ON MULTIVARIATE APPROXIMATION BY INTEGER TRANSLATES OF A BASIS FUNCTION

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Abstract

Approximation properties of the dilations of the integer translates of a smooth function, with some derivatives vanishing at infinity, are studied. The results apply to fundamental solutions of homogeneous elliptic operators and to "shifted" fundamental solutions of the iterated Laplacian.

Following the approach from spline theory, the question of polynomial reproduction by quasi-interpolation is addressed first. The analysis makes an essential use of the structure of the generalized Fourier transform of the basis function.

In contrast with spline theory, polynomial reproduction is not sufficient for the derivation of exact order of convergence by dilated quasi-interpolants. These convergence orders are established by a careful and quite involved examination of the decay rates of the basis function. Furthermore, it is shown that the same approximation orders are obtained with quasi-interpolants defined on a bounded domain.

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1. Introduction

The basic model in multivariate splines on regular grids consists of a compactly supported function $\phi : \mathbb{R}^d \to \mathbb{R}$ (or $\mathbb{C}$) and the space T_ϕ spanned by its integer translates. To establish the approximation order (for smooth functions) of the dilations of the space T_ϕ, one identifies first the maximal k for which

$$\pi_{k-1} \subset T_\phi,$$

with π_{k-1} the space of all polynomials of degree $\le k-1$. With the aid of a suitable quasi-interpolant scheme this k is then proved to characterize the approximation properties of T_ϕ. Here, a **quasi-interpolant** is a bounded linear operator Q_ϕ from $C^\infty(\mathbb{R}^d)$ (or another space of smooth functions) into T_ϕ of the form

$$Q_\phi f = \sum_{\alpha \in \mathbb{Z}^d} \lambda(f(\cdot + \alpha))\phi(\cdot - \alpha),$$

with λ being a bounded linear functional of compact support such that Q_ϕ **reproduces** π_{k-1} i.e.,

$$Q_\phi p = p, \ \forall p \in \pi_{k-1}.$$

In case the linear functional is based only on point-evaluations from $\mathbb{Z}^d$, the quasi-interpolant can also be written in the form

$$Q_\phi f = \sum_{\alpha \in \mathbb{Z}^d} f(\alpha)\psi(\cdot - \alpha),$$

where ψ is a certain compactly supported element in T_ϕ obtained by applying a finite difference operator to ϕ.

Although a discussion concerning polynomial reproduction (for univariate splines) already appeared in Schoenberg's paper [S], the characterization of the approximation order of the dilations of T_ϕ in terms of polynomial reproduction was first established in [SF] and therefore is usually referred to as "The Strang-Fix Conditions". The introduction of box splines in the early 80's renewed the interest in this issue by several authors, resulting in various refinements and extensions of these conditions (cf. [DM1], [DM2], [BJ], [B1] and the survey [B2]).

In this paper we investigate similar questions for *smooth* functions ϕ which are neither of compact support nor even vanish at infinity. Rather though the common property to all the functions examined here is that some derivatives of them vanish at infinity. More precisely, there exists a polynomial p such that $p(D)\phi$ vanishes at infinity more rapidly than some inverse power $\|x\|^{-k}$, with $p(D)$ the linear differential operator induced by p. To simplify the analysis we assume also that $p(D)$ is either an elliptic operator or is close in some sense to an elliptic operator, an assumption which allows us to deduce that $\widehat{\phi}$, the Fourier transform of ϕ, is a well defined smooth function away of the origin. To resolve the singularity of ϕ at the origin, we approximate p by a trigonometric polynomial, i.e., approximate $p(D)$ by a difference operator ∇. The function $\psi := \nabla\phi$ is then shown to have an algebraic decrease at infinity. Such construction was first suggested in [DL], and applications of this idea to the computation of scattered data interpolant by certain radial functions was carried out in [DLR]. The decrease of ψ makes available the quasi-interpolant

$$Q_\psi f = \sum_{\alpha \in \mathbb{Z}^d} f(\alpha)\psi(\cdot - \alpha),$$

which is well-defined with respect to functions of sufficient slow growth at infinity, where the sum is calculated in the topology of uniform convergence on compact sets.

This construct allows us to discuss questions of polynomial reproduction and approxmiation order from the space $\overline{T}_\phi :=$ the completion in the topology of uniform convergence on compact sets of T_ϕ, the latter consists of all finite linear combinations of the integer translates of ϕ. Indeed, using the expansion of $1/\widehat{\phi}$ around the origin, we identify in section 2 the space of polynomials that can be reproduced by the above technique.

Results on polynomial reproduction can be used to obtain approximation rates by scales of Q_ψ, hence to provide **lower bounds** for the approximation order from the dilations

$$\overline{T}_{\phi,h} := \left\{ f(h^{-1}\cdot) \mid f \in \overline{T}_\phi \right\}$$

of $\overline{T}_\phi$. For ψ which decays fast enough at infinity it is easy to get such rates. In the following (and elsewhere in the paper) A stands for a constant which may vary from one equation to the other.

Proposition 1.1. *Assume that for $j = 1, ..., d$ the series $\sum_{\alpha\in\mathbb{Z}^d} |x_j - \alpha_j|^{\ell+1}|\psi(x-\alpha)|$ is uniformely bounded (in x). Define*

$$Q_{\psi,h} : f \mapsto \sum_{\alpha\in\mathbb{Z}^d} f(h\alpha)\psi(h^{-1}\cdot -\alpha).$$

If $Q_\psi p = p$ for all $p \in \pi_\ell$, then for every f whose derivatives of order $\ell + 1$ are bounded we have

$$\|Q_{\psi,h}f - f\|_\infty \le A\|f\|_{\infty,\ell+1}h^{\ell+1},$$

where $\|f\|_{\infty,\ell+1} := \sum_{|\alpha|=\ell+1} \|D^\alpha f\|_\infty < \infty$.

Proof: By the standard quasi-interpolation argument, we may assume that for any fixed x the Taylor expansion of f around x up to degree ℓ is trivial. Since the $(\ell + 1)$-order derivatives of f are uniformely bounded, then

$$|f(z)| \le A\|f\|_{\infty,\ell+1}\|z - x\|_\infty^{\ell+1}. \tag{1.1}$$

Therefore

$$\begin{aligned}|Q_{\psi,h}f(x)| \le& A\|f\|_{\infty,\ell+1} \sum_{\alpha\in\mathbb{Z}^d} \|h\alpha - x\|_\infty^{\ell+1}|\psi(h^{-1}x - \alpha)|\\ \le& Ah^{\ell+1}\|f\|_{\infty,\ell+1} \sum_{j=1}^{d} \sum_{\alpha\in\mathbb{Z}^d} |h^{-1}x_j - \alpha_j|^{\ell+1}|\psi(h^{-1}x - \alpha)|. \qquad \spadesuit\end{aligned}$$

The following corollary provides an approximation order for ψ of sufficient fast decrease at infinity.

Corollary 1.2. *Assume that*

$$|\psi(x)| \le A(1 + \|x\|_\infty)^{-(d+k)}, \tag{1.2}$$

for $k > \ell + 1$. If $Q_\psi p = p$ for all $p \in \pi_\ell$, then, for every function f whose derivatives of order $\ell + 1$ are bounded

$$\|Q_{\psi,h}f - f\|_\infty \le A\|f\|_{\infty,\ell+1}h^{\ell+1}, \tag{1.3}$$

where A depends on ψ but not on h and f.

Proof: By (1.2) we have

$$|x_j - \alpha_j|^{\ell+1}|\psi(x-\alpha)| \le A(1+\|x-\alpha\|_\infty)^{-(d+\varepsilon)},\ j=1,...,d,\ \varepsilon > 0\ . \tag{1.4}$$

The rest follows from Proposition 1.1. ♠

In section 4 we study the case when ψ satisfies the weaker requirement

$$|\psi(x)| \le A(1+\|x\|_\infty)^{-(d+\ell+1)}. \tag{1.5}$$

While the above corollary shows that under (1.5) one obtains approximation order $O(h^\ell)$ for all functions whose ℓ'th order derivatives are bounded, we prove that if also the $(\ell+1)$th order derivatives of f are bounded then the approximation order is at least $O(h^{\ell+1}|\log h|)$, and under more restricitive assumptions on f and ψ, one may get approximation order $O(h^{\ell+1})$. This extends the result of [Bu] concerning univariate multiquadrics and the results of [J1] (see also [J2]) where the rate $O(h^{\ell+1}|\log h|)$ is obtained for a restricted family of radial functions ϕ, and the rate $O(h^{\ell+1})$ is established for $\phi(x) = \|x\|_2$ in an odd dimension d.

In section 3 we show that the model investigated here includes the fundamental solutions of homogeneous elliptic operators and also most of the examples of radial basis functions now in the literature. Finally, we show in section 5 how the convergence rates of section 4 can be obtained for functions defined only on a bounded domain.

Throughout this paper we use the multi-index notation. For $\alpha, \beta \in \mathbb{R}^d$, $\alpha^2 = \|\alpha\|^2 = \sum_{k=1}^{d} \alpha_k^2$, $\alpha\cdot\beta = \sum_{k=1}^{d}\alpha_k\beta_k$, $\alpha^\beta = \prod_{k=1}^{d}\alpha_k^{\beta_k}$, and for $\alpha \in \mathbb{Z}_+^d := \{\gamma \in \mathbb{Z}^d \mid \gamma_k \ge 0\}$, $\alpha! = \prod_{k=1}^{d}\alpha_k!$, $|\alpha| = \sum_{k=1}^{d}\alpha_k$, $D^\alpha = \prod_{k=1}^{d}\frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}}$ and $f^{(\alpha)} = D^\alpha f$. Also $\pi_m = \mathrm{span}\{x^\alpha \mid |\alpha| \le m\}$, while π stands for the space of all d-dimensional polynomials.

2. Polynomial reproduction

In this section we identify the polynomials which are spanned by the integer translates of a function $\phi : \mathbb{R}^d \to \mathbb{R}$, in terms of properties of the generalized Fourier transform $\widehat{\phi}$ of ϕ. The assumptions on ϕ which are made below, are tailored to provide a unified analysis for fundamental solutions of homogeneous eliptic operators on the one hand, and certain families of radial functions on the other hand.

