UNIVERSITY OF WISCONSIN-MADISON
CENTER FOR THE MATHEMATICAL SCIENCES

A multivariate form of Hardy's inequality and L_p-error bounds for multivariate Lagrange interpolation schemes

Shayne Waldron* (*waldron@math.wisc.edu*)

CMS Technical Summary Report #95-02
August 1994

ABSTRACT

The following *multivariate* generalisation of *Hardy's inequality*, that for $m - n/p > 0$

$$\| \, x \mapsto \int_{[\underbrace{x,\ldots,x}_{m},\Theta]} f \, \|_p \le \frac{\Gamma(m-n/p)}{\Gamma(m)\Gamma(\#\Theta + m - n/p)} \|f\|_p, \tag{1}$$

valid for $f \in L_p(\mathbb{R}^n)$ and Θ an arbitrary finite sequence of points in $\mathbb{R}^n$, is discussed.

The linear functional $f \mapsto \int_\Theta f$ was introduced by Micchelli [M80] in connection with *Kergin interpolation*. This functional also naturally occurs in other multivariate generalisations of Lagrange interpolation, including *Hakopian interpolation*, and the *Lagrange maps* of Section 5. For each of these schemes, (1) implies L_p-error bounds.

We discuss why (1) plays a crucial role in obtaining L_p-bounds from pointwise integral error formulæ for multivariate generalisations of Lagrange interpolation.

AMS (MOS) Subject Classifications: primary 41A05, 41A10, 41A63, 41A80

Key Words: Hardy's inequality, Lagrange interpolation, Kergin interpolation, Hakopian interpolation, B-spline, simplex spline, Hermite-Genocchi formula

* Supported by the Chebyshev professorship of Carl de Boor and the Center for the mathematical sciences.

1. Introduction

The central result of this paper is the inequality, that for $m - n/p > 0$

$$\| \, x \mapsto \int_{[\underbrace{x,\ldots,x}_{m},\Theta]} f \, \|_{L_p(\Omega)} \leq \frac{\Gamma(m-n/p)}{\Gamma(m)\Gamma(\#\Theta+m-n/p)} \|f\|_{L_p(\Omega)}, \quad \forall f \in L_p(\Omega), \tag{1.1}$$

where Θ is a finite sequence of points in $\mathbb{R}^n$, and Ω is a suitable domain in $\mathbb{R}^n$. This inequality is a *multivariate* generalisation of **Hardy's inequality**, that for $p > 1$

$$\| \, x \mapsto \frac{1}{x}\int_0^x f \, \|_{L_p(0,\infty)} \leq \frac{p}{p-1} \|f\|_{L_p(0,\infty)}, \quad \forall f \in L_p(0,\infty). \tag{1.2}$$

Thus, we will refer to (1.1) as the **multivariate form of Hardy's inequality**.

Our interest in (1.1) comes from a desire to obtain L_p-bounds from the many integral error formulæ for *multivariate* generalisations of Lagrange interpolation that involve the linear functional

$$f \mapsto \int_{[\underbrace{x,\ldots,x}_{m},\Theta]} f. \tag{1.3}$$

The paper is set out in the following way. In the remainder of this section, the notation, and facts about Sobolev spaces that we will need are discussed. In Section 2, some properties of the linear functional $f \mapsto \int_\Theta f$, and its connection with simplex splines are given. In Section 3, the multivariate form of Hardy's inequality is proved. In Section 4, the multivariate form of Hardy's inequality is applied to obtain L_p-bounds for the error in the scale of mean value interpolations, which includes Kergin and Hakopian interpolation. In Section 5, in a similar vein, L_p-bounds for the error in *Lagrange maps* are obtained. In Section 6, we discuss why the multivariate form of Hardy's inequality is applicable to the many error formulæ for multivariate Lagrange interpolation schemes, and is likely to be so for others obtained in the future.

Some notation

Our discussion takes place in $\mathbb{R}^n$, with the following definitions holding throughout. The space of n-variate polynomials of degree k will be denoted by $\Pi_k(\mathbb{R}^n)$, and the homogeneous polynomials of degree k by $\Pi_k^0(\mathbb{R}^n)$. The differential operator induced by $q \in \Pi_k(\mathbb{R}^n)$ will be written $q(D)$. Let $\|\cdot\|$ be the *Euclidean norm* on $\mathbb{R}^n$, and let $\Omega \subset \mathbb{R}^n$, with $\bar{\Omega}$ is closure. The letters i, j, k, l, m, n will be reserved for integers, and $1 \leq p \leq \infty$. We use standard multivariate notation, so, e.g., $\{|\alpha| = k\}$ is the set of multi-indices α of length k.

We find it convenient to make no distinction between the matrix $[\theta_1, \ldots, \theta_k]$, and the k-**sequence** $\theta_1, \ldots, \theta_k$ of its columns. Since $[\theta_1, \ldots, \theta_k]f$ is a standard notation for the *divided difference* of f at $\Theta = [\theta_1, \ldots, \theta_k]$, we use for the latter the nonstandard notation

$$\delta_\Theta f = \delta_{[\theta_1,\ldots,\theta_k]} f.$$

Note the special case

$$\delta_{[x]} f = f(x).$$

Similarly, to avoid any confusion, the closed interval with endpoints a and b will be denoted by $[a\,..\,b]$.

The derivative of f in the directions Θ is denoted

$$D_\Theta f := D_{\theta_1} \cdots D_{\theta_k} f.$$

The notation $\tilde{\Theta} \subset \Theta$ means that $\tilde{\Theta}$ is a subsequence of Θ, $\Theta \backslash \tilde{\Theta}$ denotes the complementary subsequence. The subsequence consisting of the first j terms of Θ is denoted Θ_j, and

$$x - \Theta := [x - \theta_1, \ldots, x - \theta_k].$$

Thus, with $\Theta := [\theta_1, \ldots, \theta_7]$, we have, for example, that

$$D_{[x-\Theta\backslash\Theta_5, x-\theta_3]} f = D_{x-\theta_6} D_{x-\theta_7} D_{x-\theta_3} f.$$

The diameter and convex hull of a sequence Θ will be that of the corresponding set and will be denoted by $\operatorname{diam}\Theta$ and $\operatorname{conv}\Theta$ respectively.

Many of the constants in this paper involve the **Gamma function** Γ. Each can be calculated from the relation: $\Gamma(a+1) = a\Gamma(a)$, $\forall a > 0$, and the fact that $\Gamma(1) = 1$. Some of our calculations require the **Beta integrals**

$$\int_0^1 t^{a-1}(1-t)^{b-1}\,dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \qquad a, b > 0; \tag{1.4}$$

see, e.g., Jones [Jo93:p200].

Geometry of the domain Ω

We say that $\Omega \subset \mathbb{R}^n$ is **starshaped with respect to** S a set (resp. sequence) in $\mathbb{R}^n$ when Ω contains the convex hull of $S \cup \{x\}$ for any $x \in \Omega$. This condition is weaker than Ω being convex.

In our results, it will be required that $\bar{\Omega}$ be starshaped with respect to $\Theta \in \mathbb{R}^{n\times k}$, where Ω is an open set in $\mathbb{R}^n$. This condition is required of $\bar{\Omega}$, rather than of Ω, so as to include cases where some points in Θ lie on the boundary of Ω. One such example of interest, is the *Lagrange* finite element given by linear interpolation at Θ, the vertices of a n-simplex, see, e.g. Ciarlet [Ci78:p46]. In this case, $\bar{\Omega} = \operatorname{conv}\Theta$ and none of the points of Θ lie in the open simplex Ω.

We now show that being starshaped with respect to a finite sequence is equivalent to being starshaped with respect to its convex hull.

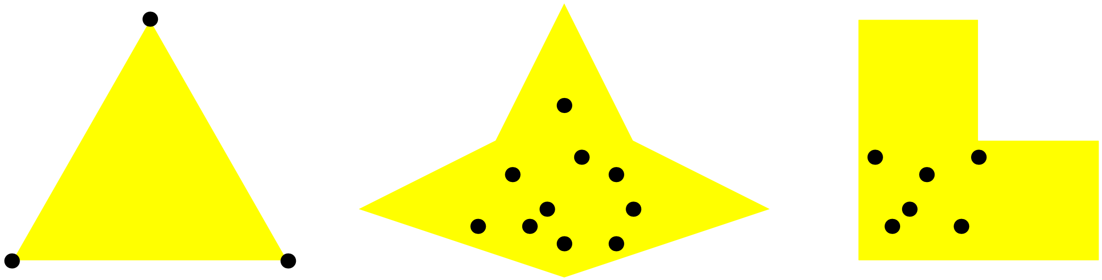

Fig 1.1 Examples of domains Ω (shaded) for which $\bar{\Omega}$ is starshaped with respect to the points in Θ ($\bullet$)

Proposition 1.5. *If $\Omega \subset \mathbb{R}^n$ and $\Theta \in \mathbb{R}^{n\times k}$, then the following are equivalent:*
(a) *Ω is starshaped with respect to Θ.*
(b) *Ω is starshaped with respect to* $\operatorname{conv}\Theta$.

Proof. Only the implication (a) $\Longrightarrow$ (b) requires proof. Suppose (a). To obtain (b) it suffices to prove that if Ω is starshaped with respect to points u and v, then $\operatorname{conv}\{u,v,x\} \subset \Omega$, $\forall x \in \Omega$, i.e., Ω is starshaped with respect to $\operatorname{conv}\{u,v\}$.

