The way things were in multivariate splines: A personal view

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Summary. A personal account of the author's encounters with multivariate splines during their early history.

1 Tensor product spline interpolation

My first contact with multivariate splines occurred in August 1960. In my first year at the Harvard Graduate School, working as an RA for Garrett Birkhoff, I had not done too well but, nevertheless, had gotten married and so needed a better income than the RAship provided. On the (very kind and most helpful) recommendation of Birkhoff who consulted for the Mathematics Department at General Motors Research in Warren MI, I had been hired in that department in order to be of assistance to Leona Junko, the resident programmer in that department.

Birkhoff and Henry L. Garabedian, the head of that department, had developed a scheme for interpolation to data on a rectangular grid meant to mimic cubic spline interpolation; see [BG]. They would use what they called "linearized spline interpolation" and what is now called cubic spline interpolation, along the meshlines in both directions, in order to obtain values of the first derivative in both directions at each meshpoint, and then fill in each rectangle by a C^1 piecewise low-degree harmonic polynomial function that would match the given information, of value and two first-order derivatives, at each corner and, thereby, match the cubic spline interpolants along the mesh-lines.

It occurred to me that the same information could be matched by a scheme that would, say, construct the cubic spline interpolants along all the mesh-lines in the x-direction, and then use cubic spline interpolation to the resulting spline coefficients as a function of y to obtain an interpolant that was a cubic spline in x for every value of y, and C^2 rather than just C^1. Of course, one could equally well start with the cubic spline interpolants along all the mesh-lines in the y-direction, and interpolate the resulting spline coefficients as a function of x and so obtain an interpolant that was a cubic spline in y for every x, and it took me some effort to convince Birkhoff that these two interpolants are the same. This is now known as bicubic spline interpolation [dB0], the tensor-product (I learned that term from Don Thomas there) of univariate cubic spline interpolation, and has become a mainstay in the construction of smooth interpolants to gridded data. I did write up a paper on n-variable tensor product interpolation, but Birkhoff thought publication of such a paper unnecessary.

Much later (see [dB4]), I realized that it is quite simple to form and use in a multivariate context tensor products of univariate programs for the approximation and evaluation of functions, provided the univariate programs can handle vector-valued functions.

Around 1960, there was related work (I learned much later) by Feodor Theilheimer of the David Taylor Model Basin, see [TS], and, in computer graphics, parametric bicubic splines were introduced around that time by J. C. Ferguson at Boeing, see [Fe], though Ferguson set the crossderivatives $D_x D_y f$ at all mesh points to zero, thereby losing C^2 and introducing flat spots.

In this connection, I completely missed out on parametric spline work, believing (incorrectly, I now know) that it is sufficient to work with spline functions on a suitably oriented domain. Nor did I get involved in the blending approach to the construction of spline surfaces, even though I was invited by Garabedian on a visit in 1962 to Coons at M.I.T. (my first plane ride) and saw there, first-hand, an ashtray being machined as a Coons' surface [C]. The paper [BdB] (which has my name on it only grace Birkhoff's generosity) contains a summary of what was then known about multivariate splines. I had left General Motors Research by the time that Bill Gordon did his work on spline-blended surfaces there; see, e.g., [G].

2 Quasiinterpolation

My next foray into multivariate splines occurred in joint work with George Fix, though my contribution to [dBF] was the univariate part (Birkhoff objected to the publication of two separate papers). Fix had worked out the existence of a local linear map into the space of tensor-product splines of (coordinate-)degree $< k$ for a given mesh, which depended only on the value of derivatives of order $< k_0 \leq k$ at all the mesh points but did not necessarily reproduce those values (hence Fix' name "quasi-interpolate" for the resulting approximation) but did reproduce all polynomials of (total) degree $< k_0$ in such a way that the approximation error can be shown to be of order k_0 in the mesh size. However, there was an unresolved argument between Fix and his thesis advisor, Garrett Birkhoff, about whether, in the univariate case, Fix' scheme was "better" than Birkhoff's "Local spline approximation by moments" [B], and Birkhoff had invited me to Cambridge MA for July 1970 to settle the matter, perhaps. ([B] started out as a joint paper but, inexplicably, did not so end up; I published the case of even-degree splines later on in [dB1].) Fortunately, once I had derived an explicit formula for Fix' map, the two methods could easily be seen to be identical.

For $k_0 = k$, Fix's univariate scheme amounted to interpolation in the sense that it was a linear projector; nevertheless, it was called "quasi-interpolation" in the spirit of finite elements of that time since its purpose was not to match given function values but, rather, to match some suitable linear information in such a way that the process was local, stable, and reproduced all polynomials of order k, thus ensuring approximation order k. In this sense, Birkhoff's local spline approximation by moments is the first quasi-interpolation spline scheme I am aware of (with [dB2] a close and derivative-free second).

Unfortunately, it was only ten years later that I became aware of Frederickson's immediate reaction [Fr1] to [dBF] in which he constructed quasi-interpolant schemes onto smooth piecewise polynomials on what we now would call the 3-direction mesh, using bump functions obtained from the characteristic function of a triangle in the same way we now obtain a bivariate box spline from the characteristic function of a square; see [Fr2].

