Refinable subspaces of a refinable space

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Abstract: Local refinable finitely generated shift-invariant spaces play a significant role in many areas of approximation theory and geometric design. In this paper we present a new approach to the construction of such spaces. We begin with a refinable function $\psi : \mathbb{R} \to \mathbb{R}^m$ which is supported on $[0, 1]$. We are interested in spaces generated by a function $\phi : \mathbb{R} \to \mathbb{R}^n$ built from the shifts of ψ.

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1. Introduction

Local refinable finitely generated shift-invariant spaces naturally arise in the theory of (multi)wavelets, splines, finite-elements, and subdivision schemes. In this paper we introduce and begin to develop a method for constructing and studying such spaces.

Let $L^1_{\rm loc}$ denote the space of all functions $f : \mathbb{R} \to \mathbb{R}$ which belong to $L^1(\mathbb{R})$ locally; that is, $f \in L^1_{\rm loc}$ provided that (f is measurable and) $\int_K |f| < \infty$ for every compact $K \subset \mathbb{R}$. This space is topologized by the family of seminorms

$$|f|_N := \int_{[-N,N]} |f|, \qquad N \in \mathbb{N}.$$

We refer to a (row) vector $\phi = [\phi_1, \dots, \phi_n]$, $n \in \mathbb{N}$, of functions in $L^1_{\rm loc}$ as a *generator*.

A generator $\phi = [\phi_1, \cdots, \phi_n]$ is said to be *refinable* if there exists a finitely supported sequence $b : \mathbb{Z} \to \mathbb{R}^{n\times n}$ (called a *mask* for ϕ) for which

$$\phi = \sum_{j\in\mathbb{Z}} \phi(2\cdot -j) b(j).$$

We begin with a generator $\psi = [\psi_1, \dots, \psi_m]$ supported in $[0, 1]$ that is refinable with a two-term mask:

$$\psi = \psi(2\cdot)a(0) + \psi(2\cdot -1)a(1); \tag{1.1}$$

and we intend to construct more useful (read “smoother”) refinable generators by using the shifts of ψ. That is, we consider generators of the form

$$\phi = \sum_{j\in\mathbb{Z}} \psi(\cdot - j)c(j),$$

for some sequence $c : \mathbb{Z} \to \mathbb{R}^{m\times n}$. The motivation for this approach is that it is much easier to study the properties of ψ since it is supported on $[0, 1]$ hence its shifts do not ‘interfere’ with each other. The crux is that ϕ constructed in this way will not, in general, be refinable.

Let V be a subspace of $L^1_{\rm loc}$. Then we say V is *shift-invariant* if

$$f \in V \Longrightarrow f(\cdot \pm 1) \in V;$$

we say V is a *finitely generated shift-invariant (FSI)* space if

$$V = S(\phi) := \operatorname{clos}_{L^1_{\rm loc}} \operatorname{span}\{\phi_i(\cdot - j) \mid i = 1, \dots, n;\ j \in \mathbb{Z}\}$$

for some generator ϕ; and we say an FSI space V is *local* if $V = S(\phi)$ for some compactly supported generator ϕ. Lastly, we say V is *refinable* if

$$f \in V \Longrightarrow f(\cdot/2) \in V.$$

Evidently, $S(\phi)$ is refinable whenever ϕ is refinable.

The main objective of this paper is:

Given a refinable generator ψ supported in $[0,1]$ with mask $\big(a(0), a(1)\big)$, characterize all local refinable FSI subspaces $V \subset S(\psi)$.

We provide such a characterization in case $a(0)$ is invertible.

We point out that every local refinable FSI space is a subspace of $S(\psi)$ for some refinable generator ψ supported in $[0,1]$. For $f \in L^1_{\text{loc}}$ and $V \subset L^1_{\text{loc}}$, define

$$f_| := f\chi_{[0,1]} \quad \text{and} \quad V_| := \{\, f_| \mid f \in V \,\}.$$

(As usual, χ_A denotes the characteristic function of a set $A \subset \mathbb{R}$.) Suppose V is a local refinable FSI space. Then $m := \dim V_|$ is finite. We refer to m as the *local dimension of* V. Let $\psi = [\psi_1, \ldots, \psi_m]$ be a basis for $V_|$. Then ψ is refinable and $V \subset S(\psi)$.