Assume wlog that u, v, x are affinely independent and $z \in \operatorname{conv}\{u,v,x\}$. Let w be the point of intersection of the line through u and z with the interval $\operatorname{conv}\{x,v\}$. Since Ω is starshaped with respect to v, one has that $w \in \Omega$. Thus, since Ω is starshaped with respect to u, one has that $z \in \operatorname{conv}\{u,w\} \subset \Omega$. $\square$

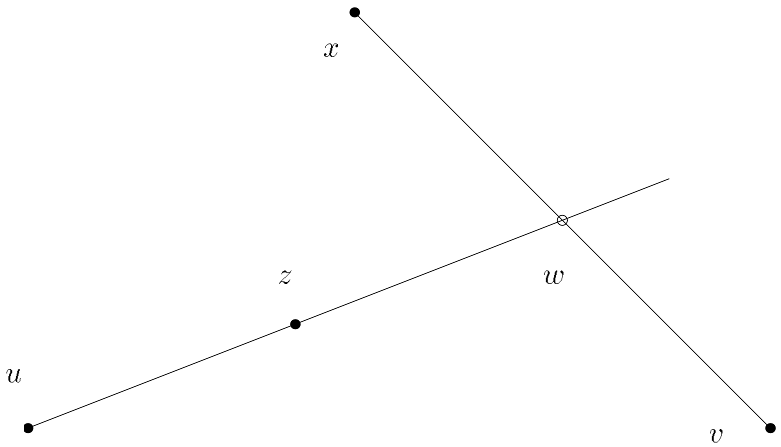

Fig 1.2 The proof of Proposition 1.5

This equivalence ensures that if $\bar{\Omega}$ is starshaped with respect to Θ, then $f \in L_p(\Omega)$ is defined over the region of integration in (1.3) for all $x \in \Omega$.

Sobolev spaces

Let $W_p^{(k)}(\Omega)$ be the **Sobolev space** consisting of those functions defined on Ω (a bounded open set in $\mathbb{R}^n$ with a *Lipschitz* boundary) with derivatives up to order k in $L_p(\Omega)$, and equipped with the usual topology; see, e.g., Adams [Ad75]. It is convenient to include the condition that Ω have a Lipschitz boundary in the definition, so that Sobolev's embedding theorem can be applied. The full statement of Sobolev's embedding theorem can be found in any text on Sobolev spaces, see, e.g., [Ad75:p97]; however we will need only the following consequence of it. If $j - n/p > 0$, then

$$W_p^{k+j}(\Omega) \subset C^k(\bar{\Omega}).$$

To measure the size of its k-th derivative, it is convenient to associate with each $f \in W_p^{(k)}(\Omega)$ the function $|D^k f| \in L_p(\Omega)$, given by the rule

$$|D^k f|(x) := \sup_{\substack{\Theta \in \mathbb{R}^{n\times k} \\ \|\theta_i\| \le 1}} |D_\Theta f(x)| = \sup_{\substack{\theta \in \mathbb{R}^n \\ \|\theta\|=1}} |D_\theta^k f(x)|, \tag{1.6}$$

where the derivatives $D_\Theta f$ are computed from any (fixed) choice of representatives for the partial derivatives $D^\alpha f \in L_p(\Omega)$, $|\alpha| = k$. The equality of the two suprema is proved in Chen and Ditzian [CD90]. This definition of $|D^k f|$ is consistent with its alternative interpretation in the univariate case. From (1.6), it is easy to see that $|D^k f|$ is well-defined and satisfies

$$|D_\Theta f| \le |D^k f| \, \|\theta_1\| \cdots \|\theta_k\|, \tag{1.7}$$

for all $\Theta \in \mathbb{R}^{n\times k}$. The inequality (1.7) holds a.e. To emphasize that $D_\Theta f$, $|D^k f| \in L_p(\Omega)$, we will say that (1.7) holds in $L_p(\Omega)$. The $L_p(\Omega)$-norm of $|D^k f|$ gives a seminorm on $W_p^{(k)}(\Omega)$,

$$f \mapsto | f |_{k,p,\Omega} := \| \, |D^k f| \, \|_{L_p(\Omega)}. \tag{1.8}$$

Because of (1.7), this coordinate-independent seminorm (1.8) is more appropriate for the analysis that follows than other equivalent seminorms, such as

$$f \mapsto \| \, (\|D^\alpha f\|_{L_p(\Omega)})_{\{|\alpha|=k\}} \, \|_p.$$

2. The linear functional $f \mapsto \int_\Theta f$

The construction of the maps of Kergin and Hakopian depends intimately on the following linear functional introduced by Micchelli [M80].

Definition 2.1. *For any $\Theta \in \mathbb{R}^{n\times(k+1)}$, let*

$$f \mapsto \int_\Theta f := \int_0^1 \int_0^{s_1} \cdots \int_0^{s_{k-1}} f(\theta_0 + s_1(\theta_1 - \theta_0) + \cdots + s_k(\theta_k - \theta_{k-1}))\, ds_k \cdots ds_2\, ds_1,$$

with the convention that $\int_{[]} f := 0$.

In addition to Kergin and Hakopian interpolation, the linear functional $f \mapsto \int_\Theta f$ naturally occurs when discussing other *multivariate* generalisations of Lagrange interpolation, e.g., the *Lagrange maps* of Section 5.

In this section we outline those properties of $f \mapsto \int_\Theta f$ needed in the subsequent sections. Many of these properties are apparent from the following observation.

Observation 2.2. *If S is any k-simplex in IR^m and $A : \mathrm{IR}^m \to \mathrm{IR}^n$ is any affine map taking the $k+1$ vertices of S onto the $k+1$ points in Θ, then*

$$\int_\Theta f = \frac{1}{k!\,\mathrm{vol}_k(S)} \int_S f \circ A,$$

with $\mathrm{vol}_k(S)$ the (k-dimensional) volume of S.

In Definition 2.1

$$A : \mathrm{IR}^k \to \mathrm{IR}^n : (s_1, \dots, s_k) \mapsto \theta_0 + s_1(\theta_1 - \theta_0) + \dots + s_k(\theta_k - \theta_{k-1}),$$

$$S := \{(s_1, \dots, s_k) \in \mathrm{IR}^k : 0 \le s_k \le \dots \le s_2 \le s_1 \le 1\}.$$

In [M80], Micchelli uses a different choice of S and A, namely

$$A : \mathrm{IR}^{k+1} \to \mathrm{IR}^n : (v_0, \dots, v_k) \mapsto v_0\theta_0 + \dots + v_k\theta_k,$$

$$S := \{(v_0, \dots, v_k) \in \mathrm{IR}^{k+1} : v_j \ge 0,\ \sum_{j=0}^{k} v_j = 1\}.$$

Properties 2.3.
(a) *The value of $\int_\Theta f$ does not depend on the ordering of the points in Θ.*
(b) *The distribution*

$$M_\Theta : C_0^\infty(\mathrm{IR}^n) \to \mathrm{IR} : f \mapsto k! \int_\Theta f$$

is the **(normalised) simplex spline** *with knots Θ.*
(c) *If $f \in C(\mathrm{conv}\,\Theta)$, then $\int_\Theta f$ is defined and, for some $\xi \in \mathrm{conv}\,\Theta$,*

$$\int_\Theta f = \frac{1}{k!} f(\xi).$$

(d) *If $g : \mathrm{IR}^s \to \mathrm{IR}$, and $B : \mathrm{IR}^n \to \mathrm{IR}^s$ is an affine map, then*

$$\int_\Theta (g \circ B) = \int_{B\Theta} g.$$

Part of Micchelli's motivation for defining $\int_\Theta f$ was the **Hermite-Genocchi formula**, namely that for $\Theta \in \mathrm{IR}^{1\times(k+1)}$ and $f \in C^k(\mathrm{conv}\,\Theta)$

$$\delta_\Theta f = \int_\Theta D^k f.$$

Some technical details

Remark 2.4. In view of Property (a),

$$\Theta \mapsto \int_\Theta f$$

could be thought of as a map defined on finite multisets in $\mathbb{R}^n$ rather than on sequences. However, adopting this definition leads to certain unnecessary complications. For example, to discuss the continuity of $\Theta \mapsto \int_\Theta f$, it would be necessary to endow the set of multisets of $k+1$ points in $\mathbb{R}^n$ with the appropriate topology. Thus, in the interest of simplicity, $\Theta \mapsto \int_\Theta f$ remains a map on sequences – but with the reader encouraged to think of it, as does the author, as a map on multisets. $\square$

Remark 2.5. The simplex spline M_Θ of (b) has support $\operatorname{conv}\Theta$. It can be represented by the nonnegative bounded function

$$\operatorname{conv}\Theta \to \mathbb{R} : t \mapsto M(t|\Theta) := \frac{\operatorname{vol}_{k-d}(A^{-1}t \cap S)}{|\det A|}, \quad d := \dim \operatorname{conv}\Theta,$$

in the sense that

$$M_\Theta f = \int_{\operatorname{conv}\Theta} M(\cdot|\Theta) f. \tag{2.6}$$

In particular, if the points of Θ are affinely independent, then

$$k! \int_\Theta f = \frac{1}{\operatorname{vol}_k(\operatorname{conv}\Theta)} \int_{\operatorname{conv}\Theta} f = \text{average value of } f \text{ on } \operatorname{conv}\Theta. \tag{2.7}$$

Thus, $\int_\Theta f$ is defined (as a real number) if and only if $M(\cdot|\Theta)f \in L_1(\operatorname{conv}\Theta)$, in which case

$$\left|\int_\Theta f\right| \le \int_\Theta |f|. \tag{2.8}$$

If f is nonnegative on $\operatorname{conv}\Theta$, then $\int_\Theta f \in [0\,..\,\infty]$ is defined (by Definition 2.1). Therefore, we will write (2.8) for all f which are defined on $\operatorname{conv}\Theta$ – with the understanding that $\int_\Theta f$ is defined if and only if $\int_\Theta |f| < \infty$ or f is nonnegative. In the univariate case, that is when $n = 1$, $M(\cdot|\Theta)$ is the **(normalised) B-spline** with knots Θ. For additional details about M_Θ and $M(\cdot|\Theta)$, see, e.g., Micchelli [M79]. $\square$

Lastly, by (2.6), we can describe the continuity of $\Theta \mapsto \int_\Theta f$ as follows.