3 Multivariate B-splines

In 1972, I moved to Madison WI, to the Mathematics Research Center (MRC) funded since 1957 by the United States Army Research Office to carry out research in applied mathematics. It had an extensive postdoc and visitors program, the only fly in the ointment its location far from the center of the University of Wisconsin-Madison because its former housing there was bombed in August 1970, as a protest against the Vietman war, by people who took the very absence of any mention of military research in the semi-annual reports of that Army-financed institution as proof of the importance of the military research supposedly going on there. I had been hired at the time of I. J. Schoenberg's retirement from MRC.

The univariate spline theory was in good shape by that time, and, thanks to my contacts with Martin Schultz and George Fix, and to having been asked to handle the MRC symposium on the "Mathematical Aspects of Finite Elements in Partial Differential Equations" in the summer of 1973, I had begun to look at smooth piecewise polynomials in two and more variables, as they were being used in finite elements. That same summer, I participated in the Numerical Analysis conference in Dundee and heard Gil Strang's talk [St] there, in which he raised the question of the dimension of the space of bivariate $C^{(1)}$-cubics on a given triangulation. I felt like a fraud for not being able to solve that problem right then and there. As it turned out, except for "nice" triangulations, this problem is still not understood in 2009, and neither is the approximation power of such spaces known, although many have worked on it; see [LS] for what was known by 2007.

At the same time, in practice, the finite element method did not work with the space of *all* piecewise polynomials of a certain degree and smoothness on a given triangulation, but with suitable subspaces, usually the linear span of suitable compactly supported more or less smooth piecewise polynomials called bump or hill functions. This, together with the essential role played by B-splines in the univariate spline theory (as summarized, e.g., in [dB3]), made me look for "B-splines", i.e., smooth compactly supported piecewise polynomials, in the multivariate setting. When discussing this issue in January 1975 with Iso Schoenberg in his home study, he went to his files and pulled out a letter [Sc] he had written to Phil Davis in 1965, with a drawing of a bivariate compactly supported piecewise quadratic function, with several planar sections drawn in as univariate quadratic B-splines; see Figure 1. The letter was in response to Davis' paper [Dav], meant to publicize the following formula, due to Motzkin and Schoenberg,

$$\frac{1}{2A}\int_T f''(z)\,\mathrm{d}x\,\mathrm{d}y = \frac{f(z_0)}{(z_0-z_1)(z_0-z_2)} + \frac{f(z_1)}{(z_1-z_0)(z_1-z_2)} + \frac{f(z_2)}{(z_2-z_0)(z_2-z_1)}, \tag{1}$$

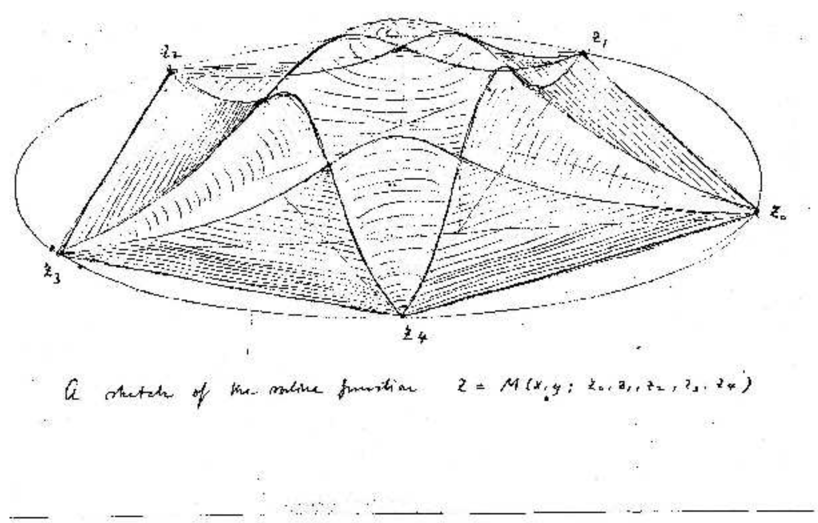


Fig. 1. Schoenberg's sketch of a bivariate quadratic B-spline

valid for all functions f regular in the triangle T in the complex plane with vertices z_0, z_1, z_2, and with A the area of T. Schoenberg points out that, in as much as the right side of (1) is the second divided difference $\Delta(z_0, z_1, z_2)f$ of f at z_0, z_1, z_2, therefore the Genocchi-Hermite formula for the nth divided difference

$$\Delta(z_0, \ldots, z_n)f = \int_0^1 \int_0^{s_1} \cdots \int_0^{s_{n-1}} f^{(n)}(z_0 + s_1 \nabla z_1 + \cdots + s_n \nabla z_n)\, \mathrm{d}s_n \cdots \mathrm{d}s_1 \quad (2)$$

provides a ready generalization of (1) to an arbitrary finite collection of z_i in the complex plane. Moreover, it is possible to write the integral as a weighted integral over the convex hull of the z_i, i.e., in the form