This has been observed and exploited already by Jia in **[7]**, **[8]**, and **[9]**, where the author studied a given function ϕ via a basis ψ for $S(\phi)_|$. The simpler structure due to the small supports of ψ and a in equation (1.1) has also been recognized by Micchelli et al. in, for example, **[11]**, **[12]**, **[13]**, and **[14]**. In particular, given a univariate refinable function ϕ with finite mask b, they define $a(0)$ and $a(1)$ by $a(\varepsilon) := [b(\varepsilon + 2j - i)]_{i,j}$ and study ϕ via the refinable function having mask a. Among other things, this was used to provide necessary and sufficient conditions for the convergence of a given subdivision scheme and, in **[14]**, to provide a fairly thorough study of regularity for refinable function vectors.

Our approach is different in that the mask a and generator ψ come first. In this paper, we identify all local refinable FSI subspaces $S(\phi)$ of $S(\psi)$. The next steps are to provide further characterizations of the properties of $S(\psi)$ in terms of a; to determine when these properties are preserved by a subspace $S(\phi)$; and to put these ideas together to construct desirable refinable generators.

2. Results

Throughout this paper, we assume that $\big(a(0), a(1)\big)$ is a mask for a refinable generator $\psi = [\psi_1, \ldots, \psi_m]$ supported in $[0,1]$ (in particular, each ψ_j is assumed to be in $L^1(\mathbb{R})$). We will show that when $a(0)$ is invertible, each local refinable FSI subspace of $S(\psi)$ corresponds to some $a(0)$-invariant space (for a matrix $a \in \mathbb{R}^{k\times k}$, a space $\Gamma \subset \mathbb{R}^k$ is *a-invariant* if $a\Gamma \subset \Gamma$). Our specific statements will require a few more definitions.

We use $\mathbb{Z}_+$ to denote the set of non-negative integers and $\mathbb{R}^k$ to denote the set of column vectors of length k. For any set $V \subset L^1_{\text{loc}}$, define

$$V^+ := \{\, f \in V \mid \operatorname{supp} f \subset [0, \infty) \,\};$$

and, for $V \subset S(\psi)$, define

$$\Sigma(V) := \{\, \sigma \in \mathbb{R}^m \mid \psi\sigma \in V_|^+ \,\}.$$

(By convention, $V_|^+ := (V^+)_|$).

Proposition 2.1. *For any refinable subspace V of $S(\psi)$, $\Sigma(V)$ is $a(0)$-invariant.*

If $\phi = [\phi_1, \ldots, \phi_k]$ is a generator supported in $[0, \infty)$, then the sum

$$\phi *' c := \sum_{j=0}^{\infty} \phi(\cdot - j)c(j)$$

is locally finite for any sequence $c : \mathbb{Z}_+ \to \mathbb{R}^k$. In particular, the set

$$R(\phi) := \{\phi *' c \mid c : \mathbb{Z}_+ \to \mathbb{R}^k\},$$

spanned by the right shifts of ϕ, is a subset of $S(\phi)^+$.

We say a subspace Λ of $\mathbb{R}^m$ is *preserved by* $a(0)$ if $a(0)\Lambda = \Lambda$, and a matrix $\lambda \in \mathbb{R}^{m\times n}$ is *preserved by* $a(0)$ if its columns form a basis for a space that is preserved by $a(0)$. Note that a matrix $\lambda \in \mathbb{R}^{m\times n}$ is preserved by $a(0)$ if and only $a(0)\lambda = \lambda\beta_\lambda$ for a unique invertible $\beta_\lambda \in \mathbb{R}^{n\times n}$. Suppose that $\lambda \in \mathbb{R}^{m\times n}$ is preserved by $a(0)$. Set

$$\ell(0) := \lambda \quad \text{and} \quad \ell(2j+\varepsilon) := a(\varepsilon)\ell(j)\beta_\lambda^{-1} \text{ for } \varepsilon \in \{0,1\},\ 2j+\varepsilon > 0. \tag{2.1}$$

We define the *generalized truncated power* e_λ by

$$e_\lambda := \psi *' \ell.$$

Proposition 2.2. *Suppose that $\lambda \in \mathbb{R}^{m\times n}$ is preserved by $a(0)$. Then*

(i) $e_\lambda = e_\lambda(2\cdot)\beta_\lambda$;

(ii) if $\lambda' \in \mathbb{R}^{m\times n}$ has the same column space as λ, then $S(e_\lambda) = S(e_{\lambda'})$; and

(iii) $S(e_\lambda)$ is a local refinable FSI subspace of $S(\psi)$.

