu}_h(w + 2\pi j/h)\right|^2 \left|\widehat{\psi}(hw + 2\pi j)\right|^2 \, dw \\
&= \int_{2\pi h^{-1}C} \left|2\pi h^{-1}k + w\right|^{-4} \left|\widehat{\nu}_h(w)\right|^2 \left|\widehat{\psi}(hw)\right|^2 \, dw \\
&\le \text{const} \int_{2\pi h^{-1}C} \left|2\pi h^{-1}k + w\right|^{-4} \frac{|w|^4}{1+|w|^3} \, dw, \qquad \text{by (2.11)}, \\
&\le \text{const}\, h^4 \int_{2\pi h^{-1}C} \frac{|w|}{|2\pi k + hw|^4} \, dw \le \text{const}\, h\, |k|^{-4} .
\end{aligned}$$

Since $\sigma \in C_c^\infty(\mathbb{R}^2)$, it follows that $\sum_{j\in\mathbb{Z}^2} \sqrt{|j|}\,\|\widehat{\sigma}\|_{L_\infty(2\pi(j+C))} < \infty$. Hence,

$$\begin{aligned}
&\left\| |\cdot|^{-2} \sum_{j\in\mathbb{Z}^2\backslash 0} \widehat{\nu}_h(\cdot + 2\pi j/h)\widehat{\psi}(h\cdot + 2\pi j) \right\|_{L_2(2\pi h^{-1}(k+C))} \\
&\le \sum_{j\in\mathbb{Z}^2\backslash 0} \left\| |\cdot|^{-2}\, \widehat{\nu}_h(\cdot + 2\pi j/h)\widehat{\psi}(h\cdot + 2\pi j) \right\|_{L_2(2\pi h^{-1}(k+C))} \\
&\le \mathrm{const}\sqrt{h}(1+|k|)^{-2} + \mathrm{const}\sqrt{h}(1+|k|)^{-2} \sum_{j\in\mathbb{Z}^2\backslash\{0,-k\}} \sqrt{|k+j|}\,\|\widehat{\sigma}\|_{L_\infty(2\pi(k+j+C))} \\
&\le \mathrm{const}\sqrt{h}(1+|k|)^{-2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^2} \left\| |\cdot|^{-2} \sum_{j\in\mathbb{Z}^2\backslash 0} \widehat{\nu}_h(\cdot + 2\pi j/h)\widehat{\psi}(h\cdot + 2\pi j) \right\|^2_{L_2(2\pi h^{-1}(k+C))} \\
&\le \mathrm{const}\, h \sum_{k\in\mathbb{Z}^2} (1+|k|)^{-4} \le \mathrm{const}\, h. \qquad (2.14)
\end{aligned}$$

And so with (2.14), (2.13), (2.12), and (2.10) in view, the lemma is proved. □

With Lemma 2.8 and Lemma 2.9 in hand we can prove the intended result.

Proof of Theorem 2.2 (ii)–(iv). Let $f \in \mathcal{F}$. Assume that $0 < h < r_0^{-1}$, and let s_h be as in Lemma 2.6 and f_h as defined in (2.7). Then

$$\begin{aligned}
|||f - T_\Xi f||| &= \min_{s\in S(\phi;\Xi)} |||f - s||| \\
&\le |||f - s_h|||, \text{ by Lemma 2.6}, \\
&\le |||f - f_h||| + |||f_h - s_h||| = O(\sqrt{h})
\end{aligned}$$

by Lemma 2.8 and Lemma 2.9. Hence (ii). Since $|||T_B f - T_\Xi f|||$ also converges to 0 (by (1.2)), it follows that $|||T_B f - f||| = 0$. Since both f and $T_B f$ belong to H and $f_{|B} = (T_B f)_{|B}$, it must be the case that $f = T_B f$. Hence (iii). Employing (1.1) we obtain

$$\begin{aligned}
\|f - T_\Xi f\|_{L_p(B)} &\le \mathrm{const}\, h^{\gamma_p} |||T_B f - T_\Xi f||| \\
&= \mathrm{const}\, h^{\gamma_p} |||f - T_\Xi f||| = O(h^{\gamma_p + 1/2})
\end{aligned}$$

which proves (iv). □

3. Proof of Theorem 1.5

With Theorem 2.2 in view, in order to prove Theorem 1.5, it suffices to prove the following

Theorem 3.1. *For all $f \in C^\infty(\mathbb{R}^2)$, there exists $\widetilde{f} \in \mathcal{F}$ such that $f_{|B} = \widetilde{f}_{|B}$.*

Recall that the form of $\widetilde{f} \in \mathcal{F}$ is $\widetilde{f} = \phi * \mu + p$ where $p \in \Pi_1$ and the distribution μ satisfies the conditions of Definition 2.1, the most restrictive of which is $|\widehat{\mu}(w)| = O(\sqrt{|w|})$ for large $|w|$. The following two lemmas display a class of distributions which satisfy this condition.

Let $y(\theta)$ denote the point $(\cos\theta, \sin\theta) \in \partial B$, and note that $y(\theta)$ is the outward unit normal to ∂B at $y(\theta)$.

Lemma 3.2. *If ν is the distribution given by*

$$\langle g, \nu \rangle := \int_{-\pi}^{\pi} D_{y(\theta)} g(y(\theta))\, d\theta,$$

then $|\widehat{\nu}(w)| \le \mathrm{const}\,(1 + \sqrt{|w|})$, $w \in \mathbb{R}^2$.