Proposition 2.9.
(a) *For $f \in C(\mathbb{R}^n)$, the map*

$$\mathbb{R}^{n\times(k+1)} \to \mathbb{R} : \Theta \mapsto \int_\Theta f$$

is continuous.
(b) *For $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, the map*

$$\{\Theta \in \mathbb{R}^{n\times(k+1)} : \operatorname{vol}_n(\operatorname{conv}\Theta) > 0\} \to \mathbb{R} : \Theta \mapsto \int_\Theta f$$

is continuous.

3. The main results: the multivariate form of Hardy's inequality and L_p-inequalities

In this section we prove the multivariate form of Hardy's inequality. This inequality is useful for obtaining L_p-bounds from integral error formulæ for various multivariate interpolation schemes.

First we need a technical lemma.

Lemma 3.1. *Let m, k be integers, and $\mu \in \mathbb{R}$. If $1 \le m \le k$ and $m + \mu > 0$, then*

$$\int_0^1 \int_0^{s_1} \cdots \int_0^{s_{k-1}} (1 - s_m)^\mu \, ds_k \cdots ds_1 = \frac{\Gamma(m+\mu)}{\Gamma(m)\Gamma(k+1+\mu)}.$$

Proof. This can be proved by successively evaluating the univariate integrals. Instead we give the following proof – a neat application of the properties of $f \mapsto \int_\Theta f$. By Observation 2.2, we see that

$$\int_0^1 \int_0^{s_1} \cdots \int_0^{s_{k-1}} (1 - s_m)^\mu \, ds_k \cdots ds_1 = \int_\Theta (\cdot)^\mu,$$

where

$$\Theta := [\underbrace{0, \ldots, 0}_{m}, \underbrace{1, \ldots, 1}_{k+1-m}].$$

For this Θ, $M(\cdot|\Theta)$ is the B-spline of order k supported on $[0\,..\,1]$, with $(m-1)$-fold, $(k-m)$-fold zeros at 0, 1 respectively, and $\int M(\cdot|\Theta) = 1$. Thus, by (1.4), we have that

$$M(t|\Theta) = \frac{\Gamma(k+1)}{\Gamma(m)\Gamma(k+1-m)} t^{m-1}(1-t)^{k-m}, \quad 0 \le t \le 1.$$

From (2.6) and (1.4) we conclude that

$$\begin{aligned}
\int_\Theta (\cdot)^\mu &= \frac{1}{\Gamma(k+1)} \int_0^1 (\cdot)^\mu M(\cdot|\Theta) \\
&= \frac{1}{\Gamma(m)\Gamma(k+1-m)} \int_0^1 t^\mu t^{m-1}(1-t)^{k-m} \, dt \\
&= \frac{\Gamma(m+\mu)}{\Gamma(m)\Gamma(k+1+\mu)}.
\end{aligned}$$

Here the condition that $m + \mu > 0$ is needed to ensure that the Beta integral is finite. □

The multivariate form of Hardy's inequality

Now we prove the multivariate form of Hardy's inequality.

Theorem 3.2. *Let Θ be a finite sequence in $\mathbb{R}^n$, and let Ω be an open set in $\mathbb{R}^n$ for which $\bar{\Omega}$ is starshaped with respect to Θ. If $m - n/p > 0$, then the rule*

$$L_{m,\Theta} f(x) := \int_{[\underbrace{x,\ldots,x}_{m},\Theta]} f \tag{3.3}$$

induces a monotone bounded linear map $L_{m,\Theta} : L_p(\Omega) \to L_p(\Omega)$ with norm

$$\|L_{m,\Theta}\| \le \frac{\Gamma(m - n/p)}{\Gamma(m)\Gamma(\#\Theta + m - n/p)} \to \infty \qquad \text{as} \qquad m - n/p \to 0^+. \tag{3.4}$$

This upper bound for $\|L_{m,\Theta}\|$ is sharp when $p = \infty$.

Proof. Suppose that $m - n/p > 0$. Then $m > 0$, and we let $k + 1 := m + \#\Theta$. Let $\mathbf{L}_p(\Omega)$ be the semi-normed linear space consisting of those (measurable) functions f defined on Ω with $\|f\|_{L_p(\Omega)} < \infty$, together with the semi-norm $\|\cdot\|_{L_p(\Omega)}$. Let Z be the set of those $f \in \mathbf{L}_p(\Omega)$ for which $\|f\|_{L_p(\Omega)} = 0$. By Proposition 1.5, the condition that $\bar{\Omega}$ be starshaped with respect to Θ ensures that it is starshaped with respect to $\operatorname{conv}\Theta$. In particular, for any $x \in \Omega$, the region of integration in (3.3) is contained within $\bar{\Omega}$ (upto a nullset:=set of measure zero). However, *a priori*, we do not know whether (3.3) defines a function $L_{m,\Theta} f \in L_p(\Omega)$ for every $f \in L_p(\Omega)$, i.e., equivalently, that the linear map $\mathcal{L}_{m,\Theta} : \mathbf{L}_p(\Omega) \to \mathbf{L}_p(\Omega)$ given by (3.3) maps Z to Z. In view of Remark 2.5, to show this, together with the bound for $\|L_{m,\Theta}\|$, it is sufficient to prove the inequality

$$\|L_{m,\Theta} f\|_{L_p(\Omega)} \le \frac{\Gamma(m - n/p)}{\Gamma(m)\Gamma(k + 1 - n/p)} \|f\|_{L_p(\Omega)}, \tag{3.5}$$

for all $f \in \mathbf{L}_p(\Omega)$ which are nonnegative. In this case, $L_{m,\Theta} f$ is a well-defined nonnegative function, which could possibly take on the value ∞.

We now prove (3.5). Let $f \in \mathbf{L}_p(\Omega)$ be nonnegative, and write

$$[\underbrace{x,\ldots,x}_{m},\Theta] = [\underbrace{x,\ldots,x}_{m},\theta_m,\theta_{m+1},\ldots,\theta_k].$$

By Definition 2.1,

$$L_{m,\Theta} f(x) = \int_S f(A_x s)\, ds, \tag{3.6}$$

where $s := (s_1, \ldots, s_k)$ and

$$\int_S := \int_0^1 \int_0^{s_1} \cdots \int_0^{s_{k-1}}, \qquad ds := ds_k \cdots ds_1,$$
$$A_x s := x + s_m(\theta_m - x) + s_{m+1}(\theta_{m+1} - \theta_m) + \cdots + s_k(\theta_k - \theta_{k-1}).$$

Applying Minkowski's inequality for integrals (see, e.g., Folland [Fo84:p186]) to the sum $\int_S$ of functions $x \mapsto f(A_x s)$ we obtain, by (3.6), that

$$\|L_{m,\Theta} f\|_{L_p(\Omega)} \le \int_S \|x \mapsto f(A_x s)\|_{L_p(\Omega)}\, ds. \tag{3.7}$$

The case $1 \le p < \infty$. We may write (3.7) as

$$\|L_{m,\Theta} f\|_{L_p(\Omega)} \le \int_S \left(\int_\Omega f(A_x s)^p \, dx \right)^{1/p} ds.$$

In the inner integral, make the change of variables $y = A_x s$. For this choice, the new region of integration is contained in Ω, and $dy = (1 - s_m)^n dx$. Thus, by the change of variables formula, see, e.g., Rudin [Ru87:p153], we obtain that

$$\int_S \left(\int_\Omega f(A_x s)^p \, dx \right)^{1/p} ds \le \int_S \left(\int_\Omega \frac{f(y)^p \, dy}{(1 - s_m)^n} \right)^{1/p} ds = \left(\int_S (1 - s_m)^{-n/p} \, ds \right) \|f\|_{L_p(\Omega)}.$$

Finally, by Lemma 3.1 with $m + \mu = m - n/p > 0$, we have

$$\int_S (1 - s_m)^{-n/p} \, ds = \frac{\Gamma(m - n/p)}{\Gamma(m)\Gamma(k + 1 - n/p)},$$

giving (3.4) for $1 \le p < \infty$.

The case $p = \infty$. Since $x \mapsto A_x s$ maps nullsets to nullsets, we obtain from (3.7) that

$$\|L_{m,\Theta} f\|_{L_\infty(\Omega)} \le \int_S \|f\|_{L_\infty(\Omega)} \, ds = \frac{1}{k!} \|f\|_{L_\infty(\Omega)},$$

with equality when f is constant. Here we used

$$\int_S ds = \frac{1}{k!} = \frac{\Gamma(m)}{\Gamma(m)\Gamma(k+1)},$$

which follows from Observation 2.2, or by Lemma 3.1 with $\mu = 0$. This completes the case $p = \infty$. $\square$

Remark 3.8. If $\mathrm{vol}_n(\mathrm{conv}\,\Theta) > 0$, then, by Remark 2.5, it follows that the value of $L_{m,\Theta} f(x)$ is the same for all representatives of $f \in L_p(\Omega)$. Indeed, by Proposition 2.9, for all $f \in L_p(\Omega)$ we have that $L_{m,\Theta} f \in C(\bar{\Omega})$, regardless of whether or not $m - n/p > 0$.

On the other hand, when $\mathrm{vol}_n(\mathrm{conv}\,\Theta) = 0$, then the function $L_{m,\Theta} f$ need not be so well-behaved. For example, if $n > 1$ and Θ consists of a single point θ, then $f \in L_p(\Omega)$ can be altered on a nullset so that $L_{m,\Theta} f$ takes on arbitrary preassigned values on any countable dense subset of Ω. For the details of one such construction, see the end of this section.