$$\Delta(z_0, \ldots, z_n)f = \int_{\mathrm{conv}(z_0, \ldots, z_n)} f^{(n)}(x + \mathrm{i}y) M(x, y; z_0, \ldots, z_n)\, \mathrm{d}x\, \mathrm{d}y,$$

with the value at (x, y) of the weight function $M(\cdot, \cdot; z_0, \ldots, z_n)$ the volume of $\sigma \cap P^{-1}\{(x, y)\}$, with P the orthogonal projector of $\mathbb{R}^n$ onto $\mathbb{C} \sim \mathbb{R}^2 \subset \mathbb{R}^n$, and σ any n-simplex of unit volume whose set of vertices is mapped by P onto $\{z_0, \ldots, z_n\}$. This makes $M(\cdot, \cdot; z_0, \ldots, z_n)$ the two-dimensional "X-ray" or "shadow" of an n-dimensional simplex. Hence, $M(\cdot, \cdot; z_0, \ldots, z_n)$ is piecewise polynomial in x, y of total degree $n - 2$, nonnegative, and nonzero only in the convex hull of the z_j, and, generically, in $C^{(n-3)}$. This is strikingly illustrated in Figure 1, which shows Schoenberg's sketch of the weight function for the case $n = 4$, with the z_j the five fifth-root of unity, giving a C^1 piecewise quadratic weight function.

I was much taken by this geometric construction since it immediately suggested a way to get a nonnegative partition of unity consisting of compactly supported smooth piecewise d-variate polynomials of order k: In $\mathbb{R}^k$, take a convex set C of unit k-dimensional volume (e.g., a simplex), and subdivide the cylinder $C \times \mathbb{R}^d$ into non-trivial ($k + d$-dimensional) simplices. Then their shadows on $\mathbb{R}^d$ under the orthogonal projection of $\mathbb{R}^{d+k}$ onto $\mathbb{R}^d$ provide that partition of unity. For the case $d = 1$, Schoenberg was very familiar with the resulting 1-dimensional shadows of $1 + k$-dimensional simplices. By the Hermite-Genocchi formula, they are univariate B-splines, a fact used by him in [CS] to prove the log-concavity of the univariate B-spline.

In a talk [dB3] at the second Texas conference in 1976, on the central role played by B-splines in the univariate spline theory, I finished with a brief discussion of what little I knew about Schoenberg's multivariate B-splines. In particular, I stressed the lack of recurrence relations to match those available for univariate B-splines, and should have pointed out that I had no idea (except when $d = 1$) how to choose the partition of $C \times \mathbb{R}^d$ into simplices in order to ensure that the linear span of the resulting d-dimensional shadows has nice properties. A very alliterative solution to this difficult problem was offered in [DMS] but, to me, the most convincing solution is the one finally given by Mike Neamtu; see [N] and the references therein (although Höllig's solution [H2] is not mentioned).

Subsequently, Karl Scherer informed me that his new "Assistent", Dr. Wolfgang Dahmen, intended to provide the missing recurrence relations. It seems that Scherer had given him [dB3] to read as an introduction to splines.

4 Kergin interpolation

In January 1978, I was asked by T. Bloom of Toronto (possibly because his colleague, Peter Rosenthal, and I had been students together at Ann Arbor) my opinion of a recent result of one of his students, Paul Kergin, and, for this purpose, was sent a handwritten draft of Kergin's Ph.D. thesis [K1]. The thesis proposed a remarkable generalization of univariate Lagrange interpolation from $\Pi_{\le k}$ at a $k + 1$-set $\mathrm{Z} = \{z_0, \ldots, z_k\}$ of sites to the multivariate setting, with the interpolant chosen uniquely from $\Pi_{\le k}$ (:= the space of polynomials in d variables of total degree $\le k$) and depending continuously on the sites even when there was coalescence and, correspondingly, Hermite interpolation. To be sure, in $d > 1$ dimensions, $\dim \Pi_{\le k} = \binom{k+d}{d}$ is much larger than $k + 1$, hence Kergin had to choose additional interpolation conditions in order to single out a particular element $Pf \in \Pi_{\le k}$ for given f. This he did in the following way. He required that P be linear and such that, for every $0 \le j \le k$ and every homogeneous polynomial q of degree j, and every $j + 1$-subset Σ of Z, $q(D)(\mathrm{id} - P)f$ should vanish at some site in $\mathrm{conv}(\Sigma)$.

The thesis (and subsequent paper [K2]) spends much effort settling the question of how all these conditions could be satisfied simultaneously, and, in discussions with members and visitors at MRC that Spring, we looked for some simplification. Michael Golomb pointed to the "lifting" Kergin used in his proof as a possible means for simplification: If the interpoland f is a "ridge function", i.e., of the form $g \circ \lambda$ with λ a linear functional on $\mathbb{R}^d$, then Pf is of the same form; more precisely, then $Pf = (Qg) \circ \lambda$, with Qg the univariate polynomial interpolant to g at the possibly coalescent sites $\lambda(\mathrm{Z})$.