Proof. Since ν is compactly supported, its Fourier transform is entire, and so it suffices to show that $|\widehat{\nu}(w)| \le \mathrm{const}\,\sqrt{|w|}$, $|w| > 2\pi$. Since ν is radially symmetric, so is its Fourier transform, and hence it suffices to consider only $w = (0, t)$, $t > 2\pi$, wherein

$$\begin{aligned}
\widehat{\nu}(w) = \langle e_{-w}, \nu \rangle &= \int_{-\pi}^{\pi} D_{y(\theta)} e_{-w}(y(\theta))\, d\theta \\
&= \int_{-\pi}^{\pi} -it \sin\theta\, e^{-it\sin\theta}\, d\theta \\
&= \int_{-\pi}^{\pi} -t \sin\theta\, \sin(t\sin\theta)\, d\theta, \qquad \text{since } -it\sin\theta\, \cos(t\sin\theta) \text{ is odd in } \theta, \\
&= -4 \int_0^{\pi/2} t\sin\theta\, \sin(t\sin\theta)\, d\theta.
\end{aligned}$$

Employing the change of variables $x = t\sin\theta$, we arrive at $\widehat{\nu}(w) = -4 \int_0^t \sin x \frac{x}{\sqrt{t^2 - x^2}}\, dx$. Let N be the largest integer for which $N\pi \le t$, and define $A_n := \int_{n\pi}^{(n+1)\pi} \sin x \frac{x}{\sqrt{t^2 - x^2}}\, dx$, for $n = 0, 1, \ldots, N-1$. Then

$$\widehat{\nu}(w) = -4 \sum_{n=0}^{N-1} A_n - 4 \int_{N\pi}^{t} \sin x \frac{x}{\sqrt{t^2 - x^2}}\, dx.$$

Since $x \mapsto x/\sqrt{t^2 - x^2}$ is increasing on $[0\,.\,.\,t)$, it follows that $|A_n| < |A_{n+1}|$ for $n = 0, 1, \ldots, N-2$, and since the A_n's are alternating in sign, it follows that $\left| \sum_{n=0}^{N-1} A_n \right| < |A_{N-1}|$. Therefore,

$$\begin{aligned}
|\widehat{\nu}(w)| &\le 4\,|A_{N-1}| + 4 \int_{N\pi}^{t} |\sin x| \frac{x}{\sqrt{t^2 - x^2}}\, dx \\
&\le 4 \int_{t-2\pi}^{t} \frac{x}{\sqrt{t^2 - x^2}}\, dx = 4\sqrt{4t\pi - 4\pi^2} \le 8\sqrt{\pi t}.
\end{aligned}$$

□

Lemma 3.3. *Let $g_1 \in L_1(B)$, let $g_2 \in L_1([-\pi \,..\, \pi])$, and let $g_3 \in C^\infty(\mathbb{R})$ be 2π-periodic. If μ is the distribution given by*

$$\langle g, \mu \rangle := \int_B g(\xi) g_1(\xi)\, d\xi + \int_{-\pi}^{\pi} g(y(\theta)) g_2(\theta) + D_{y(\theta)} g(y(\theta)) g_3(\theta)\, d\theta,$$

then $|\widehat{\mu}(w)| \le \text{const}\,(1 + \sqrt{|w|})$, $w \in \mathbb{R}^2$.

Proof. Let ν be as defined in Lemma 3.2, and let $q \in C_c^\infty(\mathbb{R}^2)$ be such that $q = 0$ on $\frac{1}{4}B$ and $q = 1$ on $\frac{5}{4}B \backslash \frac{3}{4}B$. Define $q_3 \in C_c^\infty(\mathbb{R}^2)$ by $q_3(x) := q(x) g_3(\arg(x_1 + ix_2))$. Note that

$$\begin{aligned}
\langle g, q_3\nu \rangle = \langle g q_3, \nu \rangle &= \int_{-\pi}^{\pi} D_{y(\theta)}(g q_3)(y(\theta))\, d\theta \\
&= \int_{-\pi}^{\pi} D_{y(\theta)} g(y(\theta)) q_3(y(\theta))\, d\theta, \qquad \text{since } D_{y(\theta)} q_3(y(\theta)) = 0, \\
&= \int_{-\pi}^{\pi} D_{y(\theta)} g(y(\theta)) g_3(\theta)\, d\theta.
\end{aligned}$$

Hence, μ can be written as $\mu = \mu_1 + q_3\nu$, where μ_1 is given by

$$\langle g, \mu_1 \rangle := \int_B g(\xi) g_1(\xi)\, d\xi + \int_{-\pi}^{\pi} g(y(\theta)) g_2(\theta)\, d\theta.$$

Of course, $|\widehat{\mu}_1(w)| \le \|g_1\|_{L_1(B)} + \|g_2\|_{L_1([-\pi..\pi])}$. On the other hand, since $|\widehat{\nu}(w)| \le \text{const}\,(1 + \sqrt{|w|})$ (by Lemma 3.2) and since $q_3 \in C_c^\infty(\mathbb{R}^2)$, it follows that

$$|\widehat{q_3\nu}(w)| = (2\pi)^{-2}\, |\widehat{q}_3 * \widehat{\nu}(w)| \le \text{const}\,(1 + \sqrt{|w|})$$

which completes the proof. □

The following statement of Green's second identity may be found in [5; page 17].