Assume the Fourier transform $\widehat{\phi}$ of the given function ϕ (treated as a tempered distribution, cf. [GS]) satisfies the equation

$$G\widehat{\phi} = F\ , \tag{2.1}$$

where the functions $F \in C^{m_0}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d \backslash 0)$ and $G \in C^\infty(\mathbb{R}^d)$ satisfy

(a) $G(w) \ne 0$ if $w \ne 0$,

$$
\begin{aligned}
&\text{(b)} \quad G^{(\alpha)}(0) = 0 \quad , \quad |\alpha| < m \ , \\
&\text{(c)} \quad G_m(w) := \sum_{|\alpha|=m} G^{(\alpha)}(0) \frac{w^\alpha}{\alpha!} \not\equiv 0 \ , \\
&\text{(d)} \quad F(0) \neq 0 \ , \quad F(x) - \sum_{|\alpha| \le m_0} \frac{F^{(\alpha)}(0)}{\alpha!} x^\alpha \in \mathcal{F}_{m_0+\theta} \quad \text{for some} \quad \theta > 0, \\
&\text{(e)} \quad |(F/G)^{(\alpha)}(w)| \le \frac{A_\alpha}{\|w\|^{d+\varepsilon}} \quad \text{for} \quad \|w\| \ge 1 \ , \ \varepsilon > 0 \ , \ \alpha \in \mathbb{Z}_+^d \ ,
\end{aligned}
\tag{2.2}
$$

where in (d) we have used the notation

$$
\mathcal{F}_r = \left\{ f \in C^\infty(\mathbb{R}^d \backslash 0) \mid f^{(\alpha)}(x) = O(\|x\|^{r-|\alpha|}), \text{ as } x \to 0, \ \alpha \in \mathbb{Z}_+^d \right\}.
$$

Conditions (b) and (c) are simply specifying m as the order of the zero G has at the origin, while conditions (d) and (e) are of technical nature. The crucial assumption above is (a), which implies that $\widehat{\phi}$ coincides away from the origin with the function F/G. We also assume that the order of $\widehat{\phi}$ (as a distribution) does not exceed $m-1$, namely that $\widehat{\phi}$ is a well defined linear functional on S_{m-1} given by

$$
\widehat{\phi}(s) = \int_{\mathbb{R}^d} s(w) \frac{F(w)}{G(w)} dw \ , \qquad s \in S_{m-1} \ , \tag{2.3}
$$

where

$$
S_{m-1} = \{ s \in S \mid s^{(\alpha)}(0) = 0 \ , \ |\alpha| \le m-1 \} \ ,
$$

and S is the space of C^∞ rapidly decreasing test functions.

To understand the nature of solutions of (2.1), we prove

Lemma 2.1. *Let ϕ satisfy (2.1) and (2.3). Then any other ϕ_1 that satisfies $G\widehat{\phi}_1 = F$ differs from ϕ by a polynomial. Precisely,*

$$
\widehat{\phi}_1(s) = \widehat{\phi}(s) + \big[p(D)s\big](0) \ , \qquad s \in S \ , \tag{2.4}
$$

where

$$
p \in \left\{ p \in \pi \;\middle|\; \big[p^{(\alpha)}(D)G\big](0) = 0 \ , \ \alpha \in \mathbb{Z}_+^d \right\} = \ker G(D)\big|_\pi \ . \tag{2.5}
$$

If in addition ϕ_1 satisfies (2.3) then $p \in \pi_{m-1}$.

Proof: The equation $G(\widehat{\phi}_1 - \widehat{\phi}) = 0$ is equivalent by definition to

$$
G(\widehat{\phi}_1 - \widehat{\phi})(s) = 0 \ , \qquad s \in S \ .
$$

Since $G(w) \neq 0$ for $w \neq 0$, the support of $\widehat{\phi}_1 - \widehat{\phi}$ is the origin, hence there exists $p \in \pi$ such that (2.4) holds. Furthermore,

$$
[p(D)(Gs)](0) = (\widehat{\phi}_1 - \widehat{\phi})(Gs) = G(\widehat{\phi}_1 - \widehat{\phi})(s) = 0 \ , \qquad \forall s \in S \ ,
$$

an equality which is equivalent to

$$\left[p^{(\alpha)}(D)G\right](0)=0, \quad \alpha\in\mathbb{Z}_+^d\ ,$$

and thus, indeed, p satisfies (2.5). If in addition ϕ_1 satisfies (2.3), then by (2.4) $p(D)s(0)=0$, $s\in S_{m-1}$ showing that $p\in\pi_{m-1}$. The second equality in (2.5) is a straightforward extension of Theorem 3.3. ♠

To resolve the singularity of $\widehat{\phi}$ we multiply it by a trigonometric polynomial with a high order zero at the origin, i.e., apply a finite difference operator ∇ supported on $I\subset\mathbb{Z}^d$ to ϕ, so that the resulting function ψ is in span $\{\phi(x-\alpha)\mid \alpha\in I\}$.

Lemma 2.2. *Assume F,G satisfy (2.2). Then $G/F\in C^{m+m_0}$ in a neighborhood of the origin and*

$$(G/F)^{(\alpha)}(0)=0\ ,\qquad |\alpha|<m\ . \tag{2.6}$$

Let

$$e(w)=\sum_{\alpha\in I}a_\alpha e^{-i\alpha\cdot w} \tag{2.7}$$

be a trigonometric polynomial that satisfies

$$(e-G/F)^{(\alpha)}(0)=0\ ,\qquad |\alpha|\le m+\ell\ , \tag{2.8}$$

for some integer $0\le\ell\le m_0$. Then the Fourier transform of

$$\psi:=\sum_{\alpha\in I}a_\alpha\phi(\cdot-\alpha) \tag{2.9}$$

is the **function**

$$\widehat{\psi}=e\frac{F}{G}. \tag{2.10}$$

Moreover, $\widehat{\psi}\in C^\ell(\mathbb{R}^d)$ and satisfies

$$\widehat{\psi}(0)=1\ ,\quad \widehat{\psi}^{(\alpha)}(0)=0\ ,\quad 1\le|\alpha|\le\ell\ , \tag{2.11}$$

$$p(-iD)\widehat{\psi}(2\pi\alpha)=0\ ,\quad \alpha\in\mathbb{Z}^d\backslash 0,\qquad p\in\mathcal{P}_G\cap\pi_{\ell+m}\ , \tag{2.12}$$

where

$$\mathcal{P}_G:=\left\{p\in\pi\mid \left(p^{(\alpha)}(-iD)G\right)(0)=0\ ,\quad \alpha\in\mathbb{Z}_+^d\right\}\ . \tag{2.13}$$

Proof: By (2.2) near the origin

$$\frac{G}{F}(w)=F(0)^{-1}G_m(w)+\sum_{j=m+1}^{m+m_0}q_j(w)+\widetilde{q}(w) \tag{2.14}$$

where q_j is a homogeneous polynomial of degree j and $\widetilde{q}\in\mathcal{F}_{m+m_0+\theta}$. Hence $\frac{G}{F}\in C^{m+m_0}$ in a neighborhood of the origin and (2.6) holds.

From the definition (2.9) of ψ it follows that

$$\widehat{\psi} = e\widehat{\phi} \ . \tag{2.15}$$

To establish (2.10), we note that by (2.6) and (2.8), $es \in S_{m-1}$ for any $s \in S$. Thus (2.15) together with (2.3) yields the equality

$$\widehat{\psi}(s) = \widehat{\phi}(es) = \int\limits_{\mathbb{R}^d} \frac{F}{G} es \ , \qquad s \in S \ ,$$

which is equivalent to (2.10). The structure of the zeros of $\widehat{\psi}$ at the origin is obtained from (2.10) by observing that near the origin

$$\widehat{\psi}(w) = \frac{F(w)}{G(w)} \left[e(w) - \frac{G(w)}{F(w)} \right] + 1$$

with

$$\frac{F(w)}{G(w)} = \sum_{j=-m}^{m_0-m} \eta_j(w) + \widetilde{\eta}(w) \ ,$$

where by (2.2) $\widetilde{\eta} \in \mathcal{F}_{m_0-m+\theta}$, and η_j is a homogeneous function of order j in w, namely $\eta_j(tw) = t^j \eta_j(w)$, $t \in \mathbb{R}\backslash 0$. Thus, in view of (2.8) and (2.14)

$$\widehat{\psi}(w) = 1 + \sum_{j=\ell+1}^{m_0} h_j(w) + \widetilde{h}(w) \tag{2.16}$$

with $\widetilde{h} \in \mathcal{F}_{m_0+\theta}$ and h_j a homogeneous function of order j. Since $h_j^{(\alpha)}(0) = 0$, $0 \le |\alpha| < j$, we conclude that $\widehat{\psi} \in C^\ell(\mathbb{R}^d)$ and satisfies (2.11).

Finally, since for every smooth enough f and every polynomial $p \in \mathcal{P}_G$,

$$p(-iD)(fG)(0) = \sum_{\nu \ge 0} \left[\frac{(-iD)^\nu}{\nu!} f \right](0) \left[p^{(\nu)}(-iD)G \right](0) = 0 \ ,$$

we see that

$$\mathcal{P}_G \cap \pi_{\ell+m} \subset \left\{ p \in \pi \;\middle|\; \left(p^{(\alpha)}(-iD)(G/F) \right)(0) = 0 \ , \ \alpha \in \mathbb{Z}_+^d \right\} \ .$$

Furthermore, $e(x + 2\pi\alpha) = e(x)$, $\alpha \in \mathbb{Z}^d$ while $G(w) \ne 0$ for $w \ne 0$, hence, by (2.8),

$$\left[p(-iD)\widehat{\psi} \right](2\pi\alpha) = \sum_{\nu \ge 0} \left[\frac{(-iD)^\nu}{\nu!} \left(\frac{F}{G} \right) \right] (2\pi\alpha) \left[p^{(\nu)}(-iD)e \right](0) = 0 \ ,$$

for any $p \in \pi_{\ell+m} \cap \mathcal{P}_G$ and $\alpha \in \mathbb{Z}^d \backslash 0$. This completes the proof of (2.12). ♠

Remark 2.3.

A trigonometric polynomial (2.7) satisfying (2.8) can always be constructed, by choosing $I = \{\alpha \mid 0 \le \alpha_i \le m + \ell \ , \ i = 1, \dots, d\}$, since the tensor-product interpolation problem

$$e^{(\alpha)}(0) = b_\alpha \ , \qquad 0 \le \alpha_i \le m + \ell \ , \quad i = 1, \dots, d \ ,$$

with any $\{b_\alpha \mid 0 \le \alpha_i \le \ell + m\}$ admits a unique solution from $\mathrm{span}\{e^{\alpha \cdot x}\}_{\alpha \in I}$.

Remark 2.4.

A set I supports e of the form (2.7) which satisfies (2.8), if I is *total* for $\pi_{\ell+m}$, i.e.,

$$p \in \pi_{\ell+m} \ , \ p_{|_I} = 0 \ \Rightarrow \ p = 0 \ . \tag{2.17}$$

In case $|I| = \dim \pi_{\ell+m}$, conditions for I satisfying (2.17) can be found in [GM], see also the general discussion in [BR]. For specific G, F, however, smaller sets I may be available (cf. [J2], [R2]).

The next two lemmas deal with the rate of decay of the function ψ, obtained from properties of $\widehat{\psi}$. This is a crucial step in the identification of the polynomials spanned by the integer translates of ψ.