Fortunately, C. A. Micchelli was visiting MRC that year, from 1apr to 15sep, and readily entered these ongoing discussions on Kergin interpolation (and the missing recurrence relations for multivariate B-splines). He extended (see [M1]) the linear functional occurring in the Genocchi-Hermite formula (2) to functions of d variables by setting

$$\int_{[z_0,\ldots,z_n]} h := \int_0^1 \int_0^{s_1} \cdots \int_0^{s_{n-1}} h(z_0 + s_1 \nabla z_1 + \cdots + s_n \nabla z_n)\, \mathrm{d}s_n \cdots \mathrm{d}s_1 \qquad (3)$$

for arbitrary $z_0, \ldots, z_n \in \mathbb{R}^d$, recalled the Newton form

$$Qg = \sum_{j=0}^{k} (\cdot - \lambda z_0) \cdots (\cdot - \lambda z_{j-1})\, \Delta(\lambda z_0, \ldots, \lambda z_j) g$$

for the univariate polynomial interpolant to g at the sites $\lambda(\mathrm{Z})$, and realized that, with $D_y := \sum_j y_j D_j$ the directional derivative in the direction y,

$$D_{x-z_0} \cdots D_{x-z_{j-1}} (g \circ \lambda) = \lambda(x - z_0) \cdots \lambda(x - z_{j-1}) (D^j g) \circ \lambda,$$

hence, using Genocchi-Hermite, saw that

$$\begin{aligned}(Qg) \circ \lambda &= \sum_{j=0}^{k} \lambda(\cdot - z_0) \cdots \lambda(\cdot - z_{j-1}) \int_{[\lambda z_0, \ldots, \lambda z_j]} D^j g \\ &= \sum_{j=0}^{k} \int_{[z_0, \ldots, z_j]} D_{\cdot - z_0} \cdots D_{\cdot - z_{j-1}} (g \circ \lambda),\end{aligned}$$

and so knew that the ansatz

$$Pf = \sum_{j=0}^{k} \int_{[z_0, \ldots, z_j]} D_{\cdot - z_0} \cdots D_{\cdot - z_{j-1}} f$$

for the Kergin projector was correct for all ridge functions (given Kergin's result concerning interpolation to ridge functions), hence must be correct.

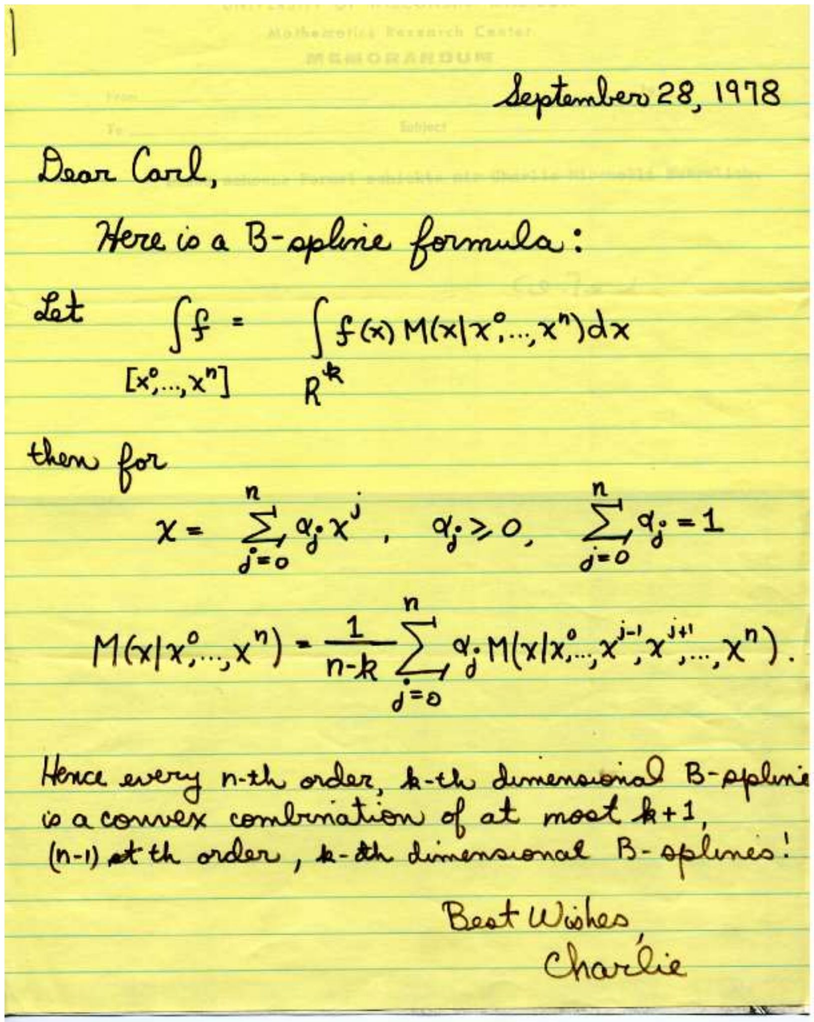
I remember the exact spot on the blackboard in the coffee room at MRC where Micchelli wrote this last formula down for me, and can still experience my astonishment and admiration. I had no inkling that this was coming, hence declined his gracious offer of making this a joint paper.