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a C^1 boundary, and let $u, v \in C^2(\overline{\Omega})$. Then*

$$\int_\Omega \Delta u\, v\, dm = \int_\Omega u \Delta v\, dm + \int_{\partial\Omega} [v D_{\overrightarrow{n}} u - u D_{\overrightarrow{n}} v]\, ds,$$

where $\overrightarrow{n}$ denotes the outward unit normal to $\partial\Omega$.

Our goal at present is to identify a distribution μ of the form described in Lemma 3.3 such that

$$f(x) = \phi * \mu(x), \quad \forall x \in B. \tag{3.5}$$

The following proposition displays (implicitly) a distribution whose convolution with ϕ agrees with f on B. This distribution, however, is not of the desired form because its third term involves the Laplacian of the test function.

Proposition 3.6. *If $f \in C^\infty(\mathbb{R}^2)$, then for all $x \in B$,*

$$8\pi f(x) = \int_B \phi(\xi - x)\,\Delta^2 f(\xi)\,d\xi + \int_{\partial B} D_{\overrightarrow{n}}\phi(s-x)\,\Delta f(s) - \phi(s-x) D_{\overrightarrow{n}}\Delta f(s)\,ds$$
$$+ \int_{\partial B} D_{\overrightarrow{n}}\Delta\phi(s-x)\,f(s) - \Delta\phi(s-x)\,D_{\overrightarrow{n}} f(s)\,ds,$$

where $\overrightarrow{n}$ is the outward unit normal to ∂B at ξ.

Proof. WLOG we may assume that f is compactly supported since otherwise we could replace f with a compactly supported C^∞ function which agrees with f on B. Let $a > 1$ be so large that $\operatorname{supp} f \subset aB$.

Claim. $\Delta^2 f * \phi = 8\pi f$.

proof. Since $\Delta^2 f \in C_c^\infty(\mathbb{R}^2)$ it follows that $\left(\Delta^2 f * \phi\right)\widehat{\ } = |\cdot|^4\,\widehat{f}\widehat{\phi}$. Hence $\left(\Delta^2 f * \phi\right)\widehat{\ } = 8\pi\widehat{f}$ on $\mathbb{R}^2\backslash 0$. Therefore $\Delta^2 f * \phi = 8\pi f + p$ for some polynomial p. In order to show that $p = 0$, it suffices to show that $\Delta^2 f * \phi(x) = 0$ for sufficiently large $|x|$. For that let $|x| > a$. Then

$$\begin{aligned}
\Delta^2 f * \phi(x) &= \int_{aB} \Delta^2 f(\xi)\phi(x-\xi)\,d\xi \\
&= \int_{aB} f(\xi)\Delta^2\phi(x-\xi)\,d\xi, \quad \text{by Lemma 3.4,} \\
&= 0, \quad \text{as } \Delta^2\phi(x-\xi) = 0 \text{ for } \xi \in aB.
\end{aligned}$$

Hence the claim.

Let $x \in B$. By the claim, we have

$$8\pi f(x) = \int_{\mathbb{R}^2} \Delta^2 f(\xi)\phi(x-\xi)\,d\xi = \int_{aB} \Delta^2 f(\xi)\phi(\xi - x)\,d\xi, \quad \text{since } \phi(-\cdot) = \phi,$$
$$= \int_B \Delta^2 f(\xi)\phi(\xi-x)\,d\xi + \int_{aB\backslash B} \Delta^2 f(\xi)\phi(\xi-x)\,d\xi. \tag{3.7}$$

Since $\phi(\cdot - x) \in C^\infty(\mathbb{R}^2\backslash B)$, we may apply Lemma 3.4 twice to the latter integral above to obtain

$$\begin{aligned}
&\int_{aB\backslash B} \Delta^2 f(\xi)\phi(\xi-x)\,d\xi \\
&= \int_{aB\backslash B} \Delta f(\xi)\Delta\phi(\xi-x)\,d\xi - \int_{\partial B}[D_{\overrightarrow{n}}\Delta f(s)\phi(s-x) - \Delta f(s) D_{\overrightarrow{n}}\phi(s-x)]\,ds \\
&= -\int_{\partial B}[D_{\overrightarrow{n}}\Delta f(s)\phi(s-x) - \Delta f(s) D_{\overrightarrow{n}}\phi(s-x)]\,ds \\
&- \int_{\partial B}[D_{\overrightarrow{n}} f(s)\Delta\phi(s-x) - f(s) D_{\overrightarrow{n}}\Delta\phi(s-x)]\,ds,
\end{aligned}$$

as $\int_{aB\setminus B} f(\xi)\Delta^2\phi(\xi-x)\,d\xi = 0$ since $\Delta^2\phi(\xi-x)=0$ for $\xi\in aB\setminus B$. With (3.7) in view, this completes the proof. $\square$

In order to obtain a distribution of the desired form and satisfying (3.5), it suffices to find C^∞ functions g_1 and g_2 such that

$$\begin{aligned}&\int_{\partial B} D_{\vec{n}}\Delta\phi(s-x)\,f(s)-\Delta\phi(s-x)\,D_{\vec{n}}f(s)\,ds\\ &=\int_{\partial B} D_{\vec{n}}\phi(s-x)\,g_1(s)+\phi(s-x)\,g_2(s)\,ds,\quad \forall x\in B.\end{aligned}\tag{3.8}$$