The function $L_{m,[\theta]}f$ is more than simply an interesting example. It occurs in the *multipoint Taylor error formula* for multivariate Lagrange interpolation given by Ciarlet and Raviart [CR72]. From the multipoint Taylor formula, Arcangeli and Gout [AG76] obtained L_p-bounds for multivariate Lagrange interpolation, long used by those working in finite elements, but known to few approximation theorists. For this reason, these bounds are discussed in some detail in Section 5. $\square$

Special case: Hardy's inequality

In the very special case $n=1$, $m=1$, and $\Theta=[0]$, one has, by (2.7), that

$$L_{m,\Theta}f(x)=\frac{1}{x}\int_0^x f. \tag{3.9}$$

With the choice $\Omega=(0,\infty)$, (3.4) is Hardy's inequality (1.2). This well-known inequality was first proved by Hardy [Ha28], see also [HLP67:§9.8].

For a comprehensive survey of the literature connected with Hardy's inequality, see Chapter IV: Hardy's, Carleman's and related inequalities, of the monograph [FMP91]. The only *multivariate* occurrence of Theorem 3.2 that the author is aware of is, implicitly, in Arcangeli and Gout [AG76] for the case when Θ consists of a single point. The bulk of the 174 references for chapter IV of [FMP91] deals with *univariate* generalisations of Hardy's inequality – some of which are special cases of Theorem 3.2.

In this paper we will not be concerned with the sharpness of (3.4). However, for those so interested we mention the following point of departure. For the map (3.9),

$$\|L_{m,\Theta}\|=\frac{p}{p-1}$$

when $\Omega=(0,\infty)$, see, e.g., [Ru87:ex.14,p72], [Jo93:p275,p289]; and also when $\Omega=(0,b)$, $b>0$, see Shum [Sh71].

Other L_p-bounds

Next we use Theorem 3.2 to give a bound particularly suited for obtaining L_p-bounds from integral error formulæ, such as those given in Sections 4 and 5.

Theorem 3.10. *Fix $a_1,\ldots,a_s\in\mathbb{R}^{k+1}\setminus 0$, where $s\ge 0$. Let $\Theta\in\mathbb{R}^{n\times k}$, and let Ω be a bounded open set in $\mathbb{R}^n$ for which $\bar\Omega$ is starshaped with respect to Θ. If $m-n/p>0$, then the rule*

$$\mathcal{L}f(x):=\int_{\underbrace{[x,\ldots,x,\Theta]}_{m}}\Bigl(\prod_{j=1}^{s}D_{[x,\Theta]a_j}\Bigr)f \tag{3.11}$$

induces a bounded linear map $\mathcal{L}:W_p^s(\Omega)\to L_p(\Omega)$, with

$$\|\mathcal{L}f\|_{L_p(\Omega)}\le\Bigl(\max_{x\in\bar\Omega}\prod_{j=1}^{s}\|[x,\Theta]a_j\|\Bigr)\frac{\Gamma(m-n/p)}{\Gamma(m)\Gamma(\#\Theta+m-n/p)}|f|_{s,p,\Omega}. \tag{3.12}$$

In addition, when $p = \infty$, we have the pointwise estimate

$$|\mathcal{L}f(x)| \le \frac{1}{(\#\Theta + m - 1)!}\left(\prod_{j=1}^{s} \|[x,\Theta]a_j\|\right) |f|_{s,\infty,\Omega}, \qquad \text{a.e. } x \in \Omega. \tag{3.13}$$

Proof. Let $x \in \Omega$ and $f \in W_p^s(\Omega)$. By (1.7),

$$\left|\left(\prod_{j=1}^{s} D_{[x,\Theta]a_j}\right) f\right| \le \left(\prod_{j=1}^{s} \|[x,\Theta]a_j\|\right) |D^s f|, \tag{3.14}$$

in $L_p(\Omega)$. Here $|D^s f| \in L_p(\Omega)$ is defined by (1.6). Thus, the rule

$$Af(x) := \left(\prod_{j=1}^{s} D_{[x,\Theta]a_j}\right) f$$

defines a bounded linear map $A : W_p^s(\Omega) \to L_p(\Omega)$, with

$$\|Af\|_{L_p(\Omega)} \le \left(\max_{x\in\bar\Omega} \prod_{j=1}^{s} \|[x,\Theta]a_j\|\right) |f|_{s,p,\Omega}. \tag{3.15}$$

Remember that $\|\,|D^s f|\,\|_{L_p(\Omega)} = |f|_{s,p,\Omega}$.

Notice that $\mathcal{L} = L_{m,\Theta} \circ A$. Thus, from (3.15) and Theorem 3.2, we obtain the bound (3.12). Similarly, from (3.14) and Theorem 3.2, we have for a.e. $x \in \Omega$, that

$$\begin{aligned}|\mathcal{L}f(x)| &\le \left(\prod_{j=1}^{s} \|[x,\Theta]a_j\|\right) \|L_{m,\Theta}(|D^s f|)\|_{L_\infty(\Omega)} \\ &\le \left(\prod_{j=1}^{s} \|[x,\Theta]a_j\|\right) \frac{1}{(\#\Theta + m - 1)!} |f|_{s,\infty,\Omega},\end{aligned}$$

which is (3.13). □

In the special case when $s = 0$, Theorem 3.10 reduces to Theorem 3.2. Theorem 3.10, together with Property 2.3 (d), can be used to obtain bounds for maps more general than (3.11). One such example is the *lift* of an *elementary liftable map*, see [Wa94].

An example

Finally, the example promised in Remark 3.8.

Let $n > 1$ and Θ consist of the single point θ. Suppose that $\bar\Omega$ is starshaped with respect to θ, and that B is a countable dense subset of Ω. It is possible to change $f \in L_p(\Omega)$ on the intersection of Ω with the cone C with vertex θ and base B, which is a nullset, so that $L_{m,[\theta]}f$, as computed from (3.3), takes on arbitrary preassigned values on B.

The cone C consists of the union of rays r emanating from θ and passing through a point $b \in B$. Let r be such a ray, and order the points from B lying on r as $b_1, b_2, \ldots$, so that b_i is closer to θ than b_{i+1}. By Remark 2.5,

$$L_{m,[\theta]} f(b_i) = \int M(\cdot \,|\, \underbrace{b_i, \ldots, b_i}_{m}, \theta)\, f$$

with the integration above being over the interval $[\theta \,..\, b_i] := \operatorname{conv}\{\theta, b_i\}$ weighted by a nonnegative polynomial. Thus, by redefining f to be an appropriate constant over each of the intervals $[\theta \,..\, b_1]$, $[b_1 \,..\, b_2]$, $[b_2 \,..\, b_3], \ldots$, one can make $L_{m,[\theta]} f(b_i)$ take on any preassigned values.

4. Application: L_p-error bounds for Kergin and Hakopian interpolation

In this section we use Theorem 3.10 to obtain L_p-error bounds for the *scale of mean value interpolations*, which includes the *Kergin* and *Hakopian maps*.

To describe the mean value interpolations, and the Lagrange maps of Section 5, we will need the following facts about linear interpolation.

Linear interpolation

Let F be a finite-dimensional space and Λ a finite-dimensional space of linear functionals defined at least on F. We say that the corresponding **linear interpolation problem**, $\mathrm{LIP}(F, \Lambda)$ for short, is **correct** if for every g upon which Λ is defined there is a unique $f \in F$ which agrees with g on Λ, i.e.,

$$\lambda(f) = \lambda(g), \quad \forall \lambda \in \Lambda.$$

The linear map $L : g \mapsto f$ is called the associated (**linear**) **projector** with **interpolants** F and **interpolation conditions** Λ. Each linear projector with finite-dimensional range F is the solution of a $\mathrm{LIP}(F, \Lambda)$ for some unique choice of the interpolation conditions Λ.

Notice that the correctness of $\mathrm{LIP}(F, \Lambda)$ depends only on the action of Λ on F.

The scale of mean value interpolations

Throughout this section, $\Theta \in \mathbb{R}^{n \times k}$. For $0 \le m < k$, we have the **mean value interpolation**

$$\mathcal{H}_\Theta^{(m)} : \{f : f \text{ is } C^{k-m-1} \text{ on } \operatorname{conv}\Theta\} \to \Pi_{k-m-1}(\mathbb{R}^n),$$

which is given by

$$\mathcal{H}_\Theta^{(m)} f(x) := m! \sum_{j=m+1}^{k} \sum_{\substack{\tilde\Theta \subset \Theta_{j-1} \\ \#\tilde\Theta = m}} \int_{\Theta_j} D_{x-\Theta_{j-1}\setminus\tilde\Theta} f.$$

$\mathcal{H}_\Theta^{(m)}$ is a linear projector, with interpolants $\Pi_{k-m-1}(\mathbb{R}^n)$ and interpolation conditions

$$\operatorname{span}\{f \mapsto \int_{\tilde\Theta} q(D)f : \tilde\Theta \subset \Theta,\ \#\tilde\Theta \geq m+1,\ q \in \Pi^0_{\#\tilde\Theta-m-1}(\mathbb{R}^n)\}.$$

The map $\mathcal{H}_\Theta^{(0)}$ is **Kergin's map**, and $\mathcal{H}_\Theta^{(n-1)}$ is **Hakopian's map**. Both of these maps interpolate function values at Θ, and so the scale $(\mathcal{H}_\Theta^{(m)} : 0 \leq m < k)$ of multivariate mean value interpolations is thought of as a multivariate generalisation of Lagrange interpolation. For more details see [Wa94].

For the remainder of this section, Ω will be a bounded open set in $\mathbb{R}^n$ with a Lipschitz boundary. From [Wa94], one obtains the following integral error formulæ for the scale of mean value interpolations.

Theorem 4.1. *Suppose that $\bar\Omega$ is starshaped with respect to Θ. If $0 \leq j < k-m$, $q \in \Pi_j^0(\mathbb{R}^n)$, $p > n$, and $f \in W_p^{(k-m)}(\Omega)$, then*

$$q(D)(f - \mathcal{H}_\Theta^{(m)} f)(x) = (m+j)! \sum_{i=k-m-j}^{k} \sum_{\substack{\tilde\Theta \subset \Theta_{i-1} \\ \#\tilde\Theta = m+j+i-k}} \int_{[\underbrace{x,\ldots,x}_{k+1-i},\Theta_i]} D_{[x-\Theta_{i-1}\setminus\tilde\Theta,\, x-\theta_i]} q(D) f. \tag{4.2}$$

This formula involves only derivatives of f of order $k-m$.