The property (2.2.ii) above allows us to unambiguously define, for any Λ preserved by $a(0)$, the space $S_\Lambda := S(e_\lambda)$ where the columns of λ form a basis for Λ.

Theorem 2.3. *Suppose V is a local refinable FSI subspace of $S(\psi)$. If $\Sigma(V)$ is preserved by $a(0)$ then $V = S_{\Sigma(V)}$.*

If $a(0)$ is invertible then every $a(0)$-invariant subspace is, in fact, preserved by $a(0)$. So we have the following corollary — one of the main results of this paper.

Corollary 2.4. *Suppose $a(0)$ is invertible. Then V is a local refinable FSI subspace of $S(\psi)$ if and only if $V = S_\Lambda$ for some $a(0)$-invariant Λ.*

So, in the case $a(0)$ is invertible, every local refinable FSI subspace of $S(\psi)$ is of the form S_Λ for some $a(0)$-invariant space Λ. The $a(0)$-invariant spaces are easily identified from the Jordan-Canonical form of $a(0)$. By Theorem 2.3, $S_\Lambda = S_{\Sigma(S_\Lambda)}$. So, if $a(0)$ is invertible and ψ *is linearly independent* (meaning the entries of ψ are linearly independent), the local refinable FSI subspaces of $S(\psi)$ are in one-to-one correspondence with those $a(0)$-invariant spaces Λ satisfying $\Lambda = \Sigma(S_\Lambda)$. Our next result provides a characterization of such Λ.

First, define

$$A_0 := \begin{bmatrix} a(1) & 0 \\ 0 & a(0) \end{bmatrix}, \quad A_1 := \begin{bmatrix} 0 & a(0) \\ 0 & a(1) \end{bmatrix}.$$

Then $\mathcal{H}_\Lambda$ is defined to be the minimal subspace of $\mathbb{R}^{2m}$ that contains

$$\begin{bmatrix} 0 \\ \Lambda \end{bmatrix} := \{ \begin{bmatrix} 0 \\ v \end{bmatrix} \mid v \in \Lambda \}$$

and is $\{A_0, A_1\}$-invariant, i.e., A_ε-invariant for $\varepsilon = 0, 1$.

Theorem 2.5. *Suppose Λ is an $a(0)$-invariant subspace of $\mathbb{R}^m$. Let the columns of $\lambda \in \mathbb{R}^{m\times n}$ form a basis for Λ. If $a(0)$ is invertible and ψ is linearly independent, then the following are equivalent.*

(i) $\Lambda = \Sigma(S_\Lambda)$.

(ii) $\Lambda = \Sigma(V)$ *for some local refinable FSI subpace* $V \subset S(\psi)$.

(iii) $S_\Lambda^+ = R(e_\lambda)$

(iv) The set $\mathcal{H}_\Lambda^0 := \{v \in \mathbb{R}^m \mid \begin{bmatrix} 0 \\ v \end{bmatrix} \in \mathcal{H}_\Lambda\}$ *is equal to* Λ.

It is clear that $S_{\Lambda|}$ is always a subset of $\operatorname{span}\{\psi_1, \ldots, \psi_m\}$. We now give a characterization of when these sets are actually equal.

Theorem 2.6. *Suppose ψ is linearly independent. Suppose $\Lambda \subset \mathbb{R}^m$ is preserved by $a(0)$. Define $\mathcal{L}_\Lambda$ to be the minimal $\{a(0), a(1)\}$-invariant subspace of $\mathbb{R}^m$ containing Λ. Then* $S_{\Lambda|} = \operatorname{span}\{\psi_1, \ldots, \psi_m\}$ *if and only if* $\mathcal{L}_\Lambda = \mathbb{R}^m$.

Among the premises of Theorems 2.5 and 2.6 is the statement that ψ is linearly independent. A characterization of this property is provided for completeness.

Define

$$T := a(0) + a(1).$$

Then a necessary condition for the generator ψ to be linearly independent is that 2 be a simple eigenvalue of the matrix T with left eigenvector $\hat{\psi}(0)$ and that all other eigenvalues have modulus strictly less than 2 (cf., e.g., **[2]**, **[3]**, **[10]**). In this case, ψ is the unique (up to constant multiple) generator satisfying Eq. (1.1). With this in mind, we offer the following theorem (**[4]** provides a generalization of this result).