It turned out that P. Milman, who is acknowledged in [K2] for many helpful discussions, also had this formula, resulting in [MM].

5 The recurrence for multivariate B-splines

Shortly after Micchelli had left MRC that fall, I received from him the one-page letter shown in Figure 2, containing the sought-after recurrence relations for multivariate B-splines, a second occasion for me to be astonished. Micchelli had not made my mistake, of concentrating on the geometric definition of the multivariate B-spline, but had stuck with the setting in which Schoenberg first thought of these multivariate B-splines, namely as the representers of the "divided difference" functionals $f \mapsto \int_{[z_0,\ldots,z_n]} f$ defined in (3).

The formula was first published in MRC TSR 1895 in November 1978, a preliminary version of [M1].



Mathematics Research Center

MEMORANDUM

September 28, 1978

Dear Carl,

Here is a B-spline formula:

Let
$$\int_{[x^0,\ldots,x^n]} f = \int_{R^k} f(x)\, M(x|x^0,\ldots,x^n)\,dx$$

then for

$$x = \sum_{j=0}^{n} \alpha_j x^j, \quad \alpha_j \geq 0, \quad \sum_{j=0}^{n} \alpha_j = 1$$

$$M(x|x^0,\ldots,x^n) = \frac{1}{n-k} \sum_{j=0}^{n} \alpha_j M(x|x^0,\ldots,x^{j-1},x^{j+1},\ldots,x^n).$$

Hence every n-th order, k-th dimensional B-spline is a convex combination of at most k+1, (n-1) st th order, k-th dimensional B-splines!

Best Wishes,

Charlie

Fig. 2. Micchelli's recurrence relation for simplex splines

The paperclip shown in the upper left corner of Figure 2 holds a copy of an MRC memo, saying: "Diese schoene Formel schickte mir Charlie Micchelli kuerzlich. Ihr Carl de Boor". The memo accompanied a copy of Micchelli's letter which I mailed to Wolfgang Dahmen, knowing from my short visit to Bonn in August 1978 that he thought he was on the track to getting recurrence relations.

Dahmen's response was swift: in a missive dated 30oct78, he submitted to me directly for possible publication in SJNA the first version of [D5], containing a proof of the recurrence relations but based on what we now call multivariate truncated powers or cone splines since they can be thought of as shadows of high-dimensional polyhedral cones. A second version reached me 14nov78 which I promptly sent to Micchelli for refereeing, who was wondering how Dahmen could have found out so quickly about his formula. In January, Micchelli asked permission (granted, of course) to contact Dahmen directly during his visit to Germany in February, and this led to Dahmen's application (granted, of course) for a research fellowship at Micchelli's home institution, the mathematics department at IBM Watson Research Center in Yorktown Heights NY, and the rest is history. While Dahmen published various results on multivariate B-splines alone, including papers in conference proceedings [D1], [D3], [D4], the construction of spaces spanned by such B-splines and their approximation order [D6], requiring the determination of the polynomials contained in such a span [D2], all leading up to his Habilitationsschrift [D7], his joint results with Micchelli on the mathematics of box splines were the pay-off of their joining forces in 1979. But, for that, the box splines had to make their appearance first.

6 Polyhedral splines

It must have happened during my visit with Ron DeVore at the University of South Carolina in April 1980 that he and I started a discussion on the relative merits in multivariate piecewise polynomial approximation of using total degree *vs.* coordinate degree whose outcome is [dBD]. The discussion was motivated by the fact that the approximation order achievable from a space A_h, of piecewise polynomials on a partition of mesh size h, is bounded by the maximum k for which $\Pi_{\leq k}$ is contained in the approximation space A_h, and it seems that a tensor product spline space of coordinate degree k employs many more degrees of freedom (involving polynomial pieces of total degree $> k$) than seem necessary to have $\Pi_{\leq k}$ contained in it.

To be sure, it is not sufficient to have $\Pi_{\leq k} \subset A_h$ (see, e.g., [dBH3]); rather, $\Pi_{\leq k}$ must be contained in A_h locally and stably, i.e., there must be a (local and stable) quasiinterpolant scheme with range A_h available that reproduces $\Pi_{\leq k}$. It is this requirement that becomes increasingly hard and eventually, impossible if one increases the required smoothness of the approximating piecewise polynomials of order $\leq k$ for a given partition or mesh. We only considered the bivariate case and considered only two partitions, a square mesh, and, in order to get some feeling for triangulations, the square mesh with all northeast diagonals drawn in (now called a 3-direction mesh or uniform type I triangulation). But how to get the smooth compactly supported piecewise polynomials needed? In the case of the 3-direction mesh, Courant's hat function offers itself for degree 1 and smoothness 0. But it was the sudden (and very pleasant) realization that this function is the 2-dimensional (skewed) shadow of a 3-cube that provided us with a recipe for the needed "bump functions" for the 3-direction mesh, as appropriate shadows of higher-dimensional cubes. We realized that other finite elements, e.g., the piecewise quadratic finite element constructed by Powell in [P], and, earlier, by Zwart in [Z], or certain elements discussed by Sablonnière, see [Sa], as well as those constructed by Sabin [S], could also be obtained as shadows of higher-dimensional cubes.