Remark 4.3. In [Wa94] the formula (4.2) was proved only for $f \in C^{k-m}(\mathbb{R}^n)$, without any reference to p. We now outline how it can be extended to $f \in W_p^{(k-m)}(\Omega)$. By Sobolev's embedding theorem, the condition $p > n$ implies that

$$W_p^{(k-m)}(\Omega) \subset C^{k-m-1}(\bar\Omega) \subset C(\bar\Omega).$$

Thus, $\mathcal{H}_\Theta^{(m)} f$ is defined for all $f \in W_p^{(k-m)}(\Omega)$. To extend (4.2) to $f \in W_p^{(k-m)}(\Omega)$ use the density of $C_0^\infty(\Omega)$ in $W_p^{(k-m)}(\Omega)$. $\square$

L_p-bounds for the scale of mean value interpolations

Next we apply Theorem 3.10 to (4.2) to obtain L_p-bounds for the scale of mean value interpolations. Let

$$h_{x,\Theta} := \sup_{\theta \in \Theta} \|x - \theta\|, \quad h_{\Omega,\Theta} := \sup_{x \in \Omega} h_{x,\Theta} \leq \operatorname{diam} \Omega.$$

Theorem 4.4. *Suppose that $\bar{\Omega}$ is starshaped with respect to Θ. If $0 \le j < k-m$, $p > n$, and $f \in W_p^{(k-m)}(\Omega)$, then*

$$|f - \mathcal{H}_\Theta^{(m)} f|_{j,p,\Omega} \le C_{n,p,j,k,m} \, (h_{\Omega,\Theta})^{k-m-j} \, |f|_{k-m,p,\Omega}, \tag{4.5}$$

where

$$C_{n,p,j,k,m} := \frac{(m+j)!}{\Gamma(k+1-n/p)} \sum_{i=k-m-j}^{k} \binom{i-1}{m+j+i-k} \frac{\Gamma(k+1-i-n/p)}{(k-i)!}.$$

The constant $C_{n,p,j,k,m} \to \infty$ as $p \to n^+$. Additionally, if $p = \infty$, then we have the pointwise estimate that, for all $x \in \bar{\Omega}$,

$$|D^j(f - \mathcal{H}_\Theta^{(m)} f)|(x) \le \frac{1}{(k-m-j)!} (h_{x,\Theta})^{k-m-j} \, |f|_{k-m,\infty,\Omega}.$$

Proof. Choose $q \in \Pi_j^0(\mathbb{R}^n)$ so that

$$q(D) = D_{u_1} \cdots D_{u_j},$$

where $u_1, \ldots, u_j \in \mathbb{R}^n$ with $\|u_i\| \le 1$. By Theorem 3.10, we have for each of the terms in (4.2) that

$$\|x \mapsto \int_{[\underbrace{x,\ldots,x}_{k+1-i},\Theta_i]} D_{[x-\Theta_{i-1}\backslash\tilde{\Theta},x-\theta_i]} q(D) f\|_{L_p(\Omega)}$$

$$\le \frac{\Gamma(k+1-i-n/p)}{\Gamma(k+1-i)\Gamma(k+1-n/p)} (h_{x,\Theta})^{k-m-j} \, |f|_{k-m,\infty,\Omega}.$$

Notice that in the above, the constants

$$\max_{x \in \bar{\Omega}} \prod_{\theta \in [\Theta_{i-1}\backslash\tilde{\Theta},\theta_i]} \|x - \theta\|$$

were replaced by the possibly larger, but far less complicated constant $(h_{\Omega,\Theta})^{k-m-j}$. This gives the first inequality.

The second, which is proved in [Wa94], follows from the pointwise estimate (3.13). □

A related result of Lai and Wang

The only related result in the literature is an L_p-bound for the error in Hakopian interpolation given by Lai and Wang [LW84]. In that paper they show the following.

Theorem 4.6 ([LW84:Th.1]). *Let $|\alpha| \le k - n$. Then for any positive integer $\ell \le k + |\alpha| - n + 1$, we have*

$$\begin{aligned}
&D^\alpha(f - \mathcal{H}_\Theta^{(n-1)})(x) \\
&=(|\alpha| + n - 1) \sum_{\mu_1=1}^{|\alpha|+n} \sum_{i_1=1}^{n} (x - \theta_{|\alpha|+n-\mu_1+1})_{i_1} \sum_{\mu_2=1}^{\mu_1} \sum_{i_2=1}^{n} (x - \theta_{|\alpha|+n-\mu_2+2})_{i_2} \times \\
&\quad \cdots \times \sum_{\mu_\ell=1}^{\mu_{\ell-1}} \sum_{i_\ell=1}^{n} (x - \theta_{|\alpha|+n-\mu_\ell+\ell})_{i_\ell} \int_{[\underbrace{x,\dots,x}_{\mu_\ell},\theta_1,\dots,\theta_{|\alpha|+n-\mu_\ell+\ell}]} D^{\alpha+\sum_{j=1}^{\ell} e^{i_j}} f \qquad (4.7) \\
&\quad - \sum_{j=|\alpha|+n-1+\ell}^{k-1} \quad \sum_{|\gamma|=j-n+1} D^\alpha \omega_\gamma(x) \int_{[\theta_1,\dots,\theta_j]} D^\gamma f.
\end{aligned}$$

The above uses standard multi-index notation. The i-th component of $x \in \mathbb{R}^n$ is x_i, and e^i is the i-th unit vector in $\mathbb{R}^n$. To (4.7), Lai and Wang apply the integral form of Minkowski's inequality in the form

$$\|x \mapsto \int_{[\underbrace{x,\dots,x}_{\mu},\theta_1,\dots,\theta_{k+1-\mu}]} D^\beta f\|_{L_p(G)} \le C_2 \, \|D^\beta f\|_{L_p(G)}, \quad \mu = 1, \dots, |\alpha| + n, \qquad (4.8)$$

to obtain the following.

Theorem 4.9 ([LW84:Th.2]). *Let G be a convex set containing Θ, with diameter h. If $p > n$, $|\alpha| \le k - n$, and $f \in W_p^{(k-n+1)}(G)$, then*

$$\|D^\alpha(f - \mathcal{H}_\Theta^{(n-1)} f)\|_{L_p(G)} \le C \, h^{k-n+1-|\alpha|} \max_{|\beta|=k-n+1} \|D^\beta f\|_{L_p(G)}, \qquad (4.10)$$

where C a constant independent of f.

Since $f \mapsto \max_{|\beta|=k+1-n} \|D^\beta f\|_{L_p(\Omega)}$, and $f \mapsto | f |_{k+1-n,p,\Omega}$ are equivalent semi-norms, Theorem 4.9 follows from Theorem 4.4. Had Lai and Wang attempted to compute the C_2 of (4.8) using the multivariate form of Hardy's inequality, they would have obtained

$$C_2 \le \frac{\Gamma(\mu - n/p)}{\Gamma(\mu)\Gamma(k+1)}.$$

Thus, their constant C in (4.10) would have the same qualitative behaviour as our own $C_{n,p,j,k,m}$ of (4.5), namely that $C \to \infty$ as $p \to n^+$.

The behaviour of $C_{n,p,j,k,m}$ as a function of its parameters

In [Wa94] it is shown that, in an appropriate sense, the constant $C_{n,p,j,k,m}$ of (4.5) is best possible when $p = \infty$. The question then arises whether or not the over-estimation committed in using the multivariate form of Hardy's inequality to obtain $C_{n,p,j,k,m}$ is significant for $p < \infty$. In particular, does the best possible constant C in the inequality

$$|f - \mathcal{H}_\Theta^{(m)} f|_{j,p,\Omega} \le C\,(h_{\Omega,\Theta})^{k-m-j}\,|f|_{k-m,p,\Omega} \tag{4.11}$$

become unbounded as $p \to n^+$? In the univariate case, at least, the answer is *no* – the best possible constant in (4.11) does not become unbounded.

Before we show this, let us clarify a little the role that the condition $p > n$ plays in Theorems 4.4 and 4.9. The condition $p > n$ is necessary if these results are to be stated in terms of the Sobolev space $W_p^{(k-m)}(\Omega)$ – in particular so that $\mathcal{H}_\Theta^{(m)} f$ is defined for $f \in W_p^{(k-m)}(\Omega)$. However, it makes good sense to ask what is the best constant C for which (4.11) holds for all sufficiently smooth functions f – say, e.g., $f \in C^{k-m}(\bar{\Omega})$. The condition $p > n$ is again needed when one seeks to apply the multivariate form of Hardy's inequality to the integral error formulæ (4.2) and (4.7).

We end this section by showing that in the univariate case, i.e., when $n = 1$, there is a best possible constant C in (4.11) for all sufficiently smooth f, which can be bounded independently of $1 \le p \le \infty$. The crucial step in the argument to follow is the use of the B-spline L_p-estimate that

$$\|M(\cdot|\Theta)\|_{L_p(\mathbb{R})} \le \left(\frac{\#\Theta - 1}{\operatorname{diam}\Theta}\right)^{1-1/p} \tag{4.12}$$

when $\operatorname{diam}\Theta > 0$, see de Boor [B73].

In line with [Wa94], the univariate case of the map $\mathcal{H}_\Theta^{(m)}$, termed the *generalised Hermite map*, will be emphasised by writing it as $H_\Theta^{(m)}$. This map has the simple form

$$H_\Theta^{(m)} f = D^m(H_\Theta D^{-m} f),$$

where H_Θ is the *Hermite interpolator* at the points Θ, and $D^{-m} f$ is any function for which $D^m(D^{-m} f) = f$.