Lemma 2.5. *Under the conditions of Lemma 2.2 for* $0 \le \ell \le m_0$

$$|\psi(x)| = o(\|x\|^{-d-\ell}) \quad \text{as} \quad \|x\| \to \infty \ . \tag{2.19}$$

Proof: For $0 \le \ell < m_0$ it follows from (2.16) that, for $\alpha \in \mathbb{Z}_+^d$,

$$\widehat{\psi}^{(\alpha)}(w) = O(\|w\|^{\ell+1-|\alpha|}) \quad \text{as} \quad w \to 0 \ ,$$

while for $\ell = m_0$

$$\widehat{\psi}^{(\alpha)}(w) = O(\|w\|^{m_0+\theta-|\alpha|}) \quad \text{as} \quad w \to 0 \ .$$

Moreover, by (2.2), (2.10) and the boundedness of all the derivatives of e,

$$\widehat{\psi}^{(\alpha)} \in L_1(\mathbb{R}^d \backslash B_\varepsilon) \ , \qquad \alpha \in \mathbb{Z}_+^d \ , \quad \varepsilon > 0 \ , \tag{2.20}$$

where $B_\varepsilon = \{w \in \mathbb{R}^d \mid \|w\| < \varepsilon\}$. Hence, for $|\alpha| \le \ell + d$, $\widehat{\psi}^{(\alpha)} \in L_1(\mathbb{R}^d)$ which implies, by the Riemann-Lebesgue lemma, that

$$\lim_{\|x\| \to \infty} x^\alpha \psi(x) = 0 \ , \qquad |\alpha| \le \ell + d \ . \qquad \spadesuit$$

For $\ell < m_0$, and under some additional conditions also for $\ell = m_0$, the decay of ψ at infinity is better than in (2.19). For $F \in C^\infty(\mathbb{R}^d)$, we can take $m_0 = \infty$, and therefore the result of the previous lemma holds for any $\ell > 0$.

Lemma 2.6. *Under the conditions of Lemma 2.2, if*

$$\widehat{\psi}(w) = \widehat{f}(w) + \widetilde{f}(w) \ , \tag{2.21}$$

with $\widetilde{f} \in \mathcal{F}_{\ell+1+\theta}$, $\theta > 0$, *and* $f(x) = O(\|x\|^{-d-\ell-1})$ *as* $\|x\| \to \infty$, *then*

$$\psi(x) = O(\|x\|^{-d-\ell-1}) \quad \text{as} \quad \|x\| \to \infty \ . \tag{2.22}$$

Proof: Consider the function

$$\widehat{R}(w) = \widehat{\psi}(w) - \rho(w)\widehat{f}(w)$$

where $\rho \in C^\infty(\mathbb{R}^d)$, $0 \le \rho(w) \le 1$ satisfies, for some $\delta > 0$,

$$\rho(w) = \begin{cases} 1, & \|w\| < \delta/2, \\ 0, & \|w\| > \delta. \end{cases} \tag{2.23}$$

Then by (2.20) and (2.23) $\widehat{R}^{(\alpha)} \in L^1(\mathbb{R}^d \backslash B_{\delta/2})$ for $\alpha \in \mathbb{Z}_+^d$, while for $w \in B_{\delta/2}$

$$\widehat{R}(w) = \widetilde{f}(w) \ \in \mathcal{F}_{\ell+1+\theta}$$

and hence

$$\widehat{R}^{(\alpha)} \in L^1(B_{\delta/2}) \quad \text{for} \quad |\alpha| \le \ell + 1 + d \ .$$

Thus

$$\widehat{R}^{(\alpha)} \in L^1(\mathbb{R}^d) \quad \text{for} \quad |\alpha| \le \ell + 1 + d$$

implying, as in the proof of Lemma 2.5, that

$$R(x) = o(\|x\|^{-d-\ell-1}) \quad \text{as} \quad \|x\| \to \infty \ .$$

To complete the proof of the lemma it remains to show that $\psi(x) - R(x) = (\rho\widehat{f})^\vee(x) = O(\|x\|^{-d-\ell-1})$ as $\|x\| \to \infty$. First, we conclude that $(\rho\widehat{f})^\vee \in C^\infty(\mathbb{R}^d)$ since $\rho\widehat{f}$ is of compact support. Writing $\rho\widehat{f} = \widehat{f} + (\rho - 1)\widehat{f}$, we observe that $(\rho - 1)\widehat{f} \in C^\infty(\mathbb{R}^d)$ and hence $\big((\rho - 1)\widehat{f}\big)^\vee$ is rapidly decaying as $\|x\| \to \infty$. This together with the assumption $f(x) = O(\|x\|^{-d-\ell-1})$ as $\|x\| \to \infty$, completes the proof of the lemma. □

Corollary 2.7. *Under the assumptions of Lemma 2.2, for* $0 \le \ell < m_0$, $\psi(x) = O(\|x\|^{-d-\ell-1})$ *as* $\|x\| \to \infty$.

Proof: By (2.16), for $0 \le \ell < m_0$, the terms $\widehat{f}, \widetilde{f}$ in (2.21) may be chosen as

$$\widehat{f}(w) = h_{\ell+1}(w) \ , \qquad h_{\ell+1}(tw) = t^{\ell+1} h_{\ell+1}(w) \ , \qquad t \in \mathbb{R}, \tag{2.24}$$

and

$$\widetilde{f}(w) = 1 + \sum_{j=\ell+2}^{m_0} h_j(w) + \widetilde{h}(w) \in \mathcal{F}_{\ell+1+\theta} \ .$$

It follows from (2.24) that f is the inverse Fourier transform of a homogeneous function of order $\ell+1$, therefore is homogeneous of order $-\ell-1-d$ away of the origin, and hence $f(x) = O(\|x\|^{-d-\ell-1})$ as $\|x\| \to \infty$. Thus $\widehat{\psi}$ satisfies the requirements of Lemma 2.6 and (2.22) holds. ♠

Remark 2.8.

The observation that $f \in \mathcal{F}_r \Rightarrow x^\alpha f \in \mathcal{F}_{r+|\alpha|}, \alpha \in \mathbb{Z}_+^d$, combined with the same type of arguments as above, proves that under the conditions of Corollary 2.7

$$D^\alpha \psi(x) = O(\|x\|^{-d-\ell-1-|\alpha|}) , \qquad \alpha \in \mathbb{Z}_+^d .$$

We can now state the main result of this section.

Theorem 2.9. *Let $\widehat{\phi}$ satisfy (2.1), (2.2) and (2.3), and let $\widehat{\psi} = e\widehat{\phi}$ where e is a trigonometric polynomial (2.7) satisfying (2.8) with $0 \le \ell < m_0$. Then for $p \in \mathcal{P}_G \cap \pi_\ell$*

$$Q_\psi p := \sum_{\alpha \in \mathbb{Z}^d} p(\alpha)\psi(\cdot - \alpha) = p , \tag{2.25}$$

and the convergence of the infinite sum to p is uniform on compact subsets of $\mathbb{R}^d$.

Proof: For $p \in \mathcal{P}_G \cap \pi_\ell$, consider the function

$$g_x(y) = p(y)\psi(x-y) . \tag{2.26}$$

By Corollary 2.7, $g_x(y) \in L^1(\mathbb{R}^d)$ for any fixed $x \in \mathbb{R}^d$, and also the sum $\sum_{\alpha \in \mathbb{Z}^d} g_x(\cdot + \alpha)$ is uniformly convergent on $\mathbb{R}^d$, since

$$|g_x(y)| = |p(y)\psi(x-y)| = O(\|y\|^\ell \|x-y\|^{-\ell-d-1}) = O(\|y\|^{-d-1}) \ , \ \text{as } \|y\| \to \infty. \tag{2.27}$$

Hence $\sum_{\alpha \in \mathbb{Z}^d} g_x(y+\alpha)$ defines a continuous function in y which is periodic. Now by (2.26)

$$\begin{aligned} \widehat{g}_x(w) &= e^{-ix\cdot w}\left[p(x-iD)\widehat{\psi}\right](-w) \\ &= e^{-ix\cdot w} \sum_{|\beta| \le \ell} \frac{x^\beta}{\beta!}\left[p^{(\beta)}(-iD)\widehat{\psi}\right](-w) \end{aligned}$$

and in view of (2.16) and (2.2) $\widehat{g}_x \in L^1(\mathbb{R}^d)$.

Now, since $p^{(\beta)} \in \pi_\ell \cap \mathcal{P}_G$ for all $\beta \ge 0$, it follows from (2.11) and (2.12) that

$$\widehat{g}_x(0) = \sum_{|\beta| \le \ell} \frac{x^\beta}{\beta!} p^{(\beta)}(0) = p(x) , \tag{2.28}$$

$$\widehat{g}_x(2\pi\alpha) = 0 \qquad \text{for} \qquad \alpha \in \mathbb{Z}^d \backslash 0 . \tag{2.29}$$

Hence $\sum_{\alpha \in \mathbb{Z}^d} \widehat{g}_x(2\pi\alpha)$ is absolutely convergent and the Poisson Summation Formula ([SW], see also [J2]) may be invoked to yield

$$\sum_{\alpha \in \mathbb{Z}^d} p(\alpha)\psi(x-\alpha) = \sum_{\alpha \in \mathbb{Z}^d} g_x(\alpha) = \sum_{\alpha \in \mathbb{Z}^d} \widehat{g}_x(2\pi\alpha) = p(x). \tag{2.30}$$

Moreover, the convergence in (2.30) is uniform in x on compact sets of $\mathbb{R}^d$, as is implied by (2.27) with $y = \alpha$. ♠

Corollary 2.10. *Under the conditions of Theorem 2.9*

$$\mathcal{P}_G \cap \pi_{m_0-1} \subset \overline{T}_\phi \ , \qquad T_\phi = \mathrm{span}\{\, \phi(x-\alpha) \mid \alpha \in \mathbb{Z}^d \} \ , \tag{2.31}$$

where the closure is taken in the topology of uniform convergence on compact sets.

It is clear from the proof of Theorem 2.9 and Lemmas 2.5, 2.6 that

Corollary 2.11. *Let ψ be as in Theorem 2.9 but with $\ell = m_0$. Then (2.25) holds for $p \in \mathcal{P}_G \cap \pi_{m_0-1}$. Moreover, if ψ satisfies the conditions of Lemma 2.6 then (2.25) holds for $p \in \mathcal{P}_G \cap \pi_{m_0}$ and*

$$\mathcal{P}_G \cap \pi_{m_0} \subset \overline{T}_\phi \tag{2.32}$$

Remark 2.12.

By (2.2) and (2.13),

$$\mathcal{P}_G \cap \pi_\ell = \pi_\ell \Longleftrightarrow \ell \le m-1 \ . \tag{2.33}$$

Thus for $\ell \le m-1$ the spaces of polynomials reproduced by sums of the type (2.25) are of total degree. The value $\ell = \min(m-1, m_0)$ is the maximal ℓ such that π_ℓ is reproduced by quasi-interpolation operators of the form (2.25), and hence determines the maximal rate of approximation by the scaled versions of Q_ψ

$$Q_{\psi,h} f(x) = \sum_{\alpha \in h\mathbb{Z}^d} f(\alpha) \psi\big(h^{-1}(x-\alpha)\big) \tag{2.34}$$

(see sections 4 and 5).

It should be noted, in view of Remarks 2.3 and 2.4, that as ℓ increases the support I of the difference operator defining $\psi \in T_\phi$ in (2.9) is likely to increase.