To do this we employ the Fourier series representations of $f_{|\partial B}$ and $D_{\vec{n}}f_{|\partial B}$, say $f(y(\theta))=\sum_n a_n e^{in\theta}$ and $D_{y(\theta)}f(y(\theta))=\sum_n b_n e^{in\theta}$. The purpose of the following proposition and two corollaries is to identify sequences $\{c_n\}$ and $\{d_n\}$ such that

$$\begin{aligned}&\int_{-\pi}^{\pi} D_{y(\theta)}\Delta\phi(y(\theta)-x)\,a_n e^{in\theta}-\Delta\phi(y(\theta)-x)\,b_n e^{in\theta}\,d\theta\\ &=\int_{-\pi}^{\pi} D_{y(\theta)}\phi(y(\theta)-x)\,c_n e^{in\theta}+\phi(y(\theta)-x)\,d_n e^{in\theta}\,d\theta,\quad \forall x\in B.\end{aligned}$$

That (3.8) holds will then follow with $g_1(y(\theta))=\sum_n c_n e^{in\theta}$ and $g_2(y(\theta))=\sum_n d_n e^{in\theta}$.

Definition. For $u\in C(\mathbb{R}^2)$ and $n\in\mathbb{Z}$, we define

$$R_n[u](t):=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}u(ty(\theta))\,d\theta,$$

where $y(\theta):=(\cos\theta,\sin\theta)$.

Proposition 3.9. *For $t>0$, $x\in tB$, and $n\in\mathbb{Z}$, $|n|\ge 2$, the following hold:*

$$\begin{aligned}&(i) && R_0[\phi(\cdot-x)](t)=t^2\log t+|x|^2(1+\log t),\\ &(ii) && R_{\pm1}[\phi(\cdot-x)](t)=-(x_1\pm ix_2)(t\log t+t/2+t^{-1}|x|^2/4),\\ &(iii) && R_n[\phi(\cdot-x)](t)=\frac{1}{2|n|}(x_1+\mathrm{sign}(n)ix_2)^{|n|}\left(\frac{t^{2-|n|}}{|n|-1}-\frac{t^{-|n|}|x|^2}{|n|+1}\right).\end{aligned}$$

Proof. (i) was proved in [6]. Since $R_{-n}[\phi(\cdot-x)](t)$ is simply the complex conjugate of $R_n[\phi(\cdot-x)](t)$, it suffices to prove (ii) and (iii) only for n positive. Since, for $t>0$ and $x\in\mathbb{R}^2$, $\phi(tx)=t^2|x|^2\log t+t^2\phi(x)$, it follows that

$$\begin{aligned}R_n[\phi(\cdot-x)](t)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{in\theta}\phi(ty(\theta)-x)\,d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{in\theta}\phi(t(y(\theta)-x/t))\,d\theta\\ &=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{in\theta}[t^2|y(\theta)-x/t|^2\log t+t^2\phi(y(\theta)-x/t)]\,d\theta\\ &=t^2(\log t)R_n[|\cdot-x/t|^2](1)+t^2R_n[\phi(\cdot-x/t)](1).\end{aligned}\tag{3.10}$$

It is a simple matter to show that

$$R_n[|\cdot - \xi|^2](1) = \begin{cases} -(\xi_1 + i\xi_2) & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{3.11}$$

Define $G \in C(\mathbb{R}^2)$ by

$$G(\xi) := R_n[\phi(\cdot - \xi)](1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta} \phi(y(\theta) - \xi)\, d\theta.$$

It is a straightforward matter to verify that $G_{|B} \in C^\infty(B)$, $\Delta^2 G = 0$ on B, and $G(\xi) = e^{in\arg(\xi_1 + i\xi_2)} G(|\xi|\, y(0))$ for $0 < |\xi| < 1$. In polar coordinates, if $\widetilde{G}(r,\theta) := G(ry(\theta))$, then this last condition can be expressed as $\widetilde{G}(r,\theta) = e^{in\theta} g(r)$, where $g(r) := \widetilde{G}(r,0)$. The equation $\Delta^2 G = 0$, written in polar coordinates, reduces to the homogeneous differential equation

$$L^2 g = 0, \text{ where}$$
$$Lg := g'' + \frac{1}{r} g' - \frac{n^2}{r^2} g.$$

It is easy to verify that on the interval $(0\,..\,1)$, this equation has the four linearly independent solutions $\{r, r^3, r^{-1}, r\log r\}$ (if $n = 1$) and $\{r^n, r^{2+n}, r^{-n}, r^{2-n}\}$ (if $n > 1$). It then follows from the classical theory of differential equations that there exist a_n, b_n, c_n, and d_n such that

$$g(r) = \begin{cases} a_n r^n + b_n r^{2+n} + c_n r^{-n} + d_n r \log r & \text{if } n = 1, \\ a_n r^n + b_n r^{2+n} + c_n r^{-n} + d_n r^{2-n} & \text{if } n > 1. \end{cases}$$

It must be the case that $c_n = d_n = 0$ since otherwise G would not be C^∞ near the origin. Therefore, $g(r) = a_n r^n + b_n r^{2+n}$ and hence,

$$\begin{aligned} G(\xi) &= e^{in\arg(\xi_1 + i\xi_2)} (a_n |\xi|^n + b_n |\xi|^{n+2}) \\ &= (\xi_1 + i\xi_2)^n (a_n + b_n |\xi|^2), \quad \xi \in B. \end{aligned} \tag{3.12}$$