Theorem 4.13. *Let Θ be a k-sequence in the interval $[a\,..\,b]$. If $0 \le j < k - m$, and $f \in C^{k-m}[a\,..\,b]$, then*

$$\|D^j(f - H_\Theta^{(m)} f)\|_{L_p[a..b]} \le \frac{(m+j)!}{(k-m-j)!}\,\frac{k^{1/q}}{k!}\,(b-a)^{k-m+\frac{1}{p}-\frac{1}{q}}\,\|D^{k-m} f\|_{L_q[a..b]}.$$

Here $1 \le p, q \le \infty$.

Proof. Fix $x \in [a\,..\,b]$. For Θ a finite sequence in $\mathbb{R}$, let

$$\omega_\Theta(x) := \prod_{\theta\in\Theta}(x - \theta).$$

With this notation, replacing each occurrence in (4.2) of a linear functional of the form $f \mapsto \int_\Theta f$ by integration against a B-spline, we obtain that

$$D^j(f - H_\Theta^{(m)} f)(x)$$
$$= (m+j)! \sum_{i=k-m-j}^{k} \sum_{\substack{\tilde\Theta \subset \Theta_{i-1} \\ \#\tilde\Theta = m+j+i-k}} \omega_{\Theta_{i-1}\backslash\tilde\Theta}(x)\,(x-\theta_i)\,\frac{1}{k!} \int D^{k-m} f\, M(\cdot|x, \Theta_i).$$

By Hölder's inequality, and (4.12), we have that

$$\left| \int D^{k-m} f\, M(\cdot|x,\Theta_i) \right| \le \left(\frac{k}{\operatorname{diam}[x,\Theta_i]} \right)^{1/q} \|D^{k-m} f\|_{L_q[a..b]}.$$

Since

$$\left| \frac{\omega_{\Theta_{i-1}\backslash\tilde\Theta}(x)\,(x-\theta_i)}{(\operatorname{diam}[x,\Theta_i])^{1/q}} \right| \le (b-a)^{k-m-1/q},$$

we obtain that

$$|D^j(f - H_\Theta^{(m)} f)(x)|$$
$$\le (m+j)! \sum_{i=k-m-j}^{k} \binom{i-1}{m+j+i-k} \frac{k^{1/q}}{k!} (b-a)^{k-m-1/q} \|D^{k-m} f\|_{L_q[a..b]}$$
$$= \frac{(m+j)!}{(k-m-j)!} \frac{k^{1/q}}{k!} (b-a)^{k-m-1/q} \|D^{k-m} f\|_{L_q[a..b]}.$$

Finally, take $\|\cdot\|_{L_q[a..b]}$ of both sides. □

To adapt this argument to the multivariate case, it is necessary to have the *simplex spline* analog of the B-spline L_p-estimate (4.12). This is provided by Dahmen [D79], who shows that when $\operatorname{vol}_n(\operatorname{conv}\Theta) > 0$,

$$\|M(\cdot|\Theta)\|_{L_p(\mathbb{R}^n)} \le \frac{k!(k+1)!}{n!(n+1)!(n-k)!} \left(\frac{1}{\operatorname{vol}_n(\operatorname{conv}\Theta)} \right)^{1-1/p}, \tag{4.14}$$

with $k+1 := \#\Theta$. Yet, with this in hand, it does not seem possible to apply the argument of Theorem 4.13 in any satifactory form.

Remark 4.15. Incidentally, the constant in (4.14) is not the best possible. Already, by using the fact that $\int M(\cdot|\Theta) = 1$, together with the case $p = \infty$ of (4.14), one obtains

$$\|M(\cdot|\Theta)\|_{L_p(\mathbb{R}^n)} \le \left(\frac{k!(k+1)!}{n!(n+1)!(n-k)!} \frac{1}{\operatorname{vol}_n(\operatorname{conv}\Theta)} \right)^{1-1/p}.$$

In the univariate case this over-estimates (4.12) by a factor of $((k+1)!/2)^{1-1/p}$.

The key step in proving (4.12) is the bound

$$M(\cdot|\Theta) \le \frac{k}{\operatorname{diam}\Theta}, \tag{4.16}$$

which follows from the partition of unity property of B-splines. Thus, a close examination of the simplex spline analog of the B-spline partition of unity, given recently by Dahmen, Micchelli and Seidel [DMS92], should give tighter bounds than those of (4.14). However, we make no attempt here to give such an argument. □

Remark 4.17. There are other integral error formulæ for the scale of mean value interpolations, to which Theorem 3.10 can be applied to give L_p-bounds. These include Lai and Wang [LW86] (Kergin interpolation), Gao [Ga88], and Hakopian [BHS93:p200] (Hakopian interpolation). See [Wa94] for a discussion of the relative merits of each of these formulæ. □

5. Application: L_p-error bounds for multivariate Lagrange interpolation

In this section we use Theorem 3.10 to obtain L_p-error bounds for *multivariate Lagrange interpolation schemes.*

Lagrange maps

A linear interpolation problem for which the space of interpolation conditions is spanned by *point evaluations* at Θ, a finite sequence in $\mathbb{R}^n$, is called a **Lagrange interpolation problem**. If P is the space of interpolants for such a problem and the problem is correct, then the associated linear projector, called the **Lagrange map**, will be denoted by $L_{P,\Theta}$. The **Lagrange form** of a Lagrange map is given by

$$L_{P,\Theta} f = \sum_{\theta\in\Theta} f(\theta)\ell_\theta. \tag{5.1}$$

Here (5.1) uniquely defines

$$\ell_\theta := \ell_{\theta,P,\Theta} \in P,$$

the **Lagrange function** for $\theta \in \Theta$. In other words, $(\delta_{[\theta]})_{\theta\in\Theta}$ is dual (bi-orthonormal) to $(\ell_\theta)_{\theta\in\Theta}$.

Lagrange maps into a space containing polynomials of degree k are frequently used to interpolate to scattered data, see, e.g., Alfeld [Al89]. Particular examples receiving much attention lately are maps where the interpolants include *radial basis functions* or *multivariate splines*, and de Boor and Ron's *least solution* for the polynomial interpolation problem [BR92]. In addition there are of course the maps of Kergin and Hakopian.

For such maps, there is the *multipoint Taylor formula* for the error. This formula was initiated by the work of Ciarlet and Wagschal [CW71]; most of the relevant papers are in French, and it is little known outside the area of finite elements. It is for these reasons, and because our Theorem 3.10 implies L_p-estimates from the multipoint Taylor formula, that we discuss the formula here.

The multipoint Taylor formula

Multipoint Taylor formula 5.2 ([CR72]). *Let Θ be a finite sequence in $\mathbb{R}^n$, and let Ω be an open set in $\mathbb{R}^n$ for which $\bar{\Omega}$ is starshaped with respect to Θ. If $L_{P,\Theta}$ is a Lagrange map with $\Pi_k(\mathbb{R}^n) \subset P \subset C^k(\bar{\Omega})$, then for $f \in C^{k+1}(\bar{\Omega})$, $q \in \Pi_k(\mathbb{R}^n)$, and $x \in \bar{\Omega}$, its error satisfies:*

$$\big(q(D)(L_{P,\Theta} f - f)\big)(x) = \sum_{\theta \in \Theta} \Big(\int_{[\underbrace{x, \ldots, x}_{k+1}, \theta]} D^{k+1}_{\theta - x} f \Big) (q(D)\ell_\theta)(x). \tag{5.3}$$

The term *multipoint Taylor formula* comes from the fact that

$$\theta \mapsto \int_{[\underbrace{x, \ldots, x}_{k+1}, \theta]} D^{k+1}_{\theta - x} f$$

is the error in *Taylor interpolation* of degree k at the point x, a special case of the error in Kergin interpolation. The proof of (5.3) further justifies the use of this term.

The region of integration in (5.3) consists of line segments from x to $\theta \in \Theta$.

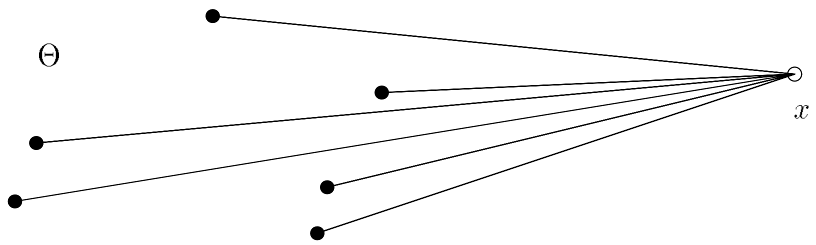

Fig 5.1 The region of integration in (5.3) for Θ consisting of 6 points

From the multipoint Taylor formula, Arcangeli and Gout [AG76] obtain L_p-bounds for the error in a Lagrange map. These bounds are precisely those obtained by applying Theorem 3.10 to (5.3). The crucial step in the argument presented in [AG76:Prop.1-1] is the use of the multivariate form of Hardy's inequality for the map

$$x \mapsto L_{k+1,[v]} f(x) := \int_{[\underbrace{x, \ldots, x}_{k+1}, v]} f. \tag{5.4}$$

This inequality is not explicitly stated, though the proof of the (weaker) Proposition 1-1 would imply it.

Remark 5.5. The key step in the proof of Proposition 1-1 in [AG76] is an application of Hölder's inequality to the splitting

$$\int_{\underbrace{[x,\ldots,x,v]}_{k+1}} f = \frac{1}{k!}\int_0^1 (1-t)^{-1/q-\varepsilon}\left((1-t)^{k+1/q-\varepsilon} f(x+t(v-x))\right) dt,$$

where $\varepsilon := (k+1-n/p)/q$, and $1/p+1/q=1$, as opposed to our use of the integral form of Minkowski's inequality. □

Having identified the precise role of the multivariate form of Hardy's inequality in [AG76] it is possible to use it to run through Arcangeli and Gout's calculation for a much more general class of norms, including those most often used in numerical analysis. The resulting bounds, given below, have smaller (and simpler) constants than those one might hope to obtain by applying the inequalities for similar norms to the results of [AG76].