3. Applications

In this section we discuss two classes of functions that satisfy the assumptions in (2.1) and (2.2): fundamental solutions of homogeneous elliptic operators, and "shifts" of the fundamental solutions of the iterated Laplacian. In both cases, we employ the results of the previous section for identification of polynomials in the associated $\overline{T}_\phi$.

Roughly speaking, the difference between these two cases can be summarized as follows: in the case of a fundamental solution ϕ of an elliptic operator $P(D)$, the distribution $P(D)\phi$ is of compact support (in effect, the Dirac distribution). Finite difference operators can then "imitate" $P(D)$ to the extent that the resulting $\nabla\phi$ decreases to 0 at infinity at any desirable algebraic rate. In the case of the "shifted" fundamental solutions of the iterated Laplacian, the associated differential operator when applied to ϕ, yields a function with only algebraic decrease at infinity. Analogously, the associated $\nabla\phi$'s decrease at infinity no faster then $P(D)\phi$. This observation will lead in the next section to a saturation result concerning the approximation order by quasi-interpolants which uses these "shifted" functions.

I. Fundamental solutions of Elliptic Operators.

Let $P_m(x) = \sum\limits_{|\alpha|=m} a_\alpha x^\alpha$, $m > d$, m even, satisfy

$$P_m(x) \neq 0 \quad , \quad x \neq 0 \ . \tag{3.1}$$

We take ϕ to be a fundamnetal solution of the operator $(-1)^{m/2}P_m(D)$, namely a solution of the equation

$$(-1)^{m/2}P_m(D)\phi = P_m(-iD)\phi = \delta \ .$$

The equivalent definition of ϕ, in terms of the generalized Fourier transform $\widehat{\phi}$ of ϕ, reads as

$$P_m\widehat{\phi} = 1 \ . \tag{3.2}$$

It is clear that the pair $G := P_m$ and $F := 1$ satisfies (2.2) with $m_0 = \infty$. Among all solutions $\widehat{\phi}$ of (3.2) we choose $\widehat{\phi}$ as the distribution

$$\widehat{\phi}(s) = \int\limits_B \frac{1}{P_m(w)} \left(s(w) - \sum_{|\alpha|<m} \frac{s^{(\alpha)}(0)}{\alpha!} w^\alpha \right) dw + \int\limits_{\mathbb{R}^d \setminus B} \frac{s(w)}{P_m(w)} dw \ , \qquad s \in S \ , \tag{3.3}$$

where $B = \{w \in \mathbb{R}^d \mid \|w\| \leq 1\}$.

Lemma 3.1. *The distribution $\widehat{\phi}$ defined by (3.3) satisfies (2.3) and is a solution of equation (3.2).*

Proof: Since $s^{(\alpha)}(0) = 0$, $|\alpha| < m$ for $s \in S_{m-1}$ then by (3.3)

$$\widehat{\phi}(s) = \int\limits_{\mathbb{R}^d} \frac{1}{P_m(w)} s(w)\, dw \ , \qquad s \in S_{m-1} \ . \tag{3.4}$$

Now, for any $s \in S$, $P_m s \in S_{m-1}$, hence by (3.4)

$$P_m\widehat{\phi}(s) = \widehat{\phi}(P_m s) = \int\limits_{\mathbb{R}^d} s(w)\, dw$$

proving that $P_m\widehat{\phi} = 1$. ♠

It is easy to check that for ϕ defined by (3.3)

$$\phi(x) = \frac{1}{(2\pi)^d} \Bigg\{ \int\limits_B \frac{1}{P_m(w)} \{e^{iw\cdot x} - \sum_{j=0}^{m-1} \frac{(iw\cdot x)^j}{j!}\} dw + \int\limits_{\mathbb{R}^d \setminus B} \frac{1}{P_m(w)} e^{iw\cdot x} dw \Bigg\} \ , \tag{3.5}$$

since

$$\phi(\widehat{s}) = \widehat{\phi}(s) \ , \qquad s \in S \ .$$

For any $\ell \geq 0$, let the trigonometric polynomial

$$e_\ell(w) = \sum_{\alpha \in I} a_\alpha e^{-i\alpha \cdot w} \tag{3.6}$$

satisfy

$$(e_\ell - P_m)^{(\alpha)}(0) = 0 \ , \qquad |\alpha| \leq m + \ell \ .$$

Then, by Lemma 2.2, the Fourier transform of

$$\psi_\ell(x) := \sum_{\alpha \in I} a_\alpha \phi(x - \alpha) \in T_\phi \tag{3.7}$$

is the *function*

$$\widehat{\psi}_\ell = \frac{e_\ell}{P_m}. \tag{3.8}$$

Further, since in this case $m_0 = \infty$, Corollary 2.7 guarantees that $\psi_\ell(x) = O(\|x\|^{-d-\ell-1})$ as $\|x\| \to \infty$, for $\ell \geq 0$. Thus we obtain from Theorem 2.9 that

$$Q_{\psi_\ell}(p) = p \ , \qquad p \in \mathcal{P}_{P_m} \cap \pi_\ell, \tag{3.9}$$

and by Corollary 2.10

$$\mathcal{P}_{P_m} \cap \pi_\ell \subset \overline{T}_\phi, \tag{3.10}$$

where

$$\mathcal{P}_{P_m} = \big\{ p \in \pi \ \big| \ \big(p^{(\alpha)}(-iD)P_m\big)(0) = 0 \quad \forall\, \alpha \in \mathbb{Z}_+^d \big\} \ .$$

Since the polynomial P_m is homogeneous, so is the associated space $\mathcal{P}_{P_m}$. Hence this space can be written in the simpler form

$$\mathcal{P}_{P_m} = \big\{ p \in \pi \ \big| \ \big(p^{(\alpha)}(D)P_m\big)(0) = 0 \ , \quad \forall\, \alpha \in \mathbb{Z}_+^d \big\}. \tag{3.11}$$

Since in (3.10) ℓ can be arbitrarily large (with the support I of ψ_ℓ in (3.7) changing with ℓ), we conclude that

Corollary 3.2. $\mathcal{P}_{P_m} \subset \overline{T}_\phi$. *In particular*

$$\pi_{m-1} \subset \overline{T}_\phi. \tag{3.12}$$

As already indicated in (2.5), spaces of the form $\mathcal{P}_G$ constitute in π the kernel of a certain differential operator. In the case in hand this space in simply the kernel in π of $P_m(D)$.

Theorem 3.3. *Let*

$$\ker P_m(D) := \big\{ p \in \pi \ \big| \ P_m(D)p \equiv 0 \big\} \ . \tag{3.13}$$

Then

$$\ker P_m(D) = \mathcal{P}_{P_m} \ , \tag{3.14}$$

and hence $\ker P_m(D) \subset \overline{T}_\phi$.

Proof: Observe first that $p \in \ker P_m(D)$ if and only if for all $\alpha \in \mathbb{Z}_+^d$

$$P_m(D)p^{(\alpha)}(0) = D^\alpha[P_m(D)p](0) = 0. \tag{3.15}$$

This means that (3.15) identifies $\ker P_m(D)$ as the largest D-invariant (namely, differentiation-closed) subspace of

$$\{p \in \pi \mid P_m(D)p(0) = 0\},$$

while, by its definition, $\mathcal{P}_{P_m}$ is the largest D-invariant subspace of

$$\{p \in \pi \mid p(D)P_m(0) = 0\}.$$

Our claim thus follows from the fact that for any two polynomials p, q $p(D)q(0) = q(D)p(0)$. ♠

It is interesting to note that the difference operator defining ψ_ℓ in (3.7)

$$\nabla_\ell f = \sum_{\alpha \in I} a_\alpha f(\cdot - \alpha) \ , \tag{3.16}$$

approximates the differential operator $(-1)^{m/2}P_m(D)$ in the following sense:

Proposition 3.4. *Let* ∇_ℓ *be given by (3.16), where* $I \subset \mathbb{Z}^d$ *and* $\{a_\alpha, \alpha \in I\}$ *are chosen so that* $e_\ell(w) = \sum_{\alpha \in I} a_\alpha e^{-i\alpha\cdot w}$ *satisfies*

$$e^{(\alpha)}(0) = P_m^{(\alpha)}(0) \quad , \quad |\alpha| \le \ell + m \ . \tag{3.17}$$

Then

$$\nabla_\ell f = (-1)^{m/2} P_m(D) f \ , \qquad f \in \pi_{\ell+m} \ .$$

Proof: Let $q \in \pi_{\ell+m}$, and consider the polynomial

$$p = (-1)^{m/2}P_m(D)q - \nabla_\ell q = P_m(-iD)q - \nabla_\ell q \ .$$

To show that $p \equiv 0$ we prove that $\widehat{p} \equiv 0$. Now

$$\widehat{p}(s) = \big[P_m - e_\ell\big]\widehat{q}(s) = \widehat{q}\big((P_m - e_\ell)s\big) \ , \qquad s \in S \ , \tag{3.18}$$

and since $q \in \pi_{\ell+m}$

$$\widehat{q}(s) = \sum_{|\alpha| \le \ell+m} c_\alpha s^{(\alpha)}(0) \ , \qquad s \in S \ .$$

On the other hand by (3.17) $\big[(P_m - e_\ell)s\big]^{(\alpha)} = 0$, $|\alpha| \le \ell + m$ proving that in (3.18) $\widehat{p}(s) = 0$, $s \in S$. ♠

Remark 3.5.

For ϕ satisfying (2.2), a result similar to Proposition 3.4 holds, namely

$$\nabla_\ell f = \frac{G}{F}(-iD)f \ , \qquad f \in \pi_{\ell+m} \ , \quad \ell \le m_0 \ ,$$

where for $f \in \pi_{\ell+m}$

$$(G/F)(-iD)f := \sum_{\substack{\alpha\in\mathbb{Z}^d \\ |\alpha|\le m\ +\ \ell}} \frac{(G/F)^{(\alpha)}(0)}{\alpha!}(-iD)^\alpha f \ .$$

The most interesting cases for P_m are when $P_m(D)$ is the iterated Laplacian, namely

$$P_m(x) = \|x\|^m \quad , \quad P_m(D) = (D\cdot D)^{m/2} \ , \quad m \text{ even} \ .$$

In these cases for $m > d$ [GS]

$$\phi_{m,d}(x) = \begin{cases} C_{m,d}\|x\|^{m-d} \ , & d \text{ odd} \ , \\ C_{m,d}\|x\|^{m-d}\log\|x\| \ , & d \text{ even} \ , \end{cases} \tag{3.19}$$

and, by Corollary 3.2, the space of total degree polynomials contained in $\overline{T}_{\phi_{m,d}}$ is π_{m-1}. Thus for fixed odd $\nu > 0$, $\pi_{\nu+d-1} \in \overline{\text{span}}\{\|x-\alpha\|^\nu \ , \ \alpha \in \mathbb{Z}^d\}$ in odd dimension, while for even $\nu > 0$, $\pi_{\nu+d-1} \in \overline{\text{span}}\{\|x-\alpha\|^\nu \log\|x-\alpha\| \ , \ \alpha \in \mathbb{Z}^d\}$ in even dimension. This observation shows that the same radial function has better approximation properties (see sections 4,5) as the dimension of the space increases.