Theorem 2.7. *Let $\mathcal{W}$ be the smallest subspace of $\mathbb{R}^m$ satisfying*

$$\hat{\psi}(0) \in \mathcal{W}, \quad \mathcal{W}a(0) \subset \mathcal{W}, \quad \mathcal{W}a(1) \subset \mathcal{W}.$$

Then the generator ψ is linearly independent if and only if

(i) 2 is a simple eigenvalue of T;

(ii) all other eigenvalues have modulus strictly less than 2; and

(iii) $\mathcal{W} = \mathbb{R}^m$.

3. Proofs

Throughout this section, we write $\phi \subset V$ to mean that the entries of the generator ϕ are elements of V.

We recall some results from **[1]**.

Lemma 3.1. *For any closed shift-invariant space V of finite local dimension, there exists $r > 0$ such that if $f \in V$ vanishes on $[-r, 0]$ then $f_| \in V_|^+$.*

Lemma 3.2. *For any closed shift-invariant space V of finite local dimension, there is a compactly supported generator $\phi = [\phi_1, \ldots, \phi_k] \subset V$ such that $\phi_|$ is basis for $V_|^+$ and $V^+ = R(\phi)$.*

Actually, the topology used in **[1]** is that of uniform convergence on compact sets. However the arguments used there also apply to the topology of L^1_{loc}.

Proof of Proposition 2.1: Suppose $\sigma \in \Sigma(V)$. Then there exists $f \in V^+$ such that $f_| = \psi\sigma$. Since V is refinable, $f(\cdot/2) \in V^+$. But, $f(\cdot/2) = \psi(\cdot/2)\sigma = \psi a(0)\sigma$ on $[0, 1]$. So $a(0)\sigma \in \Sigma(V)$. □

Proof of Proposition 2.2:

(i) $$e_\lambda\left(\frac{\cdot}{2}\right) = \sum_j \psi\left(\frac{\cdot - 2j}{2}\right)\ell(j) = \sum_{j,\varepsilon} \psi(\cdot - 2j - \varepsilon)a(\varepsilon)\ell(j)$$
$$= \sum_{j,\varepsilon} \psi\big(\cdot - (2j + \varepsilon)\big)\ell(2j + \varepsilon)\beta_\lambda = e_\lambda\beta_\lambda.$$

(ii) There exists $\gamma \in \mathbb{R}^{n\times n}$ such that $\lambda = \lambda'\gamma$. Set $\beta := \beta_\lambda$ and $\beta' := \beta_{\lambda'}$. Then

$$\lambda'\gamma\beta = \lambda\beta = a(0)\lambda = a(0)\lambda'\gamma = \lambda'\beta'\gamma.$$

Since the columns of λ' form a basis, $\gamma = \beta'\gamma\beta^{-1}$. Define ℓ by Eq. (2.1) and ℓ' similarly, but with λ' in place of λ. Then, $\ell(0) = \lambda = \lambda'\gamma = \ell'(0)\gamma$. Now, suppose $2j + \varepsilon > 0$ and $\ell(j) = \ell'(j)\gamma$. Then

$$\ell(2j + \varepsilon) = a(\varepsilon)\ell(j)\beta^{-1} = a(\varepsilon)\ell'(j)\gamma\beta^{-1} = \ell'(2j + \varepsilon)\beta'\gamma\beta^{-1} = \ell'(2j + \varepsilon)\gamma.$$

It follows that $e_\lambda = e_{\lambda'}\gamma$.

(iii) Set $V := S(e_\lambda)$ and let ϕ be as guaranteed by Lemma 3.2. Since $\phi \subset V$, we have $S(\phi) \subset S(e_\lambda)$. Conversely, since $e_\lambda \in V^+ = R(\phi) \subset S(\phi)$, we have $S(e_\lambda) \subset S(\phi)$. □

The proof of Theorem 2.3 will require the following lemma.