However, these new multivariate B-splines might not have been looked at carefully all that quickly but for the arrival at MRC, in the summer of 1980, of Klaus Höllig, for a 2-year postdoc. I had met Höllig the previous summer during an extended stay with Karl Scherer at the University of Bonn (during which Ron DeVore and I worked successfully in a local "Weinstube" on a problem of mixed-norm n-width that had arisen in Höllig's thesis work; see [dBDH]). Höllig produced in short order the two papers [H1], [H2], rederiving Micchelli's (and Dahmen's) results via Fourier transforms, and proposing a particular way of choosing a collection of simplices so that their shadows span a linear space of piecewise polynomials of order k with approximation order k. But, more than that, Höllig was swift to follow up on the suggestion that Micchelli's recurrence might be a simple consequence of Stokes' theorem, hence there is a version for shadows of cubes and, more generally, for shadows of convex polyhedra, as follows.

In the spirit of Micchelli's view of Schoenberg's multivariate B-spline, for a convex body B in $\mathbb{R}^n$ and a linear map P from $\mathbb{R}^n$ to $\mathbb{R}^d$, define the corresponding distribution M_B on $\mathbb{R}^d$ by

$$M_B\varphi := \int_B \varphi \circ P, \qquad \text{all test functions } \varphi,$$

with $\int_K$ the k-dimensional integral over the convex K in case the flat $\flat(K)$ spanned by K is k-dimensional. Assuming that $\flat(P(B)) = \mathbb{R}^d$, M_b is a nonnegative piecewise polynomial function, with $P(B)$ its support. Moreover, at each corner of its support, it agrees with one of Dahmen's truncated powers.

Assume that the boundary of B is the essentially disjoint union of finitely many $(n-1)$-dimensional convex bodies B_i. Then

$$D_{Pz}M_B = -\sum_i \langle z, n_i\rangle M_{B_i}, \qquad z \in \mathbb{R}^n, \tag{4}$$

$$(n-d)M_B(Pz) = \sum_i \langle b_i - z, n_i\rangle M_{B_i}(Pz), \qquad z \in \mathbb{R}^n, \tag{5}$$

with n_i the outside normal to $\flat(B_i)$, $\langle x, y\rangle$ the scalar product of x with y, and b_i a point in B_i, hence $\langle b_i - z, n_i\rangle$ is the signed distance of z from $\flat(B_i)$. The pointwise equality has to be taken in the sense of distributions. The proof of (5) in [dBH1] follows Hakopian's proof of (5) in [Ha] for the special case that B is a simplex. Under the assumption that B is a convex polytope, repeated application of (4) establishes that M_B is piecewise polynomial of degree at most $n-d$, and in $C^{n-\rho-2}$, with ρ the greatest integer with the property that a ρ-dimensional face of B is mapped by P into a $(d-1)$-dimensional set.

In [dBH2], we called M_B a polyhedral spline. Schoenberg's B-spline became a simplex spline, Dahmen's truncated power a cone spline, and the one introduced in [dBD] a box spline (though Micchelli prefers "cube spline"). These three examples seem, at present, the only ones carefully studied, probably because their polyhedra are the only ones whose facets are polyhedra of the same type.

7 Box splines

In contrast to the simplex splines, the construction of a collection of box splines spanning a useful space of piecewise polynomials is quite simple. If the box spline in question is

$$M := M_\Xi : \varphi \mapsto \int_{[0..1)^\Xi} \varphi(\Xi x)\,\mathrm{d}x$$

for some multiset or matrix Ξ of full rank of integer-valued nontrivial directions in $\mathbb{R}^d$, then

$$S(\Xi) := S_{M_\Xi} := \mathrm{span}(M_\Xi(\cdot - j) : j \in \mathbb{Z}^d)$$

is a cardinal, i.e., shift-invariant, spline space which contains all polynomials of (total) degree k where k is maximal with respect to the property that, for any k-subset Z of Ξ, $\Xi\backslash\mathrm{Z}$ is still of full rank. The full space of polynomials contained in $S(\Xi)$ is, in general, larger; it is denoted by $D(\Xi)$; it is the joint kernel of the differential operators $D_\mathrm{H} := \prod_{\eta\in\mathrm{H}} D_\eta$ where H ranges over the set $\mathcal{A}(\Xi)$ of all $\mathrm{H} \subset \Xi$ that intersect every basis in Ξ. In this connection, for any $\mathrm{Z} \subset \Xi$, $D_\mathrm{Z} M_\Xi = \nabla_\mathrm{Z} M_{\Xi\backslash\mathrm{Z}}$ and, in particular, $D_\Xi M_\Xi = \nabla_\Xi \delta$, with $\delta : \varphi \mapsto \varphi(0)$. Also, $M_\Xi * M_\mathrm{Z} = M_{\Xi\cup\mathrm{Z}}$. However, linear independence of $(M(\cdot - j) : j \in \mathbb{Z}^d)$ cannot hold unless Ξ is "unimodular", i.e., $|\det \mathrm{Z}| = 1$ for all bases $\mathrm{Z} \subset \Xi$. Nevertheless, even when there is no linear independence, one can construct, for $k_0 \le k$, a quasi-interpolant scheme Q into $S(\Xi)$ whose dilation $Q_h : f \mapsto Qf(\cdot/h)(\cdot h)$ provides approximation of order h^{k_0} for every smooth enough f. It is also clear that Schoenberg's theory of univariate cardinal spline interpolation (see, e.g., [Sc2]) can be extended to multivariate box spline interpolation in case of linear independence of $(M(\cdot - j) : j \in \mathbb{Z}^d)$ (a beginning is made in [dBHR]), and that the Strang-Fix theory [FS] of the approximation order of spaces spanned by the shifts of one function is applicable here.