The functions $\psi = \nabla_0\phi_{m,d}$, $m > d$, are studied in [R1], and shown to have properties similar to the univariate B-splines. Explicit construction and properties of $\psi = \nabla_\ell\phi_{m,d}$, $\ell > 0$, are discussed in [R2].

II. Radial functions.

The fundamental solutions of the iterated Laplacian (3.19) are radial functions, namely $\phi(x) = f(\|x\|)$, for some univariate f. Here we investigate radial functions which are not fundamental solutions of elliptic operators, but are obtained from fundamental solutions of the iterated Laplacian by the change $\|x\| \longrightarrow \sqrt{\|x\|^2+c^2}$ with $c > 0$. Thus we consider the following functions

$$\begin{aligned} \phi(x) &= f_\lambda(x) = (\|x\|^2+c^2)^{\lambda/2}, \\ \phi(x) &= g_\lambda(x) = (\|x\|^2+c^2)^{\lambda/2}\log(\|x\|^2+c^2)^{1/2}, \end{aligned} \tag{3.20}$$

without restricting initially the values of λ. The Fourier transforms of these functions satisfy the equation [GS]:

$$\|w\|^{d+\lambda}\widehat{f}_\lambda(w) = c_\lambda \widetilde{K}_{(d+\lambda)/2}(\|cw\|), \tag{3.21}$$

$$\|w\|^{d+\lambda}\widehat{g}_\lambda(w) = \left(\frac{\partial}{\partial\lambda}c_\lambda + c_\lambda\log c\right)\widetilde{K}_{\frac{d+\lambda}{2}}(\|cw\|) + c_\lambda\|w\|^{d+\lambda}\frac{\partial}{\partial\lambda}\left[\|w\|^{-d-\lambda}\widetilde{K}_{\frac{d+\lambda}{2}}(\|cw\|)\right] \tag{3.22}$$

where $c_\lambda = \frac{(2\pi)^{d/2}2^{\lambda/2+1}}{\Gamma(-\frac{\lambda}{2})}$ and $\widetilde{K}_\nu(\|w\|) = \|w\|^\nu K_\nu(\|w\|)$ with $K_\nu(t)$ the modified Bessel function [AS]. The expression (3.22) may be obtained from (3.21) with the aid of the relation $\frac{\partial}{\partial\lambda} f_\lambda = g_\lambda$. The properties of $\widetilde{K_\nu}(t)$ needed in our analysis are:

$$\widetilde{K}_0(t) \sim -\log t\ , \qquad \widetilde{K}_\nu(t) \sim 2^{\nu-1}\Gamma(\nu)\ , \qquad \nu > 0\ , \quad t \to 0^+\ , \tag{3.23}$$

and for integer values of ν

$$\widetilde{K}_\nu(t) = \sum_{k=0}^{\infty} a_{k,\nu} t^{2k} + (\log t) t^{2\nu} \sum_{k=0}^{\infty} b_{k,\nu} t^{2k}, \quad a_{0,\nu} = 2^{\nu-1}(\nu-1)! \text{ as } t \to 0^+, \quad b_{0,\nu} \neq 0. \tag{3.24}$$

Defining $G(w) := \|w\|^{d+\lambda}$, we see that G satisfies (2.2) only when $d+\lambda$ is an even positive integer. Furthermore, with this choice and with $F(w) := c_\lambda \widetilde{K}_{(d+\lambda)/2}(\|cw\|)$, $F(0) \neq 0$ whenever $c_\lambda \neq 0$. The interesting choices of (d, λ) here are:

$$\begin{aligned} \phi(x) &= (\|x\|^2 + c^2)^{\lambda/2} \qquad d, \lambda \text{ odd},\ -d < \lambda, \\ \phi(x) &= (\|x\|^2 + c^2)^{\lambda/2} \qquad d, \lambda \text{ even},\ -d < \lambda < 0\ . \end{aligned} \tag{3.25}$$

The choice λ, d even $\lambda > 0$ corresponds to $c_\lambda = 0$ and is of no interest since then ϕ is a polynomial of degree λ, hence its translate generate a finite dimensional polynomial space. For the ranges of λ in (3.25), (3.24) indicates that (2.2) is satisfied with $m := d + \lambda$, $m_0 := m - 1$,

$$G(w) := \|w\|^m\ , \qquad F(w) := c_{m-d}\widetilde{K}_{m/2}(\|cw\|)\ . \tag{3.26}$$

Note that ϕ is infinitely differentiable, hence its Fourier transform is rapidly decreasing at infinity. In fact, $\widehat{\phi}$ decays exponentially at infinity, [AS].

Restricting $\lambda + d$ to positive even integers is also necessary for equation (3.22) to satisfy conditions (2.2). Furthemore, choosing λ to be even and non-negative, we get $c_\lambda = 0$, hence resulting (up to a constant factor) in the same Fourier transform as in the previous case. Thus we choose the range of d, λ there to be

$$\phi(x) = (\|x\|^2 + c^2)^{\lambda/2} \log(\|x\|^2 + c^2)^{1/2}\ , \qquad \lambda, d \text{ even}\ , \lambda \geq 0. \tag{3.27}$$

This range complements that in (3.25) for even d. For this range, G and F in (2.2) are of the form (3.26) with c_{m-d} being replaced by $\frac{\partial}{\partial m} c_{m-d}$, and with the same choice of m and m_0.

Thus for the functions in (3.25) and (3.27) the ratio G/F, can be expanded near the origin in the form:

$$\frac{G}{F}(w) = \|w\|^m \left\{ \sum_{\nu \geq 0} a_\nu \|w\|^{2\nu} + \|w\|^m \log\|w\| \sum_{\nu \geq 0} b_\nu \|w\|^{2\nu} + \|w\|^{2m} (\log\|w\|)^2 \sum_{\nu \geq 0} c_\nu \|w\|^{2\nu} + \cdots \right\} \tag{3.28}$$

In view of this expansion, a trigonometric polynomial $e_\ell(x)$ satisfying

$$D^\alpha \left(e_\ell - \frac{G}{F} \right)(0) = 0\ , \qquad |\alpha| \leq m + \ell \tag{3.29}$$

exists only for $\ell \leq m - 1$, and by Theorem 2.9 with $\widehat{\psi}_\ell = e_\ell \widehat{\phi}$

$$Q_{\psi_\ell}(p) = p\ , \qquad p \in \pi_\ell\ , \quad 0 \leq \ell \leq m - 2\ . \tag{3.30}$$

This result can be extended to $\ell = m - 1$ by showing that $\widehat{\psi}_{m-1}$ satisfies the conditions of Lemma 2.6 and then applying Corollary 2.11.

Lemma 3.5. *Let e_{m-1} be a trigonometric polynomial satisfying (3.29) with $\ell = m-1$, and let*

$$\widehat{\psi}_{m-1} = e_{m-1}F/G$$

where $G(w) = \|w\|^m$ and $F(w) = \widetilde{c}\widetilde{K}_{m/2}(\|cw\|)$ with $\widetilde{c} \in \mathbb{R}$. Then $\widehat{\psi}$ satisfies the conditions of Lemma 2.6 with $\ell = m-1$.

Proof: By (3.28),(3.29) and (3.24), near the origin $\widehat{\psi}$ has the expansion

$$\begin{aligned}\widehat{\psi}_{m-1}(w) &= 1 + \frac{F(w)}{G(w)}\left[e(w) - \frac{G(w)}{F(w)}\right] \\ &= 1 + h_m(w) + \widetilde{a}\|w\|^m \log\|w\| + \widetilde{h}(w)\ ,\end{aligned} \tag{3.31}$$

with $h_m(tw) = t^m h_m(w)$, $h_m \in C^\infty(\mathbb{R}^d\backslash 0)$, $\widetilde{h} \in \mathcal{F}_{m+\theta}$ for any $\theta \in (0,1)$, and $\widetilde{a} \in \mathbb{R}\backslash 0$. By the homogeneity of h_m its inverse generalized Fourier transform $\overset{\vee}{h}_m(x)$ is homogeneous of order $-m-d$ away of the origin. Hence

$$\overset{\vee}{h}_m(x) = O(\|x\|^{-m-d})\ , \qquad \text{as} \quad \|x\| \to \infty\ .$$

Moreover, since m is even, $(\|w\|^m \log\|w\|)^\vee$ is infinitely differentiable away of the origin and a direct calculation yields

$$(\|w\|^m \log\|w\|)^\vee(tx) = t^{-(m+d)}(\|w\|^m \log\|w\|)^\vee(x) - t^{-(m+d)}\log|t| L,\ t \in \mathbb{R}\backslash 0,$$

where L is a distribution supported at the origin. Hence $\widehat{\psi}$ satisfies the requirements of Lemma 2.6 with $\ell = m-1$. ♠

Remark 3.6.

The same type of arguments together with the two observations:

$$f \in \mathcal{F}_r \Rightarrow x^\alpha f \in \mathcal{F}_{r+|\alpha|}, \qquad \alpha \in \mathbb{Z}_+^d\ ,$$

$$(w^\alpha\|w\|^m \log\|w\|)^\vee(tx) = t^{-(|\alpha|+m+d)}(w^\alpha\|w\|^m \log\|w\|)^\vee(x) - t^{-(|\alpha|+m+d)}\log|t| L_\alpha,\ t \in \mathbb{R}\backslash 0,$$

for some L_α supported at the origin, prove that under the conditions of Lemma 3.5

$$D^\alpha\psi_{m-1}(x) = O(\|x\|^{-d-m-|\alpha|}) \quad \text{as} \quad \|x\| \to \infty\ , \qquad \alpha \in \mathbb{Z}_+^d\ .$$

Corollary 3.7. *Let ϕ_λ be one of the functions in (3.25) or (3.27), and let $\widehat{\psi}_\lambda = e_\ell\widehat{\phi}_\lambda$ where e_ℓ satisfies (3.29) with $m = \lambda + d$, $\ell = m-1$, and F, G as above. Then*

$$Q_{\psi_\lambda}p = p\ , \qquad p \in \pi_\ell = \pi_{\lambda+d-1}\ ,$$

and hence

$$\pi_{\lambda+d-1} \subset \overline{T}_{\phi_\lambda}\ .$$

Remark 3.8.

The radial functions in (3.25), (3.27) are all obtained from fundamental solutions of the iterated Laplacian by the change $\|x\| \to \sqrt{\|x\|^2 + c^2}$ with $c > 0$. This "shifted" version of the fundamental solutions of $P_m(D) = (D{\cdot}D)^{m/2}$ correspond to all positive values of the even integer m $(= \lambda{+}d > 0)$, while in the case of the fundamental solutions themselves, (3.19), the possible range for the even integer m is restricted to $m > d$. It is clear that this range of m cannot be extended since the fundamental solutions are singular at the origin for $0 < m \le d$.