Lemma 3.3. *Let V be a local FSI space. Suppose $\phi = [\phi_1, \ldots, \phi_n] \subset V^+$ is such that* $\mathrm{span}\{\phi_{1|}, \ldots, \phi_{n|}\} = V_|^+$. *Then $V^+ = S(\phi)^+ = R(\phi)$.*

Proof: Let $f \in V^+$. We recursively construct a sequence $c : \mathbb{Z}_+ \to \mathbb{R}^n$ so that

$$f = f_N := \sum_{j=0}^{N} \phi(\cdot - j)c(j) \text{ on } [0, N].$$

This is the so-called "peeling-off argument" from **[1]**. Since $f \in V^+$ and $\phi_|$ spans $V_|^+$, $f = \phi c(0)$ on $[0,1]$ for some $c(0) \in \mathbb{R}^n$. Now suppose we have $c(0), \ldots, c(N)$ such that $f = f_N$ on $[0, N]$. Then $(f - f_N)(\cdot + N) \in V^+$. So there exists $c(N+1)$ such that $(f - f_N)(\cdot + N) = \phi c(N+1)$ on $[0,1]$. For this value for $c(N+1)$, $f = f_{N+1}$ on $[0, N+1)$. So V^+ is contained in $R(\phi)$ which is a subset of $S(\phi)^+$.

Since $\phi \subset V$ and $S(\phi)$ is the smallest closed shift-invariant space containing ϕ, $S(\phi)$ is a subspace of V. This, in turn, implies that $S(\phi)^+ \subset V^+$. □

Proof of Theorem 2.3: Let the columns of λ form a basis for $\Sigma(V)$. We first show that $V^+ = S(e_\lambda)^+$. By Lemma 3.3, it is sufficient to show that $e_\lambda \subset V^+$ since $e_{\lambda|} = \psi\lambda$, which spans $V_|^+$.

Let $\phi = [\phi_1, \ldots, \phi_n] \subset V^+$ be such that $\phi_|$ is a basis for $V_|^+$. Then $e_{\lambda|} = \phi_| \gamma$ for some $\gamma \in \mathbb{R}^{n \times n}$. Since $e_\lambda = e_\lambda(2\cdot)\beta$, $e_\lambda = e_\lambda(2^{-k}\cdot)\beta^{-k} = \phi(2^{-k}\cdot)\gamma\beta^{-k}$ on $[0, 2^k]$. Since V is refinable, $\phi(2^{-k}\cdot)\gamma\beta^{-k} \subset V^+$. And since $\phi_|$ is a basis for $V_|^+$, it follows that $V^+ = R(\phi)$. So, for each $n \in \mathbb{N}$, there exists a sequence c_k such that

$$e_\lambda = \sum_{j=0}^{2^k} \phi(\cdot - j) c_k(j) \quad \text{on} \quad [0, 2^k].$$

Since $\phi_|$ is a basis, the set $\{\, \phi(\cdot - j)_{|_{[0,2^k]}} \mid j = 0, 1, \ldots, 2^k - 1 \,\}$ is linearly independent. It follows that the sequence

$$c(j) := c_k(j) \text{ for } j \in \mathbb{Z}_+,\ 2^k > j$$

is well-defined and satisfies $e_\lambda = \phi *' c$.

Since V is a local FSI space, it follows that $V = S(\nu)$ for some compactly supported generator ν. Without loss of generality, $\operatorname{supp} \nu \subset [0, \infty)$. Since $V^+ = S(e_\lambda)^+$, we have $\nu \in S(e_\lambda)$ and $e_\lambda \in V$. Thus, $V = S(e_\lambda)$. □

Proof of Theorem 2.5: First note that, since $a(0)$ is invertible, Λ is preserved by $a(0)$. We show that property (i) is equivalent to each of the others.

(i) $\Longrightarrow$ (ii) is obvious. To see that (ii) $\Longrightarrow$ (i), let V be a local refinable FSI subspace of $S(\psi)$ such that $\Lambda = \Sigma(V)$. By Theorem 2.3, $V = S_\Lambda$. So $\Lambda = \Sigma(S_\Lambda)$.

(iii) $\Longrightarrow$ (i) is obvious. To see that (i) $\Longrightarrow$ (iii), by Lemma 3.3, it is enough to point out that $e_{\lambda|} = \psi\lambda$ is a basis for $S_{\Lambda|}^+ = \psi\Lambda$.