In order to find a_n note that if $\xi = \tau y(0)$, then $G(\tau y(0)) = a_n \tau^n + b_n \tau^{n+2}$ and hence

$$a_n = \frac{1}{n!} \frac{d^n}{d\tau^n} G(\tau y(0))_{|\tau = 0}.$$

Put $k(\tau,\theta) := |y(\theta) - \tau y(0)|^2 = 1 - 2\tau\cos\theta + \tau^2$, $K(\tau,\theta) := \log(k(\tau,\theta))$, $F(\tau,\theta) := k(\tau,\theta)K(\tau,\theta)$, and let $'$ denote differentiation with respect to τ so that $k'(\tau,\theta) = -2\cos\theta + 2\tau$ and $k''(\tau,\theta) = 2$. Since $\phi(y(\theta) - \tau y(0)) = (1/2)F(\tau,\theta)$, it follows that

$$G(\tau y(0)) = \frac{1}{4\pi} \int_{-\pi}^{\pi} e^{in\theta} F(\tau,\theta)\, d\theta, \qquad \text{and}$$
$$a_n = \frac{1}{4\pi n!} \int_{-\pi}^{\pi} e^{in\theta} F^{(n)}(0,\theta)\, d\theta.$$

Let $\mathbb{T}_l$ denote the space of univariate trigonometric polynomials of degree $\le l$; that is, the space of functions f which can be written as $f(\theta) = \sum_{j=-l}^{l} a_j e^{ij\theta}$, $\theta \in \mathbb{R}$, and let us agree that $\mathbb{T}_{-1} = \{0\}$.

Claim. There exists $q_n \in \mathbb{T}_{n-2}$ such that

$$K^{(n)}(0,\theta) = -(n-1)!2^n \cos^n\theta + q_n(\theta).$$

proof. Note that $K(0,\theta)=0$. Since $K'(\tau,\theta) = \dfrac{k'(\tau,\theta)}{k(\tau,\theta)}$, we have $K'(0,\theta) = -2\cos\theta$. A second differentiation shows that $K''(0,\theta) = -4\cos^2\theta + 2$. Thus the claim is true for $n = 1,2$. Proceeding by induction, assume the claim for $n' \le n$ and consider $n+1 \ge 3$. Using Leibniz formula to differentiate the equality $K'(\tau,\theta)k(\tau,\theta) = k'(\tau,\theta)$ n times yields $\sum_{l=0}^{n} \binom{n}{l} K^{(n+1-l)}k^{(l)} = k^{(1+n)} = 0$; hence

$$\begin{aligned}
K^{(n+1)}(0,\theta) &= -\sum_{l=1}^{n} \binom{n}{l} K^{(n+1-l)}(0,\theta)k^{(l)}(0,\theta)\\
&= -nK^{(n)}(0,\theta)k'(0,\theta) - 2\binom{n}{2} K^{(n-2)}(0,\theta)\\
&= -n!2^{n+1}\cos^{n+1}\theta + 2nq_n(\theta)\cos\theta - 2\binom{n}{2}K^{(n-2)}(0,\theta), \qquad \text{by induction hyp.}
\end{aligned}$$

Thus the claim is true for $n+1$ with $q_{n+1}(\theta) := 2nq_n(\theta)\cos\theta - 2\binom{n}{2}K^{(n-2)}(0,\theta)$.

Since $F'(0,\theta) = k'(0,\theta)K(0,\theta) + k(0,\theta)K'(0,\theta) = -2\cos\theta$ we have

$$a_1 = \frac{1}{4\pi}\int_{-\pi}^{\pi} e^{i\theta}(-2\cos\theta)\,d\theta = -\frac{1}{2}.$$

If $n > 1$, then

$$\begin{aligned}
F^{(n)}(0,\theta) &= \sum_{l=0}^{n}\binom{n}{l} k^{(l)}(0,\theta)K^{(n-l)}(0,\theta)\\
&= K^{(n)}(0,\theta) + nk'(0,\theta)K^{(n-1)}(0,\theta) + 2\binom{n}{2}K^{(n-2)}(0,\theta)\\
&= -(n-1)!2^n\cos^n\theta + q_n(\theta) + n(n-2)!2^n\cos^n\theta - 2n\cos\theta q_{n-1}(\theta) + 2\binom{n}{2}K^{(n-2)}(0,\theta)\\
&= (n-2)!2^n\cos^n\theta + \widetilde{q}_n(\theta), \qquad \text{where}\\
\widetilde{q}_n(\theta) &:= q_n(\theta) - 2n\cos\theta q_{n-1}(\theta) + 2\binom{n}{2}K^{(n-2)}(0,\theta).
\end{aligned}$$

Note that $\widetilde{q}_n \in \mathbb{T}_{n-2}$. Since $\int_{-\pi}^{\pi} e^{in\theta}q(\theta)\,d\theta = 0$ for all $q \in \mathbb{T}_{n-2}$, it follows that

$$\begin{aligned}
a_n &= \frac{1}{4\pi n!}\int_{-\pi}^{\pi} e^{in\theta}(n-2)!2^n\cos^n\theta\,d\theta\\
&= \frac{2^n}{2n(n-1)}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\cos^n\theta\,d\theta = \frac{1}{2n(n-1)}, \qquad n>1.
\end{aligned}$$

In order to determine the b_n's, it is simpler if we first apply the Laplacian to (3.12) to obtain

$$\Delta G(\xi) = b_n 4(n+1)(\xi_1 + i\xi_2)^n, \quad \xi \in B.$$

If $\xi = \tau y(0)$, then $G(\tau y(0)) = b_n 4(n+1)\tau^n$ and hence

$$\begin{aligned} b_n &= \frac{1}{4(n+1)!}\frac{d^n}{d\tau^n}\Delta G(\tau y(0))_{|\tau=0} \\ &= \frac{1}{4(n+1)!}\frac{d^n}{d\tau^n} R_n[\Delta\phi(\cdot - \tau y(0))](1)_{|\tau=0} \end{aligned}$$