4. Approximation order by quasi-interpolation

In the introduction it is shown that the conversion of the polynomial reproduction of Q_ψ into approximation rates of $Q_{\psi,h}$ may be done exactly as in the compactly supported case, if ψ decreases fast enough at infinity, namely if $Q_\psi p = p$, $p \in \pi_\ell$ and $\psi(x) = O\big(\|x\|^{-(d+k)}\big)$ as $\|x\| \to \infty$, with $k > \ell + 1$. Here we investigate the more subtle case when it is known that

$$|\psi(x)| \le A(1 + \|x\|_\infty)^{-(d+\ell+1)}, \tag{4.1}$$

while

$$Q_\psi p = p, \ \forall p \in \pi_\ell. \tag{4.2}$$

These conditions are satisfied by most of the models considered in Section 2. We assume throughtout this section that (4.1) and (4.2) hold, and that all functions f approximated by $Q_{\psi,h}$ are *admissible* in the sense that the partial sums of $Q_{\psi,h} f$ converge absolutely and uniformly on compact sets. We look here for conditions on f and ψ that allow the improvement of the approximation rate $O(h^\ell)$ provided by Corollary 1.2.

The outline of the discussion is as follows: we show first that for a function f with bounded derivatives of order ℓ and $\ell + 1$, the approximation order to f by $Q_{\psi,h}$ depends on the behaviour of the sum

$$\sum_{\alpha \in [-h^{-1},h^{-1}]^d \cap \mathbb{Z}^d} f(h\alpha)\psi(h^{-1} \cdot - \alpha). \tag{4.3}$$

Following [J1], we then prove that for such f the approximation rate is (at least) $O(h^{\ell+1}|\log h|)$. Aiming at achieving better rates, we assume that the $(\ell+2)$-order derivatives of f are bounded as well. That latter case is treated by a sequence of reductions: first it is shown that approximation order $O(h^{\ell+1})$ is equivalent to the uniform boundeness of the sums

$$\sum_{\alpha \in [-h^{-1},h^{-1}]^d \cap \mathbb{Z}^d} \alpha^\beta \psi(h^{-1} \cdot - \alpha), \ h > 0, \ |\beta| = \ell + 1.$$

Under further assumptions on the behaviour of the first order derivatives of ψ, the boundedness of these sums is converted in the usual way to the uniform boundedness of the integrals

$$\int_{[-h^{-1},h^{-1}]^d} ()^\beta \psi, \ h > 0, \ |\beta| = \ell + 1, \tag{4.4}$$

where $()^\beta$ stands for the monomial of power β. We then exploit the precise connection between the uniform boundedness of (4.4) and the behaviour of $D^\beta \widehat{\psi}$ near the origin, showing that this uniform boundedness is equivalent to the boundedness of the integrals $(D^\beta \widehat{\psi} * u_h)(0)$ for a suitable approximate identity $\{u_h\}$. Using the decay of $D^\beta \widehat{\psi}$ at infinity, the boundedness of (4.4) is reduced to the boundedness of $D^\beta \widehat{\psi}$ around the origin, a property which is valid in the case of fundamental solutions of homogeneous elliptic operators. On the other hand, if $D^\beta \widehat{\psi}$ admits a log singularity at the origin (as in the case of the "shifted" fundamental solution), the boundedness condition is violated, hence the approximation order in general is necessarily $O(h^{\ell+1}|\log h|)$. The following theorem summarizes the resulting consequences with regard to the examples considered in the previous section.

Theorem 4.1. *Let m be an even integer and $\ell < m$.*

(a) Assume that ϕ is a fundamental solution of a homogeneous elliptic operator of order m and $\psi = \nabla\phi$ satisfies (4.1) and (4.2). Then, for f with bounded derivatives of order ℓ and $\ell+1$,

$$\|Q_{\psi,h} f - f\|_\infty = O(h^{\ell+1}|\log h|). \tag{4.5}$$

If in addition the derivatives of f of order $\ell+2$ are bounded, then

$$\|Q_{\psi,h} f - f\|_\infty = O(h^{\ell+1}). \tag{4.6}$$

(b) Assume that ϕ is a "shifted" fundamental solution of the $(m/2)$th iterated Laplacian, and that $\psi = \nabla\phi$ satisfies (4.1) and (4.2). Then for f with bounded derivatives of order ℓ and $\ell+1$, (4.5) holds. If, in addition, the derivatives of order $\ell+2$ of f are bounded then (4.6) holds for $\ell < m-1$, yet for $\ell = m-1 = m_0$ the $|\log h|$ factor cannot be removed and (4.5) gives the best rate: precisely, there exists an infinitely differentiable compactly supported f for which $\|Q_{\phi,h} f - f\|_\infty \neq o(h^d|logh|)$.

We now commence on the detailed analysis. The first two lemmas will be used as a simple technical device for the analysis to follow. We use the notation

$$S_{x,h} := \mathbb{Z}^d \cap h^{-1}(x + [-1,1]^d).$$

Lemma 4.2. *With $S_{x,h}$ as above*

(a) $$G_{x,h} := \sum_{\alpha \in S_{x,h}} (1 + \|\alpha - h^{-1}x\|_\infty)^{-(d+k)} \le A \begin{cases} |\log h|, & k = 0, \\ h^k, & -d < k < 0. \end{cases}$$

(b) $$\sum_{\alpha \in \mathbb{Z}^d \setminus S_{x,h}} (1 + \|\alpha - h^{-1}x\|_\infty)^{-(d+k)} \le A h^k, \ k > 0.$$

Proof: We treat first the case $x = 0$ in (a). For $d = 1$ all the above results can be obtained say, by an integral test. The multivariate case is then reduced to the univariate one, since

$$\sum_{\alpha \in S_{x,h}} (1 + \|\alpha\|_\infty)^{-(d+k)} \le 2d \sum_{j=0}^{[h^{-1}]} (2j+1)^{d-1}(1+j)^{-(d+k)} \le 2^d d \sum_{j=0}^{[h^{-1}]} (1+j)^{-(k+1)}.$$

For the case of general x in (a), we first note that $G_{x,h} \le \sum_{\alpha\in(\delta+\mathbb{Z}^d)\cap[-h^{-1},h^{-1}]^d}(1+\|\alpha\|_\infty)^{-(d+k)}$, with $\delta \in [0,1]^d$. Using the estimate

$$\|\delta+\alpha\|_\infty \ge \|\nu+\alpha\|_\infty \ , \qquad \nu_i = \begin{cases} 0 & \alpha_i \ge 0 \\ 1 & \alpha_i < 0 \ , \end{cases}$$

we may divide the cube $[-h^{-1},h^{-1}]^d$ by the coordinate hyperplanes, while on each subcube the corresponding partial sum is bounded by

$$\sum_{\alpha\in\mathbb{Z}^d\cap\widetilde{C}}(1+\|\alpha\|_\infty)^{-(d+k)} \ ,$$

where $\widetilde{C}$ is a cube with a main diagonal connecting the origin with a vertex of $[-h^{-1},h^{-1}]^d$. Summing up over all the 2^d subcubes with this property, we conclude that $G_{x,h} \le 2^d G_{0,h}$.
Part (b) is proved similarly. ♠

Lemma 4.3. *Let γ be a continuous function which satisfies the following conditions:*
(a) $|\gamma(x)| \le A_1(1+\|x\|_\infty)^{-d}$.
(b) $|\gamma(x)-\gamma(x+\delta)| \le A_2(1+\|x\|_\infty)^{-(d+1)}$, for all $\delta\in[-1/2,1/2]^d$.
Then

$$|\sum_{\alpha\in S_{0,h}}\gamma(\alpha) - \int_{[-h^{-1},h^{-1}]^d}\gamma(t)\,dt| = O(1). \tag{4.7}$$

Proof: For each $\alpha\in S_{0,h}$ let $C_\alpha := \alpha+[-1/2,1/2]^d$, $C := \cup_{\alpha\in S_{0,h}}C_\alpha$. Note first that (a) implies that

$$\begin{aligned}&|\int_{[-h^{-1},h^{-1}]^d}\gamma(t)\,dt - \int_C\gamma(t)\,dt| \\ \le & 2^d d(1+h^{-1})^{d-1}\max\{|\gamma(t)| :\ h^{-1}-1 \le \|t\|_\infty \le h^{-1}+1\} \\ \le & 2^d dA_1(1+h^{-1})^{d-1}h^d = O(h),\end{aligned}$$

hence we may replace the domain of integration in (4.7) by C. The claim now easily follows, since by (b) and the continuity of γ

$$|\gamma(\alpha) - \int_{C_\alpha}\gamma(t)\,dt| \le A_2(1+\|\alpha\|_\infty)^{-(d+1)},$$

and the series $\sum_{\alpha\in\mathbb{Z}^d}(1+\|\alpha\|_\infty)^{-(d+1)}$ is convergent. ♠

Next, we employ the quasi-interpolation argument in

Proposition 4.4. *For a given non-negative integer j and a smooth function f whose derivatives of order $\ell,\ \ell+1,...,\ell+j$ are all bounded, set $K_{f,j} := \sum_{s=0}^{j}\|f\|_{\infty,\ell+s}$. Then $Q_{\psi,h}$ with ψ satisfying (4.1) and (4.2) has the property that for every smooth function f with bounded derivatives of order $\ell,\ell+1,...,\ell+j$,*

$$\|Q_{\psi,h}f - f\|_\infty \le AK_{f,j}\ h^{\ell+1} \text{ resp. } AK_{f,j}\ h^{\ell+1}|\ \log h|\ ,$$

if for every function g with bounded derivatives of order $\ell, \ell+1, ..., \ell+j$, which satisfies $D^\alpha g(x) = 0$, $|\alpha| \le \ell$ for some $x \in \mathbb{R}^d$,

$$|\sum_{\alpha \in S_{x,h}} g(h\alpha)\psi(h^{-1}x - \alpha)| \le AK_{g,j}\ h^{\ell+1} \text{ resp. } AK_{g,j}\ h^{\ell+1}|\log h|\ .$$

Proof: For $x \in \mathbb{R}^d$ and a smooth f as above let $T_{\ell,x}f$ be the Taylor expansion of degree ℓ of f around x. Then, with $g := f - T_{\ell,x}f$ one has $K_{g,j} \le 2K_{f,j}$, and $D^\alpha g(x) = 0$, $|\alpha| \le \ell$. Thus, since $Q_{\psi,h}$ reproduces π_ℓ,

$$(Q_{\psi,h}f - f)(x) = Q_{\psi,h}g(x) = \sum_{\alpha \in \mathbb{Z}^d} g(h\alpha)\psi(h^{-1}x - \alpha)\ ,$$

and the claim of the proposition follows if for $g = f - T_{\ell,x}f$,

$$|\sum_{\alpha \in \mathbb{Z}^d \setminus S_{x,h}} g(h\alpha)\psi(h^{-1}x - \alpha)| \le AK_{g,j}h^{\ell+1}.$$

For that, we first observe that the assumptions on g provide the estimate

$$|g(h\alpha)| \le A\|g\|_{\infty,\ell}\|h\alpha - x\|_\infty^\ell = Ah^\ell\|g\|_{\infty,\ell}\|h^{-1}x - \alpha\|_\infty^\ell. \tag{4.8}$$

Therefore,

$$|\sum_{\alpha \in \mathbb{Z}^d \setminus S_{x,h}} g(h\alpha)\psi(h^{-1}x - \alpha)| \le Ah^\ell\|g\|_{\infty,\ell} \sum_{\alpha \in \mathbb{Z}^d \setminus S_{x,h}} (1 + \|h^{-1}x - \alpha\|_\infty)^{-(d+1)} \le A\|g\|_{\infty,\ell}h^{\ell+1},$$

where in the last inequality Lemma 4.2(b) has been employed. ♠

Following [J1], we improve the approximation order $O(h^\ell)$ implied by Corollary 1.2, for ψ satisfying (4.1) and (4.2).