To deal with property (iv), we define

$$h(0) := \begin{bmatrix} 0 \\ \lambda \end{bmatrix} \quad \text{and} \quad h(2j + \varepsilon) := A_\varepsilon h(j) \text{ for } \varepsilon \in \{0, 1\},\ 2j + \varepsilon > 0.$$

Then $\mathcal{H}_\Lambda$ is the column space of $[h(0), h(1), h(2), \ldots]$. Also, with $\ell(-1) := 0$ for consistency,

$$h(j) = \begin{bmatrix} \ell(j-1) \\ \ell(j) \end{bmatrix} \text{ for all } j \in \mathbb{Z}_+$$

by Eq. (2.1). It follows that $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H}_\Lambda$ if and only there exists an $f \in S_\Lambda$ which agrees with $u\psi(\cdot+1)+v\psi$ on $[-1,1]$.

We show that (iv) implies (i) by contraposition. Suppose $\Lambda \neq \Sigma(S_\Lambda)$. Then there exists $f \in S_\Lambda^+$ such that $f_| \notin \psi\Lambda$. That is, f agrees with $u\psi(\cdot+1)+v\psi$ on $[-1,1]$ where $u=0$ and $v \notin \Lambda$. It follows from the above remarks that $v \in \mathcal{H}_\Lambda^0 \backslash \Lambda$.

Finally, suppose there is some $v \in \mathcal{H}_\Lambda^0 \backslash \Lambda$. Then there exists $f \in S_\Lambda$ such that f vanishes on $[-1,0]$ and $f_| = \psi\sigma$ for some $\sigma \notin \Lambda$. By Lemma 3.1, there is an $n \in \mathbb{N}$ such that if $g \in S_\Lambda$ vanishes on $[-2^k, 0]$, then $g_| \in S_{\Lambda|}^+$. We show $a(0)^k\sigma \in \Sigma(S_\Lambda)\backslash\Lambda$ for this n. First, note that

$$f(2^{-k}\cdot)_| = \psi(2^{-k}\cdot)_|\sigma = \psi a(0)^k\sigma.$$

Since $f(2^{-k}\cdot)$ vanishes on $[-2^k, 0]$, it follows that $a(0)^k\sigma \in \Sigma(S_\Lambda)$. But, $a(0)^k\sigma$ is not in Λ since $\sigma \notin \Lambda$, Λ is $a(0)$-invariant, and $a(0)$ is invertible. $\square$

Proof of Theorem 2.6: Let the columns of λ form a basis for Λ and recall that $e_\lambda = \psi *' \ell$ where ℓ is given by Eq. (2.1). Then $e_\lambda(\cdot+j)_| = \psi\ell(j)$. Let L be the column space of $[\ell(0), \ell(1), \ell(2), \ldots]$. Then $S(e_\lambda)_| = \operatorname{span}\{\psi_1, \ldots, \psi_m\}$ if and only if $L = \mathbb{R}^m$. We show that $L = \mathcal{L}_\Lambda$.

Clearly $\Lambda \subset L$, since $\lambda = \ell(0)$. Also, by Eq. (2.1) and since β is invertible, $a(\varepsilon)L \subset L$ for $\varepsilon = 0, 1$. So $\mathcal{L}_\Lambda \subset L$.

Now, the columns of $\ell(0) = \lambda$ are obviously in $\mathcal{L}_\Lambda$. And if $\mathcal{L}_\Lambda$ contains the columns of $\ell(m)$ then it must contain the columns of $\ell(2m+\varepsilon)$ for $\varepsilon = 0, 1$. Hence $L \subset \mathcal{L}_\Lambda$. $\square$

Proof of Theorem 2.7: Let the columns of $w \in \mathbb{R}^{k\times m}$ form a basis for $\mathcal{W}$. Then there exists $\tilde{v} \in \mathbb{R}^{1\times k}$ and $\tilde{a}(\varepsilon) \in \mathbb{R}^{k\times k}$ such that $v = \tilde{v}w$ and $wa(\varepsilon) = \tilde{a}(\varepsilon)w$ for $\varepsilon = 0, 1$. With $\tilde{T} := \tilde{a}(0) + \tilde{a}(1)$, it follows that $\tilde{v}\tilde{T} = 2\tilde{v} \neq 0$, since the columns of w are linearly independent and

$$\tilde{v}\tilde{T}w = \tilde{v}wT = vT = v = \tilde{v}w.$$

So there exists a unique $\tilde{\psi} \subset \mathcal{D}'(\mathbb{R})$ supported in $[0,1]$ satisfying

$$\widehat{\tilde{\psi}}(0) = \tilde{v} \quad \text{and} \quad \tilde{\psi} = \tilde{\psi}(2\cdot)\tilde{a}(0) + \tilde{\psi}(2\cdot-1)\tilde{a}(1).$$

Multiplying each of these equations on the right by w, we see that $\widehat{\tilde{\psi}w}(0) = v$ and $\tilde{\psi}w$ satisfies Eq. (1.1). Hence $\tilde{\psi}w = \psi$. It follows that $\sigma \in \mathcal{W}^\perp \Longrightarrow \psi\sigma = 0$.