Since $\Delta\phi = 4 + 4\log|\cdot|$, it follows that $\Delta\phi(y(\theta) - \tau y(0)) = 4 + 2K(\tau,\theta)$ and hence

$$\begin{aligned} b_n &= \frac{1}{4(n+1)!}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}(4 + 2K^{(n)}(0,\theta))\,d\theta \\ &= \frac{1}{4(n+1)!}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}(4 - (n-1)!2^{n+1}\cos^n\theta + 2q_n(\theta))\,d\theta \\ &= -\frac{2^n}{2n(n+1)}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\cos^n\theta\,d\theta, \qquad \text{since } 4 + 2q_n \in \mathbb{T}_{n-2}, \\ &= -\frac{1}{2n(n+1)}. \end{aligned}$$

Considering first the case $n = 1$, we have by (3.12) that $G(\xi) = (\xi_1 + i\xi_2)(-\frac{1}{2} - \frac{1}{4}|\xi|^2)$, and hence by (3.10) and (3.11) it follows that

$$\begin{aligned} R_1[\phi(\cdot - x)](t) &= -t^2(\log t)(x_1/t + ix_2/t) + t^2 G(x/t) \\ &= -(x_1 + ix_2)t\log t + t^2(x_1/t + ix_2/t)(-1/2 - |x/t|^2/4) \\ &= -(x_1 + ix_2)(t\log t + t/2 + t^{-1}|x|^2/4). \end{aligned}$$

which proves (ii). Assume now that $n > 1$. Then by (3.12), $G(\xi) = (\xi_1 + i\xi_2)^n(\frac{1}{2n(n-1)} - \frac{1}{2n(n+1)}|\xi|^2)$, and by (3.10) and (3.11) we obtain

$$\begin{aligned} R_n[\phi(\cdot - x)](t) &= t^2 G(x/t) = t^2(x_1/t + ix_2/t)^n\left(\frac{1}{2n(n-1)} - \frac{1}{2n(n+1)}|x/t|^2\right) \\ &= \frac{1}{2n}(x_1 + ix_2)^n\left(\frac{t^{2-n}}{n-1} - \frac{t^{-n}|x|^2}{n+1}\right) \end{aligned}$$

which proves (iii). $\square$

Corollary 3.13. *For all $x \in B$ and $n \in \mathbb{Z}$, $|n| \geq 2$, the following hold:*

(1) $$\begin{aligned} R_0[D_{\vec{n}}\Delta\phi(\cdot - x)](1) &= R_0[\Delta\phi(\cdot - x)](1) = 4 \\ &= 4(R_0[D_{\vec{n}}\phi(\cdot - x)](1) - R_0[\phi(\cdot - x)](1)), \end{aligned}$$

(2) $$\begin{aligned} R_{\pm1}[D_{\vec{n}}\Delta\phi(\cdot - x)](1) &= -R_{\pm1}[\Delta\phi(\cdot - x)](1) = 2(x_1 \pm ix_2) \\ &= -(R_{\pm1}[D_{\vec{n}}\phi(\cdot - x)](1) + R_{\pm1}[\phi(\cdot - x)](1)), \end{aligned}$$

(3) $$\begin{aligned} R_n[D_{\vec{n}}\Delta\phi(\cdot - x)](1) &= -|n|\,R_n[\Delta\phi(\cdot - x)](1) = 2(x_1 + \operatorname{sign}(n)ix_2)^{|n|} \\ &= 2|n|(|n|-1)(R_n[D_{\vec{n}}\phi(\cdot - x)](1) + |n|\,R_n[\phi(\cdot - x)](1)). \end{aligned}$$

where for brevity we have written $R_n[D_{\vec{n}}u](1)$ in place of $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta} D_{\vec{n}} u(y(\theta))\, d\theta$, $\vec{n}$ being the outward unit normal to ∂B at $y(\theta)$.

Proof. Let Δ_x denote the Laplacian with respect to x, ie. $\Delta_x := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$. It is a straightforward matter to verify that

$$R_n[\Delta u(\cdot - x)](t) = \Delta_x R_n[u(\cdot - x)](t), \quad \text{and} \tag{3.14}$$

$$R_n[D_{\vec{n}} u(\cdot - x)](1) = \frac{\partial}{\partial t} R_n[u(\cdot - x)](t)_{|t=1}. \tag{3.15}$$

Evaluating (i) at $t = 1$ yields $R_0[\phi(\cdot - x)](1) = |x|^2$; while it follows from (i) and (3.15) that $R_0[D_{\vec{n}}\phi(\cdot - x)](1) = 1 + |x|^2$. Hence, $4R_0[D_{\vec{n}}\phi(\cdot - x)](1) - 4R_0[\phi(\cdot - x)](1) = 4$. It follows from (i) and (3.14) that