Theorem 4.5. *Under conditions (4.1) and (4.2), one has*

$$\|Q_{\psi,h}f - f\|_\infty = O(h^{\ell+1}|\log h|),$$

for every smooth function f whose derivatives of order ℓ and $\ell+1$ are all bounded.

Proof: By Proposition 4.4 (for the choice $j = 1$ there), it suffices to prove an inequality

$$|\sum_{\alpha \in S_{x,h}} g(h\alpha)\psi(h^{-1}x - \alpha)| \le A\|g\|_{\infty,\ell+1}h^{\ell+1}|\log h|,$$

for any g of the form $g = f - T_{\ell,x}f$. As in (4.8), we estimate

$$|g(h\alpha)| \le Ah^{\ell+1}\|g\|_{\infty,\ell+1}\|h^{-1}x - \alpha\|_\infty^{\ell+1},$$

and hence

$$|g(h\alpha)\psi(h^{-1}x - \alpha)| \le Ah^{\ell+1}\|g\|_{\infty,\ell+1}(1 + \|h^{-1}x - \alpha\|_\infty)^{-d}.$$

An application of Lemma 4.2(a) thus completes the proof. ♠

In order to identify situations where the above $|\log h|$ term can be removed, we assume hereafter that ψ satisfies also the additional requirement

$$|D^\beta\psi(x)| \le A(1 + \|x\|_\infty)^{-(d+\ell+2)}, \qquad |\beta| = 1\ . \tag{4.9}$$

Under this further assumption, the problem of obtaining the exact order of $Q_{\psi,h}$ can be further reduced:

Proposition 4.6. *Suppose that ψ satisfies (4.1), (4.2) and (4.9). Then the approximation order*

$$\|Q_{\psi,h}f - f\|_\infty = O(h^{\ell+1})$$

holds for every smooth function with bounded derivatives of order ℓ, $\ell+1$ and $\ell+2$ if and only if the integrals

$$I_{\eta,h} := \int_{[-h^{-1},h^{-1}]^d} t^\eta \psi(t)\, dt \ , \qquad |\eta| = \ell+1 \ ,$$

are uniformly bounded in h. Moreover, if $I_{\eta,h} \neq o(|\log h|)$ for some $|\eta| = \ell+1$, then for any compactly supported infinitely differentiable f, which coincides with t^η on the cube $[-1,1]^d$,

$$|(Q_{\psi,h}f - f)(0)| \neq o(h^{\ell+1}|\log h|) \ .$$

Proof: By Proposition 4.4 (with $j = 2$), it is sufficient to show that the above integrals are bounded if and only if the inequality

$$|\sum_{\alpha\in S_{x,h}} g(h\alpha)\psi(h^{-1}x-\alpha)| \leq AK_{g,2}h^{\ell+1} \tag{4.10}$$

holds for every g whose derivatives of order $\ell+s$, $s = 0,1,2$ are all bounded, and whose derivatives of order $\leq \ell$ vanish at x.

By our assumptions, $T_{\ell+1,x}g$, the Taylor expansion of degree $\ell+1$ of g around x, is a homogeneous polynomial of degree $\ell+1$, and

$$|(g - T_{\ell+1,x}g)(z)| \leq A\|g\|_{\infty,\ell+2}\|z-x\|_\infty^{\ell+2}. \tag{4.11}$$

Now, by (4.1), (4.11), and Lemma 4.2(a),

$$\begin{aligned}
&|\sum_{\alpha\in S_{x,h}} [g(h\alpha) - T_{\ell+1,x}g(h\alpha)]\psi(h^{-1}x-\alpha)| \\
\leq &Ah^{\ell+2}\|g\|_{\infty,\ell+2} \sum_{\alpha\in S_{x,h}} (1+\|h^{-1}x-\alpha\|_\infty)^{-(d-1)} \\
\leq &A_1 h^{\ell+1}\|g\|_{\infty,\ell+2},
\end{aligned}$$

which reduces (4.10) to the behaviour of the sum

$$\sum_{\alpha\in S_{x,h}} T_{\ell+1,x}g(h\alpha)\psi(h^{-1}x-\alpha).$$

Thus (4.10) is satisfied for all admissible functions g if and only if the sums

$$\sum_{\alpha\in S_{x,h}} (\alpha - h^{-1}x)^\eta \psi(h^{-1}x-\alpha), \quad |\eta| = \ell+1 \ , \tag{4.12}$$

are uniformly bounded in h and x. To proceed, we define $\gamma(x) := x^\eta\psi(-x)$, $|\eta| = \ell+1$. From (4.9) we conclude that

$$|D^\beta\gamma(x)| \leq A(1+\|x\|_\infty)^{-(d+1)}, \qquad |\beta| = 1 \ , \tag{4.13}$$

showing that the boundedness of the sums in (4.12) is equivalent to that of the sums

$$\sum_{\alpha\in S_{0,h}} \alpha^\eta \psi(-\alpha) \ , \qquad |\eta| = \ell+1 \ .$$

To complete the proof of the first claim, it remains to show that γ satisfies the requirements in Lemma 4.3. Yet this is evident: condition (a) of that lemma is a direct consequence of (4.1), while condition (b) follows from (4.13).

For the second claim, choose $x = 0$ to obtain

$$\sum_{\alpha\in S_{0,h}} f(h\alpha)\psi(-\alpha) = \sum_{\alpha\in S_{0,h}} (h\alpha)^\eta \psi(-\alpha).$$

Since the argument used in the proof of Proposition 4.4 shows that $\sum_{\mathbb{Z}^d\setminus S_{0,h}} |f(h\alpha)\psi(-\alpha)| = O(h^{\ell+1})$, Lemma 4.3 provides the desired result. ♠

At this point we wish to connect the behaviour of the integrals $I_{\eta,h}$, $|\eta| = \ell+1$ with the behaviour of $D^\beta\widehat{\psi}$, with $|\beta| = \ell+1$. For this, we take ρ to be as in (2.23) with $\delta = 1$ and $\|\cdot\|_2$ being replaced by $\|\cdot\|_\infty$ (any rapidly decreasing C_0^∞ function which is 1 in a neighborhood of the origin will do as well). Defining $\rho_h := \rho(h\cdot)$, we see that for an arbitrary measurable function f,

$$|\int_{[-h^{-1},h^{-1}]^d} f - \int_{\mathbb{R}^d} \rho_h f| \le \int_{1/2h^{-1}\le\|t\|_\infty\le h^{-1}} |f(t)|\, dt,$$

which is bounded independently of h provided that

$$|f(t)| = O(\|t\|_\infty^{-d}), \text{ as } t\to\infty. \tag{4.14}$$

As for the integral

$$J := \int_{\mathbb{R}^d} \rho_h f,$$

the definition of the generalized Fourier transform of $\widehat{f}$ implies that

$$J = \widehat{f}(\widehat{\rho}_h).$$

For the case of interest, viz, when $f := ()^\beta\psi$, $|\beta| = \ell+1$, (4.14) is satisfied and $\widehat{f}$ coincides, up to a constant, with $D^\beta\widehat{\psi}$. We thus conclude

Proposition 4.7. *The integrals $I_{\beta,h}$, $|\beta| = \ell+1$, are uniformly bounded in h if and only if $J_{\beta,h} := (D^\beta\widehat{\psi})(u_h)$, $|\beta| = \ell+1$, are uniformly bounded in h, with $u_h = \widehat{\rho}_h = h^{-d}\widehat{\rho}(h^{-1}\cdot)$. Furthermore, for each $|\beta| = \ell+1$, $J_{\beta,h} \ne o(|\log h|)$ if and only if $I_{\beta,h} \ne o(|\log h|)$.*

We are now in a position to improve the result of Theorem 4.5.

Theorem 4.8. *Let ψ satisfy (4.1),(4.2),(4.9) and the additional condition*

$$\|D^\beta \widehat{\psi}\|_\infty \leq A \ , \qquad |\beta| = \ell + 1 \ . \tag{4.15}$$

Then

$$\|Q_{\psi,h} f - f\|_\infty = O(h^{\ell+1}) \ , \tag{4.16}$$

for f with bounded derivatives of order $\ell, \ell+1$ and $\ell+2$.

Proof: By Propositions 4.6 and 4.7, the proof of the theorem is reduced to the proof of the boundedness in h of

$$J_{\beta,h} = \int_{\mathbb{R}^d} u_h D^\beta \widehat{\psi} \ , \qquad |\beta| = \ell + 1 \ . \tag{4.17}$$

Now, by the boundedness of $D^\beta \widehat{\psi}$, and since $u_h = h^{-d} \widehat{\rho}(h^{-1} \cdot)$

$$\left| \int_{\mathbb{R}^d} u_h D^\beta \widehat{\psi} \right| \leq A \int_{\mathbb{R}^d} |u_h| = A \int_{\mathbb{R}^d} |\widehat{\rho}| \ . \qquad \spadesuit$$

This result applies to most of the cases considered in Section 2.