Now, let the entries of $\tilde{\psi} = [\tilde{\psi}_1, \ldots, \tilde{\psi}_k]$ form a basis for $\operatorname{span}\{\psi_1, \ldots, \psi_m\}$. Then there exists $w \in \mathbb{R}^{k\times m}$ such that $\psi = \tilde{\psi}w$. Let $\operatorname{row} w$ denote the row space of w, that is, $\operatorname{row} w := \{uw \mid u \in \mathbb{R}^{1\times k}\}$. Evidently, $v \in \operatorname{row} w$. We will show that $(\operatorname{row} w)a(\varepsilon) \subset \operatorname{row} w$ for $\varepsilon = 0, 1$. Consequently, $\mathcal{W} \subset \operatorname{row} w$. So $\psi\sigma = 0 \Longrightarrow \sigma \in \mathcal{W}^\perp$.

For any $\sigma \in \mathbb{R}^m$,

$$\tilde{\psi}w\sigma = \psi\sigma = \psi(2\cdot)a(0)\sigma + \psi(2\cdot-1)a(1)\sigma = \tilde{\psi}(2\cdot)wa(0)\sigma + \tilde{\psi}(2\cdot-1)wa(1)\sigma.$$

And, since the entries of $\tilde{\psi}$ are linearly independent, $w\sigma = 0 \Longrightarrow wa(\varepsilon)\sigma = 0$ for $\varepsilon = 0, 1$. Since $\sigma \in \mathbb{R}^m$ was arbitrary, it follows that $(\operatorname{row} w)a(\varepsilon) \subset \operatorname{row} w$ for $\varepsilon = 0, 1$. $\square$

4. Examples

Example 4.1. *In this example, we present all local refinable FSI spaces of piecewise polynomials with integer breakpoints and show that the list is complete.*

For any $r, m \in \mathbb{Z}$ satisfying $-1 \le r < m$, the space $\mathcal{S}_r^m$ of all r times continuously differentiable piecewise polynomials of degree at most m with integer breakpoints is defined by

$$\mathcal{S}_r^m := \{\, f \in C^r(\mathbb{R}) \mid f|_{(j,j+1)} \text{ is polynomial of degree at most } m \text{ for all } j \in \mathbb{Z}\,\}.$$

Note that $\mathcal{S}_r^m = \sum_{j=r+1}^m \mathcal{S}_{j-1}^j$. In fact, we will show that every local refinable shift-invariant subspace of $\mathcal{S}_{-1}^m$ is of the form

$$\sum_{j\in J} \mathcal{S}_{j-1}^j \text{ for some } J \subset \{0,\dots,m\}.$$

In particular, every local refinable shift-invariant subspace of $\mathcal{S}_{-1}^m$ is a sum of refinable PSI spaces. This is not true of shift-invariant spaces in general. For example, the only refinable PSI subspace of the space generated by $\chi_{[0,1)}$ and $\chi_{[0,1/2)}$ is (the proper subspace) $\mathcal{S}_{-1}^0$.

Define $\psi := [\pi_{0|}, \dots, \pi_{m|}]$, where $\pi_j(x) := x^j$. Then the elements of ψ are linearly independent, $\mathcal{S}_{-1}^m = S(\psi)$, and ψ is refinable with mask $a(0) = d$, $a(1) = cd$, where

$$c := \left[\binom{j-1}{i-1}\right]_{i,j=0}^m \quad \text{and} \quad d := \operatorname{diag}(2^{-j})_{j=0}^m.$$

Since $a(0)$ is diagonal with distinct eigenvalues, the eigenvectors are

$$\lambda_0 := [1,0,\dots,0]^T,\ \lambda_1 = [0,1,0,\dots,0]^T,\ \dots,\ \lambda_m = [0,\dots,0,1]^T$$

and the $a(0)$-invariant spaces are $\Lambda_J := \operatorname{span}\{\lambda_j \mid j \in J\}$, $J \subset \{0,\dots,m\}$. It is easy to verify that, for each j, the function e_{λ_j} is the well-known truncated power function

$$e_{\lambda_j} : x \mapsto x_+^j := \big(\max(0,x)\big)^j$$

and $S(e_{\lambda_j}) = \mathcal{S}_{j-1}^j$. It follows that for any $J \subset \{0,\dots,m\}$,

$$S_{\Lambda_J} = \sum_{j\in J} S(e_{\lambda_j}) = \sum_{j\in J} \mathcal{S}_{j-1}^j.$$