For the remainder of this section Ω will denote a bounded open set in $\mathbb{R}^n$ with a Lipschitz boundary, and Θ a finite sequence in $\mathbb{R}^n$. Recall

$$h_{\Omega,\Theta} = \sup_{\theta\in\Theta}\sup_{x\in\Omega} \|x-\theta\| \le \operatorname{diam}\Omega.$$

Corollary 5.6. *Suppose that $\bar\Omega$ is starshaped with respect to Θ, and that $L_{P,\Theta}$ is a Lagrange map with $\Pi_k(\mathbb{R}^n)\subset P\subset C^k(\Omega)$. If $k+1-n/p>0$, and $f\in W_p^{(k+1)}(\Omega)$, then*

$$|f-L_{P,\Theta}f|_{p,\Omega} \le \frac{1}{k!(k+1-n/p)}\left(\sum_{\theta\in\Theta}|\ell_\theta|_{\infty,\Omega}\right) |f|_{k+1,p,\Omega}\,(h_{\Omega,\Theta})^{k+1}. \tag{5.7}$$

Here $|\cdot|_{p,\Omega}$ is any seminorm on $W_p^k(\Omega)$ of the form

$$|f|_{p,\Omega} := \|\,(\|g_i(D)f\|_{L_p(\Omega)})_{i=1}^m\,\|_{\mathbb{R}^m},$$

where the $g_i\in\Pi_k(\mathbb{R}^n)$ are fixed, and $\|\cdot\|_{\mathbb{R}^m}$ is any norm on $\mathbb{R}^m$ – or $|\cdot|_{p,\Omega}$ is $|\cdot|_{i,p,\Omega}$ for some $0\le i\le k$.

Proof. By Sobolev's embedding theorem, the condition $k+1-n/p>0$ implies

$$W_p^{(k+1)}(\Omega)\subset C(\bar\Omega),$$

and so the Lagrange map $L_{P,\Theta}$ is well defined. As in Remark 4.3, (5.3) can be extended to $f\in W_p^{(k+1)}(\Omega)$. Fix $f\in W_p^{(k+1)}(\Omega)$, and $x\in\Omega$. Let $h:=h_{\Omega,\Theta}$. By (1.7),

$$|D_{\theta-x}^{k+1}f| \le |D^{k+1}f|\,\|\theta-x\|^{k+1} \le |D^{k+1}f|\,h^{k+1},$$

in $L_p(\Omega)$. Thus, with $g_i \in \Pi_k(\mathbb{R}^n)$, we have for a.e. $x \in \Omega$ that

$$|(g_i(D)(f - L_{P,\Theta}f))(x)| \le \sum_{\theta\in\Theta} \Big(\int_{[\underbrace{x,\ldots,x}_{k+1},\theta]} |D^{k+1}f| \Big) \|g_i(D)\ell_\theta\|_{L_\infty(\Omega)}\, h^{k+1}.$$

To this, the condition $k+1-n/p>0$ allows us to apply the multivariate form of Hardy's inequality to obtain

$$\|g_i(D)(f - L_{P,\Theta}f)\| \le \frac{1}{k!(k+1-n/p)} \Big(\sum_{\theta\in\Theta} \|g_i(D)\ell_\theta\|_{L_\infty} \Big) |f|_{k+1,p,\Omega}\, h^{k+1}.$$

Finally, take the $\|\cdot\|_{\mathbb{R}^m}$ norm of the inequality (for m-vectors) given above. □

In [AG76:Th.1-1] Corollary 5.6 is proved only in the case when $|\cdot|_{p,\Omega}$ is of the form $|f|_{i,p,\Omega}$ for some $0 \le i \le k$, with $h_{\Omega,\Theta}$ replaced by $\operatorname{diam}\Omega$. In that paper some bounds on the size of the Lagrange functions ℓ_θ, together with relevant applications are given. One application is bounding the error in a *finite element scheme*, see also Ciarlet [Ci78:p128]. Another, of interest to approximation theorists, is to estimate the distance of smooth functions from $\Pi_k(\mathbb{R}^n)$, and to give the corresponding *constructive* version of the Bramble-Hilbert Lemma, see [BH70].

The condition in Corollary 5.6 that $k+1-n/p>0$ plays an analogous role to the condition in Theorem 4.4 that $n>p$. Namely, it is required so that the results can be stated in terms of Sobolev spaces, and to apply the multivariate form of Hardy's inequality. Additionally, by Theorem 4.13, the unboundedness of the constant in (5.7) as $k+1-n/p \to 0^+$ is, in the univariate case, not a true reflection of the behaviour of the error.

With the multivariate form of Hardy's inequality in hand, it is also possible to obtain pointwise error bounds for Lagrange maps.

Corollary 5.8. *Suppose that $\bar\Omega$ is starshaped with respect to Θ, and that $L_{P,\Theta}$ is a Lagrange map with $\Pi_k(\mathbb{R}^n) \subset P \subset C^k(\Omega)$. With $f \in W_\infty^{(k+1)} \subset C(\bar\Omega)$, and $x \in \bar\Omega$ we have the (coordinate-independent) pointwise error bound*

$$|f(x) - L_{P,\Theta}f(x)| \le \frac{1}{(k+1)!} |f|_{k+1,\infty,\Omega} \sum_{\theta\in\Theta} \|\theta - x\|^{k+1} |\ell_\theta(x)|, \tag{5.9}$$

and the (coordinate-dependent) pointwise error bound

$$|f(x) - L_{P,\Theta}f(x)| \le \sum_{\theta\in\Theta} \sum_{|\alpha|=k+1} \frac{1}{\alpha!} \|D^\alpha f\|_{L_\infty(\Omega)}\, |(\theta - x)^\alpha \ell_\theta(x)|. \tag{5.10}$$

Proof. The proof runs along the same lines as that for Corollary 5.6, except that for (5.10) we first expand $D_{\theta-x}^{k+1}f$ as

$$D_{\theta-x}^{k+1}f = \sum_{|\alpha|=k+1} \frac{(k+1)!}{\alpha!} (\theta - x)^\alpha D^\alpha f,$$

by using the multinomial identity. $\square$

Neither of (5.9) or (5.10) occurs in the literature. For $f \in C^{k+1}(\Omega)$, they can be obtained more simply, by applying the mean value theorem, as given by Properties 2.3 (c), to the integrals occurring in (5.3).

Remark 5.11. The results of [AG76] have been extended in the following ways. In [Go77], Gout treats the error in certain forms of *Hermite interpolation* – that is where, in addition to function values, certain derivatives are matched at the points in Θ. In [AS84], Arcangeli and Sanchez bound the error in a Lagrange map for functions from *fractional order* Sobolev spaces. $\square$

The error formula of Sauer and Xu

There is another error formula, for the error in a Lagrange map with range (interpolants) $\Pi_k(\mathbb{R}^n)$, that has been given recently by Sauer and Xu, see [SX94].

Sauer and Xu order the $\dim \Pi_k(\mathbb{R}^n)$ points in Θ so that each Lagrange interpolation problem with points Θ^j (by definition the initial segment of Θ consisting of the first $\dim \Pi_j(\mathbb{R}^n)$ terms) and interpolants $\Pi_j(\mathbb{R}^n)$ is correct for $j = 0, \ldots, k$. They consider the collection $\boldsymbol{\Psi}$ of all $(k+1)$-sequences $\Psi = [\psi_0, \ldots, \psi_k]$, called *paths* by them, with $\psi_j \in \Theta^j \backslash \Theta^{j-1}$, all j. Given this notation, Sauer and Xu state their result in the following form.

Theorem 5.12 ([SX94:Th.3.6]). *Suppose that $L_{P,\Theta} := L_{\Pi_k(\mathbb{R}^n),\Theta}$ is a Lagrange map, and $f \in C^{k+1}(\mathbb{R}^n)$. Then*

$$L_{P,\Theta} f(x) - f(x) = \sum_{\Psi \in \boldsymbol{\Psi}} p_\Psi(x) \int_{[x,\Psi]} D_{x-\psi_k} D_{\psi_k - \psi_{k-1}} \cdots D_{\psi_2 - \psi_1} D_{\psi_1 - \psi_0} f, \tag{5.13}$$

where $p_\Psi \in \Pi_k(\mathbb{R}^n)$ is given by

$$p_\Psi(x) := (k+1)!\, \ell_{\psi_k, \Pi_k(\mathbb{R}^n),\Theta}(x) \prod_{i=1}^{k} \ell_{\psi_i, \Pi_i(\mathbb{R}^n),\Theta^i}(\psi_{i+1}).$$

The region of integration in each term of (5.13) is the convex hull of x and Ψ.

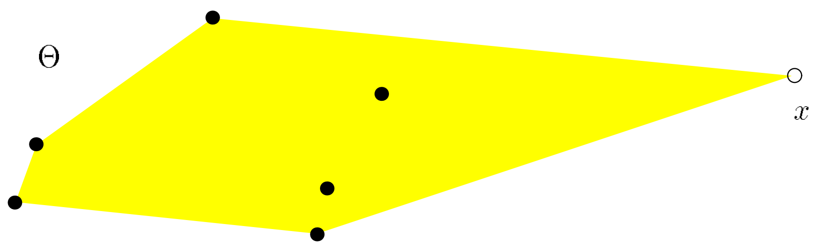

Fig 5.2 The region of integration in (5.13) for Θ consisting of 6 points

From (5.13) the following pointwise estimate is obtained.