$$R_0[\Delta\phi(\cdot - x)](t) = 4(1 + \log t). \tag{3.16}$$

Evaluating (3.16) at $t = 1$ yields $R_0[\Delta\phi(\cdot - x)](1) = 4$; while it follows from (3.16) and (3.15) that $R_0[D_{\vec{n}}\Delta\phi(\cdot - x)](1) = 4$. Hence (1). In order to prove (2) and (3), it suffices to consider only $n \geq 1$ as the remaining cases follow simply by complex conjugation. In a similar manner to the above, it can be deduced from (ii), (3.14), and (3.15) that $R_1[\phi(\cdot - x)](1) = -(x_1 + ix_2)(\frac{1}{2} + \frac{1}{4}|x|^2)$, $R_1[D_{\vec{n}}\phi(\cdot - x)](1) = -(x_1 + ix_2)(\frac{3}{2} - \frac{1}{4}|x|^2)$, $R_1[\Delta\phi(\cdot - x)](1) = -2(x_1 + ix_2)$, and $R_1[D_{\vec{n}}\Delta\phi(\cdot - x)](1) = 2(x_1 + ix_2)$ from which (2) readily follows. Similarly, it follows from (iii), (3.14), and (3.15) that $R_n[\phi(\cdot - x)](1) = \frac{1}{2n}(x_1 + ix_2)^n(\frac{1}{n-1} - \frac{|x|^2}{n+1})$, $R_n[D_{\vec{n}}\phi(\cdot - x)](1) = \frac{1}{2n}(x_1 + ix_2)^n(\frac{2-n}{n-1} + \frac{n|x|^2}{n+1})$, $R_n[\Delta\phi(\cdot - x)](1) = -\frac{2}{n}(x_1 + ix_2)^n$, and $R_n[D_{\vec{n}}\Delta\phi(\cdot - x)](1) = 2(x_1 + ix_2)^n$ from which (3) readily follows. □

Corollary 3.17. *Let $f \in C^\infty(\mathbb{R}^2)$, and define sequences $\{c_n\}_{n\in\mathbb{Z}}$, $\{d_n\}_{n\in\mathbb{Z}}$ by*

$$c_n := \begin{cases} 4(a_0 - b_0) & \text{if } n = 0, \\ -(a_n + b_n) & \text{if } |n| = 1, \\ 2(|n| - 1)(|n|\, a_n + b_n) & \text{if } |n| > 1; \end{cases} \qquad d_n := \begin{cases} -c_0 & \text{if } n = 0, \\ |n|\, c_n & \text{if } n \neq 0, \end{cases}$$

where

$$a_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} f(y(\theta))\, d\theta \quad \text{and } b_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} D_{y(\theta)} f(y(\theta))\, d\theta.$$

Then for all $x \in B$,

$$8\pi f(x) = \int_B \phi(\xi - x)\Delta^2 f(\xi)\, d\xi + \int_{-\pi}^{\pi} D_{y(\theta)}\phi(y(\theta) - x)[\Delta f(y(\theta)) + \sum_{n\in\mathbb{Z}} c_n e^{in\theta}]\, d\theta$$
$$+ \int_{-\pi}^{\pi} \phi(y(\theta) - x)[-D_{y(\theta)}\Delta f(y(\theta)) + \sum_{n\in\mathbb{Z}} d_n e^{in\theta}]\, d\theta.$$

Proof. In light of Proposition 3.6, it suffices to show that for all $x \in B$,

$$\int_{-\pi}^{\pi} D_{y(\theta)}\Delta\phi(y(\theta)-x)f(y(\theta)) - \Delta\phi(y(\theta)-x)D_{y(\theta)}f(y(\theta))\,d\theta$$
$$= \int_{-\pi}^{\pi} D_{y(\theta)}\phi(y(\theta)-x)\sum_{n\in\mathbb{Z}} c_n e^{in\theta} + \phi(y(\theta)-x)\sum_{n\in\mathbb{Z}} d_n e^{in\theta}\,d\theta.$$

Since $f \in C^\infty(\mathbb{R}^2)$, the following hold:

$$\lim_{|n|\to\infty} (|a_n| + |b_n|)\,|n|^m = 0, \quad \forall m > 0,$$
$$\lim_{|n|\to\infty} (|c_n| + |d_n|)\,|n|^m = 0 \quad \forall m > 0,$$
$$f(y(\theta)) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}, \quad \forall \theta \in [-\pi\,..\,\pi],$$
$$D_{y(\theta)}f(y(\theta)) = \sum_{n\in\mathbb{Z}} b_n e^{in\theta}, \quad \forall \theta \in [-\pi\,..\,\pi].$$

It follows from Corollary 3.13 that

$$a_n R_n[D_{\vec{n}}\Delta\phi(\cdot - x)](1) - b_n R_n[\Delta\phi(\cdot - x)](1)$$
$$= c_n R_n[D_{\vec{n}}\phi(\cdot - x)](1) + d_n R_n[\phi(\cdot - x)](1) \quad \forall n \in \mathbb{Z}, x \in B.$$

Hence, for $x \in B$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_{y(\theta)}\Delta\phi(y(\theta)-x)f(y(\theta)) - \Delta\phi(y(\theta)-x)D_{y(\theta)}f(y(\theta))\,d\theta$$
$$= \sum_{n\in\mathbb{Z}} (a_n R_n[D_{\vec{n}}\Delta\phi(\cdot - x)](1) - b_n R_n[\Delta\phi(\cdot - x)](1))$$
$$= \sum_{n\in\mathbb{Z}} (c_n R_n[D_{\vec{n}}\phi(\cdot - x)](1) + d_n R_n[\phi(\cdot - x)](1))$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_{y(\theta)}\phi(y(\theta)-x)\sum_{n\in\mathbb{Z}} c_n e^{in\theta} + \phi(y(\theta)-x)\sum_{n\in\mathbb{Z}} d_n e^{in\theta}\,d\theta.$$