Example 4.2. *We consider the case of local dimension $m = 2$ with $a(0)$ invertible in order to illustrate the main results of this paper.*

Let $\psi = [\psi_1, \psi_2]$ be a linearly independent generator supported in $[0,1]$ which is refinable with mask $\big(a(0), a(1)\big)$. Then $S(\psi)$ must contain all constant functions and we can assume, without loss of generality, that $\psi_1 = \chi_{[0,1)}$ (cf. **[3]**, **[5]**, **[6]**). It is also assumed that $a(0)$ is invertible.

First, suppose $a(0)$ is diagonalizable, in which case we may assume (by a change of basis for ψ) that $a(0)$ and $a(1)$ are of the form

$$a(0) = \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix}, \qquad a(1) = \begin{bmatrix} 1 & u \\ 0 & t \end{bmatrix},$$

where $s \neq 0$, since $a(0)$ is invertible. Then $T = a(0) + a(1) = \begin{bmatrix} 2 & u \\ 0 & s+t \end{bmatrix}$.

Since ψ is linearly independent, Theorem 2.7 implies $s + t < 2$. Then the left 2-eigenspace of T is spanned by $[2-s-t, u]$. If $u = 0$ then the invariant space $\mathcal{W}$ is spanned by $[1, 0]$; and if $s = 1$, then $\mathcal{W}$ is spanned by $[1-t, 1]$. In either case, Theorem 2.7 implies that ψ is linearly dependent. So $S(\psi) = S(\psi_1)$ which has no proper local refinable FSI subspaces. So we assume $s \neq 0$, $s \neq 1$, and $u \neq 0$. By rescaling ψ_2, we may assume $u = 1$.

There are three possible choices for an $a(0)$-invariant space Λ:

1. $\Lambda := \mathrm{span}\{\lambda := [1,0]^T\}$. Then $e_\lambda = \chi_{[0,\infty)}$ and $S_\Lambda = S(\psi_1)$ is the space of piecewise constant polynomials with integer breakpoints.
2. $\Lambda := \mathbb{R}^2$. Then $S_\Lambda = S(\psi)$.
3. (The interesting case) $\Lambda := \mathrm{span}\{\lambda := [0,1]^T\}$. Calculating $h(0) = [0,0,0,1]^T$, $h(1) = A_1 h(0)$, $h(2) = A_0 h(1)$, and $h(3) = A_1 h(1)$, we find that the span of $h(0), \ldots, h(3)$ is $\{A_0, A_1\}$-invariant and so equals $\mathcal{H}_\Lambda$. By a simple reduction, we find that $\mathcal{H}_\Lambda$ is also spanned by the four vectors

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ s+t-1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ s \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Hence $\mathcal{H}^0_\Lambda$ is $\mathrm{span}\{[0,1]^T, [s+t-1, 0]^T\}$. By Theorem 2.5, Λ is a proper subset of $\Sigma(S_\Lambda)$ whenever $s + t \neq 1$. It follows that $S(\psi)$ and $S(\psi_1)$ are the only local refinable FSI subspaces of $S(\psi)$ when $s + t \neq 1$; but, when $s + t = 1$, there is a third local refinable FSI subspace, $S(e_\lambda)$. Lastly, since $a(1)\lambda = [1, t]^T$, we see that $\mathcal{L}_\Lambda = \mathbb{R}^2$ and so, by Theorem 2.6, $S_{\Lambda|} = \mathrm{span}\{\psi_1, \psi_2\}$ for any values of s and t.

When $a(0)$ is not diagonalizable, we may assume (by a change of basis for ψ_2) that

$$a(0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The only choices for Λ, in this case, are $\Lambda = \mathrm{span}\{[1,0]^T\}$, and $\Lambda = \mathbb{R}^2$. So the only local refinable FSI spaces are $S(\psi_1)$ (which is the space of all piecewise constant polynomials with integer breakpoints) and $S(\psi)$.

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