Corollary 5.14 ([SX94:Cor.3.11]). *Suppose, in addition to the hypotheses of Theorem 5.12, that $\bar{\Omega}$ is starshaped with respect to Θ. Then for all $x \in \bar{\Omega}$*

$$|f(x) - L_{P,\Theta} f(x)| \le \frac{1}{(k+1)!} \sum_{\Psi \in \mathbf{\Psi}} \|D_{x-\psi_k} D_{\psi_k - \psi_{k-1}} \cdots D_{\psi_2 - \psi_1} D_{\psi_1 - \psi_0} f\|_{L_\infty(\Omega)} |p_\Psi(x)|. \tag{5.15}$$

The bound (5.15) is of a similar form to those of (5.9) and (5.10). For a more direct comparison, one obtains from (5.3) the bound

$$|f(x) - L_{P,\Theta} f(x)| \le \frac{1}{(k+1)!} \sum_{\theta \in \Theta} \|D_{\theta - x}^{k+1} f\|_{L_\infty(\Omega)} |\ell_\theta(x)|. \tag{5.16}$$

This bound has $\#\Theta = \sum_{j=0}^{k} \#\Theta^j$ terms, as opposed to $\#\mathbf{\Psi} = \prod_{j=0}^{k} \#\Theta^j$ for (5.15), and requires no ordering of Θ. For the purposes of comparison, in the bivariate case, i.e., when $n = 2$, one has that $\#\Theta = (k+2)(k+1)/2$, while $\#\mathbf{\Psi} = (k+1)!$. In addition, analogous bounds to (5.16) can be obtained, from (5.3), for the derivatives of the error in $L_{P,\Theta}$.

To obtain L_p-bounds from (5.13) it is necessary to bound

$$x \mapsto L_{1,\Psi} f(x) := \int_{[x,\Psi]} f \tag{5.17}$$

in terms of $\|f\|_{L_p(\Omega)}$. This can be done by using the multivariate form of Hardy's inequality. Thus, we have the following instance of Theorem 3.10.

Corollary 5.18. *Suppose the hypotheses of Corollary 5.14. If $1 - n/p > 0$, then*

$$\|f - L_{P,\Theta} f\|_{L_p(\Omega)} \le \frac{\Gamma(1-n/p)}{\Gamma(k+2-n/p)} \Bigl(\sum_{\Psi \in \mathbf{\Psi}} \|p_\Psi\|_{L_\infty(\Omega)} \Bigr) |f|_{k+1,p,\Omega} (h_{\Omega,\Theta})^{k+1}.$$

The condition $1 - n/p > 0$ is needed so that the multivariate form of Hardy's inequality can be applied to (5.17). By comparison, to obtain (5.7) from (5.4), only the weaker condition that $k + 1 - n/p > 0$ was needed.

6. Other error bounds

All of the integral error formulæ for Lagrange maps given in the literature, including those of Section 5, can be obtained from

$$f(x) - L_{P,\Theta} f(x) = \sum_{\theta \in \Theta} \Bigl(\int_{[x]} f - \int_{[\theta]} f \Bigr) \ell_\theta(x),$$

which is valid whenever P contains the constants, by appropriately using the identity

$$\int_{[\Theta,v]} f - \int_{[\Theta,w]} f = \int_{[\Theta,v,w]} D_{v-w} f, \tag{6.1}$$

and the integration by parts formula.

For example, in Gregory [Gr75] the integration by parts formula is used to give a *Taylor type* expansion for f. From this is obtained an integral error formula for *linear interpolation* on a triangle, i.e., when Θ consists of 3 affinely independent points in $\mathbb{R}^2$, and the interpolants are the linear polynomials $P := \Pi_1(\mathbb{R}^2)$. Such an argument is frequently referred to as a *Sard kernel theory* argument, as developed by Sard [Sa63]. The resulting formula is complicated – it has 4 line integrals and 5 area integrals. Another example is given by Hakopian [H82], who uses (6.1) to obtain an integral error formula for *tensor product* Lagrange interpolation.

In view of their derivations, all of these integral error formulæ involve terms which consist of a function (obtained appropriately from the Lagrange functions) multiplied by the integral of some derivative against a simplex spline. Thus, it is possible to apply the multivariate form of Hardy's inequality to all such formulæ (and those likely to be obtained in the future) to obtain L_p-bounds – with the caution that, as pointed out for the examples in Sections 4 and 5, for small p this may not accurately reflect the behaviour of the error.

Exactly how to use (6.1) and the integration by parts formula to obtain the best possible error formula for a given purpose is far from clear. In a future paper the author considers the simplest case, that of linear interpolation on a triangle. There, the formulæ of Ciarlet and Wagschal [CW71], Gregory [Gr75], Sauer and Xu [SX94], amongst others, are discussed.

References

[Ad75] Adams, R. A. (1975), *Sobolev spaces*, Academic Press (New York).

[Al89] Alfeld, P. (1989), "Scattered data interpolation in three or more variables", in *Mathematical Methods in Computer Aided Geometric Design* (T. Lyche and L. Schumaker, eds), Academic Press (New York), 1–33.

[AG76] Arcangeli, R., and J. L. Gout (1976), "Sur l'evaluation de l'erreur d'interpolation de Lagrange dans un ouvert de $\mathbb{R}^n$", *Rev. Française Automat. Informat. Rech. Opér., Anal. Numer.* **10(3)**, 5–27.

[AS84] Arcangeli, R., and A. M. Sanchez (1984), "Estimations des erreurs de meilleure approximation polynomiale et d'interpolation de Lagrange dans les espaces de Sobolev d'ordre non entier", *Numer. Math.* **45**, 301–321.

[BHS93] Bojanov, B. D., H. A. Hakopian, and A. A. Sahakian (1993), *Spline functions and multivariate interpolations*, Kluwer Academic Publishers.

[B73] Boor, C. de (1973), "The quasi-interpolant as a tool in elementary spline theory", in *Approximation Theory* (G. G. Lorentz *et al.*, eds), Academic Press (New York), 269–276.

[BR92] Boor, C. de, and A. Ron (1992), "The least solution for the polynomial interpolation problem", *Math. Z.* **210**, 347–378.

[BH70] Bramble, J. H., and S. R. Hilbert (1970), "Estimation of linear functionals on Sobolev spaces with applications to Fourier transforms and spline interpolation", *SIAM J. Numer. Anal.* **7(1)**, 112–124.

[CD90] Chen, W., and Z. Ditzian (1990), "Mixed and directional derivatives", *Proc. Amer. Math. Soc.* **108**, 177–185.

[Ci78] Ciarlet, P. G. (1978), *The finite element method for elliptic problems*, North Holland.

[CR72] Ciarlet, P. G., and P. A. Raviart (1972), "General Lagrange and Hermite interpolation in $\mathbb{R}^N$ with applications to finite element methods", *Arch. Rational Mech. Anal.* **46**, 177–199.

[CW71] Ciarlet, P. G., and C. Wagschal (1971), "Multipoint Taylor formulas and applications to the finite element method", *Numer. Math.* **17**, 84–100.

[D79] Dahmen, W. (1979), "Multivariate B-splines - recurrence relations and linear combinations of truncated powers", in *Multivariate Approximation Theory* (W. Schempp and K. Zeller, eds), Birkhäuser (Basel), 64–82.

[DMS92] Dahmen, W., C. A. Micchelli, and H.-P. Seidel (1992), "Blossoming begets B-spline bases built better by B-patches", *Math. Comp.* **59(199)**, 97–115.

[FMP91] Fink, A. M., D. S. Mitrinović, and J. E. Pečarić (1991), *Inequalities involving functions and their integrals and derivatives*, Kluwer Academic Pub..

[Fo84] Folland, G. B. (1984), *Real analysis, modern techniques and their applications*, Wiley.

[Ga88] Gao, J. B. (1988), "Multivariate quasi-Newton interpolation", *J. Math. Res. Exposition* **8(3)**, 447–453.

[Go77] Gout, J. L. (1977), "Estimation de l'erreur d'interpolation d'Hermite dans $\mathbb{R}^n$", *Numer. Math.* **28**, 407–429.

[Gr75] Gregory, J. A. (1975), "Error bounds for linear interpolation on triangles", in *Mathematics of Finite Elements and Applications* (J. Whiteman, ed), Academic Press (London), 163–170.

[H82] Hakopian, H. (1982), "Integral remainder formula of the tensor product interpolation", *Bull. Pol. Acad. Math.* **31(5-8)**, 267–272.

[Ha28] Hardy, G. H. (1928), "Notes on some points in the integral calculus LXIV", *Messenger of Math.* **57**, 12–16.

[HLP67] Hardy, G. H., J. E. Littlewood, and G. Polya (1967), *Inequalities*, Cambridge University Press.

[Jo93] Jones, F. (1993), *Lebesgue integration on Euclidean space*, Jones and Bartlett Pub..

[LW84] Lai, Mingjun, and X. Wang (1984), "A note to the remainder of a multivariate interpolation polynomial", *Approx. Theory Appl.* **1(1)**, 57–63.

[LW86] Lai, Mingjun, and X. Wang (1986), "On multivariate Newtonian interpolation", *Sci. Sinica. Ser. A* **29(1)**, 23–32.

[M79] Dahmen, W. (1979), "On a numerically efficient method for computing multivariate B-splines", in *Multivariate Approximation Theory* (W. Schempp and K. Zeller, eds), Birkhäuser (Basel), 211–248.

[M80] Micchelli, C. A. (1980), "A constructive approach to Kergin interpolation in $\mathbb{R}^k$: multivariate B-splines and Lagrange interpolation", *Rocky Mountain J. Math.* **10**, 485–497.

[Ru87] Rudin, W. (1987), *Real and Complex analysis*, McGraw-Hill.

[Sa63] Sard, A. (1963), *Linear approximation*, AMS.

[SX94] Sauer, T., and Yuan Xu (1994), "On multivariate Lagrange interpolation", ms.

[Sh71] Shum, D. T. (1971), "On integral inequalities related to Hardy's", *Canad. Math. Bull.* **14(2)**, 225–230.

[Wa94] Waldron, S. (1994), "Integral error formulæ for the scale of mean value interpolations which includes Kergin and Hakopian interpolation", submitted to Numer. Math..