□

Proof of Theorem 3.1. Let $f \in C^\infty(\mathbb{R}^2)$, and let $\{c_n\}_{n\in\mathbb{Z}}$ and $\{d_n\}_{n\in\mathbb{Z}}$ be as defined in Corollary 3.17. Define the distribution μ by

$$\langle g, \mu\rangle := \frac{1}{8\pi}\int_B g(\xi)\Delta^2 f(\xi)\,d\xi + \frac{1}{8\pi}\int_{-\pi}^{\pi} D_{y(\theta)}g(y(\theta))(\Delta f(y(\theta)) + \sum_{n\in\mathbb{Z}} c_n e^{in\theta})\,d\theta$$
$$+ \frac{1}{8\pi}\int_{-\pi}^{\pi} g(y(\theta))(-D_{y(\theta)}\Delta f(y(\theta)) + \sum_{n\in\mathbb{Z}} d_n e^{in\theta})\,d\theta.$$

It follows from Corollary 3.17 (and from the fact that $\phi = \phi(-\cdot)$) that $\phi * \mu(x) = f(x)$ for all $x \in B$. Define distributions $\mu_{(0,0)}$, $\mu_{(1,0)}$, and $\mu_{(0,1)}$ by

$$\begin{aligned}
\langle g, \mu_{(0,0)} \rangle &:= \frac{1}{2\pi} \int_{-\pi}^{\pi} D_{y(\theta)} g(y(\theta)) - g(y(\theta))\, d\theta, \\
\langle g, \mu_{(1,0)} \rangle &:= -\frac{1}{4\pi} \int_{-\pi}^{\pi} \left(D_{y(\theta)} g(y(\theta)) + g(y(\theta)) \right) \cos\theta\, d\theta, \\
\langle g, \mu_{(0,1)} \rangle &:= -\frac{1}{4\pi} \int_{-\pi}^{\pi} \left(D_{y(\theta)} g(y(\theta)) + g(y(\theta)) \right) \sin\theta\, d\theta.
\end{aligned}$$

It follows from Corollary 3.13 that for all $x \in B$, $\phi * \mu_{(0,0)}(x) = 1$, $\phi * \mu_{(1,0)}(x) = x_1$, and $\phi * \mu_{(0,1)}(x) = x_2$. And it can be shown with a simple integration that $\langle 1, \mu_{(0,0)} \rangle = -1$, $\langle ()^{(1,0)}, \mu_{(1,0)} \rangle = \langle ()^{(0,1)}, \mu_{(0,1)} \rangle = -1/2$, and $\langle ()^{\alpha}, \mu_{\beta} \rangle = 0$ whenever $\alpha, \beta \in \{(0,0), (1,0), (0,1)\}$ and $\alpha \neq \beta$. Define the distribution $\widetilde{\mu}$ and the polynomial $\widetilde{p} \in \Pi_1$ by

$$\begin{aligned}
\widetilde{\mu} &:= \mu + \langle 1, \mu \rangle \mu_{(0,0)} + 2\langle ()^{(1,0)}, \mu \rangle \mu_{(1,0)} + 2\langle ()^{(0,1)}, \mu \rangle \mu_{(0,1)}, \\
\widetilde{p} &:= -\langle 1, \mu \rangle - 2\langle ()^{(1,0)}, \mu \rangle ()^{(1,0)} - 2\langle ()^{(0,1)}, \mu \rangle ()^{(0,1)},
\end{aligned}$$

and note that $\operatorname{supp} \widetilde{\mu} \subset \overline{B}$. Put $\widetilde{f} := \phi * \widetilde{\mu} + \widetilde{p}$. Then, for $x \in B$,

$$\begin{aligned}
\widetilde{f}(x) =& \phi * \mu(x) + \langle 1, \mu \rangle (\phi * \mu_{(0,0)}(x) - 1) + 2\langle ()^{(1,0)}, \mu \rangle (\phi * \mu_{(1,0)}(x) - x_1) \\
&+ 2\langle ()^{(0,1)}, \mu \rangle (\phi * \mu_{(0,1)} - x_2) = f(x).
\end{aligned}$$

Since $f \in C^{\infty}(\mathbb{R}^2)$, it follows by Lemma 3.3 that $\left| \widehat{\widetilde{\mu}}(w) \right| \leq \operatorname{const}(1 + \sqrt{|w|})$, $w \in \mathbb{R}^2$. In the definition of $\widetilde{\mu}$, the coefficients of $\mu_{(0,0)}$, $\mu_{(1,0)}$, and $\mu_{(0,1)}$ were chosen to ensure that $\langle 1, \widetilde{\mu} \rangle = \langle ()^{(1,0)}, \widetilde{\mu} \rangle = \langle ()^{(0,1)}, \widetilde{\mu} \rangle = 0$. It follows from this that $\widehat{\widetilde{\mu}}(0) = D^{(1,0)} \widehat{\widetilde{\mu}}(0) = D^{(0,1)} \widehat{\widetilde{\mu}}(0) = 0$ from which we conclude $\left| \widehat{\widetilde{\mu}}(w) \right| \leq \operatorname{const} |w|^2$. Hence, $\left| \widehat{\widetilde{\mu}}(w) \right| \leq \operatorname{const} \dfrac{|w|^2}{1 + |w|^{3/2}}$, $w \in \mathbb{R}^2$. Therefore $\widetilde{f} \in \mathcal{F}$. $\square$

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