While Höllig and I derived such basic results, eventually published in [dBH2], Dahmen and Micchelli pursued, unknown to us, vigorously much bigger game. We first learned details of their remarkable results from their survey [DM3] in the proceedings of the January 1983 Texas conference and from their summary [DM2] submitted in August 1983, with the former the only reference in the latter, and from reading [DM1], [DM4] and [DM5] for the details of some of the results announced in [DM2].

Not only did they prove that $(M_\Xi(\cdot - j) : j \in \mathbb{Z}^d)$ is (globally or locally) linearly independent iff Ξ is unimodular (something proved independently by Jia [J1], [J2]), they showed that the volume of the support of M_Ξ equals the number of $j \in \mathbb{Z}^d$ for which the support of $M_\Xi(\cdot - j)$ has a nontrivial intersection with the support of M_Ξ, and showed that support to be the essentially disjoint union of $\tau_\mathrm{Z} + \mathrm{Z}[0\,..\,1]^\mathrm{Z}$ for suitable τ_Z as Z runs over the set $\mathcal{B}(\Xi)$ of bases in Ξ, hence $\mathrm{vol}_d(M_\Xi[0\,..\,1]^\Xi) = \sum_{\mathrm{Z}\in\mathcal{B}(\Xi)} |\det \mathrm{Z}|$. They also completely characterized the space $E(\Xi)$ of linear dependence relations for $(M_\Xi(\cdot - j) : j \in \mathbb{Z}^d)$, i.e., the kernel of the linear map $M_\Xi^* : \mathbb{C}^{\mathbb{Z}^d} \to S(\Xi) : c \mapsto \sum_j M_\Xi(\cdot - j)c(j)$ (with the sum well-defined pointwise), and showed the space of polynomials in $S(\Xi)$, i.e., the joint kernel $D(\Xi)$ of the differential operators D_H, $\mathrm{H} \in \mathcal{A}(\Xi)$, to have dimension equal to $\#\mathcal{B}$. Remarkably, this last assertion holds even without the restriction that Ξ be an integer matrix.

But there is more. Recall the truncated power $T_\Xi : \varphi \mapsto \int_{\mathbb{R}_+^\Xi} \varphi(\Xi x)\,\mathrm{d}x$ introduced by Dahmen in [D5] for the case that $0 \notin \mathrm{conv}(\Xi)$, i.e., the shadow of a cone. Already in [DM2], Dahmen and Micchelli define, under the assumption that $0 \notin \mathrm{conv}(\Xi)$, the discrete truncated power $t(\cdot|\Xi)$ associated with Ξ as the map on $\mathbb{Z}^d$ for which

$$\sum_{\alpha\in\mathbb{Z}_+^\Xi} \varphi(\Xi\alpha) =: \sum_{j\in\mathbb{Z}^d} t(j|\Xi)\varphi(j)$$

for any finitely supported φ, hence $t(j|\Xi) = \#\{\alpha \in \mathbb{Z}^{\Xi} : \Xi\alpha = j\}$. In other words, $t(\cdot|\Xi)$ counts the number of nonnegative integer solutions for the linear system $\Xi? = j$ with integer coefficients. They prove $T_\Xi = \sum_{j\in\mathbb{Z}^d} t(j|\Xi)M_\Xi(\cdot - j)$, and so obtain the remarkable formula $\nabla_\Xi T_\Xi = M_\Xi$. Their subsequent study of the discrete truncated power enabled them, as reported in [DM6], to reprove certain conjectures concerning magic squares, thus opening up a surprising application of box spline theory.

On the other hand, box splines have had some difficulty in being accepted in areas of potential applications. A particularly striking example is Rong-Qing Jia's beautiful paper [J3] which contains a carefully crafted account of the relevant parts of the theory used in his proof of a long-outstanding conjecture of Stanley's concerning the number of symmetric magic squares. Referees from Combinatorics seemed unwilling to believe that such conjectures could be successfully tackled with spline theory.

In good part because of these (and other) results of Dahmen and Micchelli, there was a great outflow of work on box splines in the 80s, and it was hard to keep up with it. For this reason, Höllig, Riemenschneider and I decided to try to tell the whole story in a cohesive manner, resulting in [dBHR2].

I now wish we had included in the book the exponential box splines of Amos Ron [R] (followed closely by [DM7]). For, as Amos Ron has pointed out to me since (and is made clear in [BR]), the (polynomial) box splines can be understood as a limiting situation of the much simpler setup of exponential box spline. Here is an example.