Corollary 4.9. *Let ϕ satisfy conditions (2.2) and let $\psi = \nabla_\ell \phi$ satisfy the conditions of Lemma 2.2 with $\ell < \min(m_0, m)$. Then for f with bounded derivatives of order $\ell, \ell+1$ and $\ell+2$,*

$$\left\| Q_{\psi,h} f - f \right\|_\infty = O(h^{\ell+1}).$$

Proof: To show that ψ satisfies the conditions of Theorem 4.8, observe that (4.1) and (4.9) follow from Lemma 2.7 and Remark 2.8, and that (4.2) is guaranteed by Theorem 2.9. Finally condition (4.15) follows from (2.2)(e) and expression (2.16) with $\ell < m_0$. $\spadesuit$

Theorem 4.5 and Corollary 4.9 when specialized to a fundamental solution of a homogeneous elliptic operator ϕ, yield part (a) of Theorem 4.1, since $m_0 = \infty$. For ϕ a "shifted" fundamental solution of the $(m/2)$th-iterated Laplacian, Theorem 4.5 and Corollary 4.9, yield part (b) of Theorem 4.1 for $\ell < m - 1$, since in this case $m_0 = m - 1$. To complete the proof of Theorem 4.1. it is sufficient, in view of Propositions 4.6 and 4.7, to show that

Lemma 4.10. *Let ϕ be a "shifted" fundamental solution of the $(m/2)$th iterated Laplacian and let $\psi = \nabla_\ell \phi$ satisfy the conditions of Lemma 2.2 with $\ell = m - 1$. Then ψ satisfies conditions (4.1),(4.2),(4.9), and for $\beta = (m, 0, ..., 0)$*

$$J_{\beta,h} = \int_{\mathbb{R}^d} u_h D^\beta \widehat{\psi} \neq o(|\log h|). \tag{4.18}$$

Proof: By Lemma 3.5, Corollary 2.11, and Remark 3.6, ψ satisfies (4.1),(4.2) and (4.9), while, by (3.26), $D^\beta \widehat{\psi}$ decays to zero at infinity. The behavior near $w = 0$ of $D^\beta \widehat{\psi}(w)$ is obtained from (3.31). For $w \in B_\varepsilon$, application of D^β to (3.31) yields

$$D^\beta \widehat{\psi}(w) = \widetilde{a} m! \log \|w\| + O(1) \ , \ \widetilde{a} \neq 0. \tag{4.19}$$

Now, for any f with at most polynomial growth at infinity

$$\lim_{h\to 0} \int\limits_{\|w\|>1} f(w)u_h(w)dw = \lim_{h\to\infty} \int\limits_{\|\lambda\|\geq h^{-1}} f(\lambda h)\widehat{\rho}(\lambda)d\lambda = 0 \tag{4.20}$$

since $\widehat{\rho}$ decays faster than any polynomial at infinity, being the Fourier transform of a C_0^∞-function [GS].

Thus for β as above, we obtain in view of (4.20) (when applied to $f = D^\beta\widehat{\psi} - \widetilde{a}m!\log\|w\|$) and (4.19),

$$\int\limits_{\mathbb{R}^d} u_h D^\beta\widehat{\psi} = A\int\limits_{\mathbb{R}^d} u_h(w)\log\|w\|dw + O(1)\ .$$

The claim (4.18) now follows from the observation that

$$\int\limits_{\mathbb{R}^d} u_h(w)\log\|w\|dw = \log h\int\limits_{\mathbb{R}^d}\widehat{\rho}(\lambda)d\lambda + O(1)\ ,$$

in which we have used the fact that $\widehat{\rho}\log\|\cdot\|$ is integrable, as the product of a rapidly decreasing function by a tempered one. $\square$

5. Approximation order by quasi-interpolation over a bounded domain

We now come to the more practical question of determining the rate of convergence for quasi-interpolation over a bounded region. We take Ω to be an open bounded region of $\mathbb{R}^d$ and suppose that we have a function $f \in C^{\ell+1}(\Omega)$. We now define the quasi-interpolant to f on Ω by

$$Q_{\psi,h,\Omega}f(x) = \sum_{\alpha\in\mathbb{Z}^d\cap h^{-1}\Omega} f(h\alpha)\psi(h^{-1}x-\alpha)\ . \tag{5.1}$$

Assuming as before that ψ satisfies conditions of the form

$$Q_\psi p = p\ ,\ p\in\pi_\ell\ ,\quad |\psi(x)| \leq A(1+\|x\|_\infty)^{-(d+k)}\ ,\quad k\geq \ell+1\ , \tag{5.2}$$

we cannot expect convergence on the whole of Ω, but look for convergence on a domain smaller by size $\delta := \delta(h)$:

$$\Omega_\delta = \{y\in\Omega\ :\ \|y-z\|_\infty \leq \delta \Rightarrow z\in\Omega\}. \tag{5.3}$$

We think of δ as being fixed or going to 0, as h goes to 0. In the compactly supported case one may define $\delta(h) = ch$, where c depends on the $\operatorname{diam}\operatorname{supp}\psi$, to get the same approximation order as obtained with respect to the whole domain $\mathbb{R}^d$. However, in our analysis we will have to impose slower decrease on δ in order to preserve the approximation orders established in the previous section. In case δ is fixed, we take it small enough so that $\Omega_\delta \neq \emptyset$. It is possible to attempt to establish the rate of convergence by first extending the function f to a function f_E over the whole of $\mathbb{R}^d$. If f_E satisfies e.g., the conditions of Proposition 1.1 or Theorem 4.5 then the rate of convergence can be deduced from an estimate of the error between $Q_{\psi,h,\Omega}f$ and $Q_{\psi,h}f_E$.

Suitable f_E can be provided in many cases by the Whitney Extension Theorem [H], although this cannot be used for all domains Ω. We take here a different approach: since the error is measured in the ∞-norm, we must treat the worst case, occuring when approximating next to the boundary of Ω_δ, where the contribution then to the approximant may be based only on a small cube (of size 2δ) centered at the point in question. Therefore, we consider, instead of the above $Q_{\psi,h,\Omega}$, a quasi-interpolant $Q_{\psi,h,\Omega,\delta}$ of the form

$$Q_{\psi,h,\Omega,\delta}f(x) := \sum_{\{\alpha\in\mathbb{Z}^d:\ \|h\alpha-x\|_\infty\le\delta(h)\}} f(h\alpha)\psi(h^{-1}x-\alpha). \tag{5.4}$$

Such an approach is independent of the topology of Ω: it requires, in the quasi-interpolation argument, an estimation of a new term in the error, associated with the error obtained when approximating $p \in \pi_\ell$ by $Q_{\psi,h,\Omega,\delta}p$. Such term did not occur in the case $\Omega = \mathbb{R}^d$, since $Q_{\psi,h}$ reproduces π_ℓ. The admissible functions that are being approximated here, are always assumed to lie in $C^{\ell+1}(\Omega)$ with their derivatives up to order $\ell+1$ bounded in Ω. As a preparation we first sketch the approach taken here.

For $x \in \Omega_\delta$, we estimate $|(Q_{\psi,h,\Omega}f - f)(x)|$ by writing

$$|(Q_{\psi,h,\Omega}f - f)(x)| \le$$

$$|Q_{\psi,h,\Omega,\delta}(f-Tf)(x)| + |(Q_{\psi,h,\Omega,\delta}Tf - Tf)(x)| + |(Q_{\psi,h,\Omega} - Q_{\psi,h,\Omega,\delta})f(x)| + |(Tf-f)(x)|\ , \tag{5.5}$$

where $Tf := T_{x,\ell}f$, namely the Taylor polynomial of degree ℓ of f at x. We then estimate each summnand on the right-hand side of (5.5) separately.

The last term in (5.5) is evidently 0. Bounding expressions like the first term in (5.5) was the focal point in Section 4. This was done, under various conditions on ψ and f, in Proposition 1.1/Corollary 1.2 and in Theorems 4.5, 4.8, and 4.1. We therefore need to bound the two middle terms in (5.5). Our first lemma treats the third term there.

Lemma 5.1. *For an admissible function f and $x \in \Omega_\delta$*

$$|(Q_{\psi,h,\Omega} - Q_{\psi,h,\Omega,\delta})f(x)| \le A\|f\|_\infty (h/\delta)^k, \tag{5.6}$$

where A is independent of h, f and $x \in \Omega_\delta$, and k is as in (5.2).

Proof: Since f is bounded, it is sufficient to prove the claim for the choice $f = 1$. In this case, by (5.2), the left hand of (5.6) is dominated by the sum

$$A \sum_{\{\alpha\in\mathbb{Z}^d:\ \|x-h\alpha\|_\infty>\delta\}} (1+\|\alpha-h^{-1}x\|_\infty)^{-(d+k)}.$$

The required estimate now follows from Lemma 4.2(b). ♠

The next lemma treats the second term in (5.5).

Lemma 5.2. *For an admissible function f and $x \in \Omega_\delta$*

$$|(Q_{\psi,h,\Omega,\delta}Tf - Tf)(x)| \le A(h/\delta)^k, \tag{5.7}$$

where A is independent of x, δ, and h, and k is as in (5.2).

Proof: Using the polynomial reproduction property of $Q_{\psi,h}$, we have

$$(Q_{\psi,h,\Omega,\delta}Tf - Tf)(x) = \sum_{\{\alpha\in\mathbb{Z}^d:\ \|x-h\alpha\|_\infty>\delta\}} Tf(h\alpha)\psi(h^{-1}x-\alpha). \tag{5.8}$$

We may assume without loss that Tf is a monomial of degree $j \le \ell$, which in turn can be bounded by $\|\cdot - x\|_\infty^j$. Now, (5.2) with Lemma 4.2(b) provide the estimate

$$\begin{aligned} &\sum_{\{\alpha\in\mathbb{Z}^d:\ \|x-h\alpha\|_\infty>\delta\}} \|h\alpha - x\|_\infty^j |\psi(h^{-1}x-\alpha)| \\ \le &\sum_{\{\alpha\in\mathbb{Z}^d:\ \|x-h\alpha\|_\infty>\delta\}} h^j\|\alpha - h^{-1}x\|_\infty^j (1+\|h^{-1}x-\alpha\|_\infty)^{-(d+k)} \\ \le &\sum_{\{\alpha\in\mathbb{Z}^d:\ \|x-h\alpha\|_\infty>\delta\}} h^j(1+\|h^{-1}x-\alpha\|_\infty)^{j-(d+k)} \le h^j(h/\delta)^{k-j}\ . \end{aligned} \tag{5.9}$$

Thus (5.7) records the worst case in (5.9), which corresponds to the choice $j=0$. ♠

The above lemmas show that for approximation order $O(h^{\ell+1})$ one should restrict $\delta(h)$ by assuming

$$\delta(h) \ge ch^{1-(\ell+1)/k},$$

for some positive c, while for approximation order $O(h^{\ell+1}|\log h|)$ in the case $k = \ell+1$, it is even sufficient to take

$$\delta(h) \ge c|\log h|^{-1/(\ell+1)}.$$

We refer to $\delta(h)$ which satisfies the restrictions above as admissible. For an admissible δ, the approximation order on $\mathbb{R}^d$ by $Q_{\psi,h}$ established in sections 1 and 4, can be converted to approximation orders by $Q_{\psi,h,\Omega}$ on $\Omega_{\delta(h)}$, provided that f, in addition to other relevant requirements (as specified in each theorem) has bounded derivatives in Ω up and including order ℓ. We summarize this as follows

Theorem 5.3. *Assume that $\Omega \subset \mathbb{R}^d$ is open and bounded, and $\delta = \delta(h)$ is admissible in the above meaning. Then, under the various conditions required from ψ and f in Proposition 1.1, Corollary 1.2, Theorem 4.1, Theorem 4.5, Theorem 4.8, and Corollary 4.9, the approximation rates established there for $\|Q_{\psi,h}f - f\|_\infty$ are valid for $\|Q_{\psi,h,\Omega}f - f\|_{\infty,\Omega_\delta}$, provided that in addition all the derivatives of f of order up to ℓ are bounded in Ω.*

Note that for the case studied in Theorem 4.8, $k = \ell+1$, and thus, in contrast with all other cases, δ must be held fixed, so that the approximation rate $\ell+1$ is proved only for fixed closed subsets of Ω.

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