Recall the Dahmen-Micchelli result that the dimension of the space $D(\Xi)$ of polynomials in the span $S(\Xi)$ of the shifts of the box spline M_Ξ equals the number $\#\mathcal{B}(\Xi)$ of bases in Ξ (provided Ξ is of full rank). This is (II.32)Theorem in the book, and its proof (a version of the Dahmen-Micchelli proof) is inductive and takes about three pages, with the main issue the claim that $\dim D(\Xi) \ge \#\mathcal{B}(\Xi)$. However, this inequality is almost immediate along the following lines suggested by Amos Ron: Choose, as we may, $\lambda : \Xi \to \mathbb{R}$ so that $(p_\xi : x \mapsto \langle x, \xi\rangle - \lambda(\xi))$ is generic, meaning that the unique common zero, v_B say, of $(p_\xi : \xi \in B)$ is different for different $B \in \mathcal{B}(\Xi)$. Consider $\mathrm{H} \in \mathcal{A}(\Xi)$. Since H intersects each $B \in \mathcal{B}(\Xi)$, the polynomial $p_{\mathrm{H}} := \prod_{\eta\in\mathrm{H}} p_\eta$ vanishes on $V := \{v_B : B \in \mathcal{B}(\Xi)\}$. Let $\mathrm{e}_v : x \mapsto \exp(\langle v, x\rangle)$. Then, for arbitrary $y \in \mathbb{R}^d$, $D_y\mathrm{e}_v = \langle v, y\rangle \mathrm{e}_v$, hence, for $p \in \Pi$, $p(D)\mathrm{e}_v = p(v)\mathrm{e}_v$. In particular, $p_{\mathrm{H}}(D)\mathrm{e}_v = 0$ for $v \in V$, hence $p_{\mathrm{H}}(D)f = 0$ for arbitrary $f = \sum_\alpha \widehat{f}(\alpha)()^\alpha \in \mathrm{Exp}(V) := \mathrm{span}\{\mathrm{e}_v : v \in V\}$. But $p(D)f = 0$ implies $p_\uparrow(D)f_\downarrow = 0$, with $p_\uparrow$ the "leading term" of p, i.e., the homogeneous polynomial for which $\deg(p - p_\uparrow) < \deg p$ and, correspondingly, $f_\downarrow$ the "least term" of f, i.e., the homogeneous polynomial for which $\mathrm{ord}(f - f_\downarrow) > \mathrm{ord} f := \min\{|\alpha| : \widehat{f}(\alpha) \ne 0\}$. Since $(p_{\mathrm{H}})_\uparrow(D) = D_{\mathrm{H}}$ and H was an arbitrary element of $\mathcal{A}(\Xi)$, it follows that $D(\Xi) = \cap_{\mathrm{H}\in\mathcal{A}(\Xi)} \ker D_{\mathrm{H}} \supset \mathrm{Exp}(V)_\downarrow := \mathrm{span}\{f_\downarrow : f \in \mathrm{Exp}(V)\}$. However, $\mathrm{Exp}(V)_\downarrow$ has dimension $\ge \#V = \#\mathcal{B}(\Xi)$, since $(\delta_v : v \in V)$ is linearly independent on $\mathrm{Exp}(V)_\downarrow$. Indeed, for any $v \in \mathbb{R}^d$ and $p \in \Pi$, $p(v) = (p(D)\mathrm{e}_v)(0)$, hence if $\sum_{v\in V} c(v)\delta_v = 0$ on $\mathrm{Exp}(V)_\downarrow$ yet $(c(v) : v \in V) \ne 0$, then $f := \sum_{v\in V} c(v)\mathrm{e}_v \ne 0$ and so $0 = f_\downarrow(D)f = \sum_{|\alpha|=\mathrm{ord} f} \widehat{f}(\alpha)^2 \alpha! \ne 0$ which is nonsense.

8 Smooth multivariate piecewise polynomials and the B-net

I had given up quite early on the study of the space of all piecewise polynomials of a given order and smoothness on a given partition in more than one variable, preferring instead the finite element method approach of seeking suitable spaces of smooth piecewise polynomials spanned by bump functions. This was surely quite narrow-minded of me as, starting in the 70's, a very large, interesting and often challenging literature developed whose results are very well reported in the recent comprehensive book [LS] by Ming-Jun Lai and Larry Schumaker.

However, in the early 80's, Peter Alfeld, as a visitor at MRC, introduced me to the wonderful tool of what is now called the B-form. In this representation, the elements of the space $S_k^{(\rho)}(\Delta)$ of piecewise polynomials of degree $\le k$ on the given triangulation Δ and in $C^{(\rho)}$ are represented, on each triangle $\tau = \mathrm{conv}(V)$ in Δ, in the form

$$p = \sum_{|\alpha|=k} c(\alpha)\binom{|\alpha|}{\alpha}\ell^\alpha, \tag{6}$$

with $\alpha = (\alpha(v) : v \in V) \in \mathbb{Z}_+^V$, $\binom{|\alpha|}{\alpha} := |\alpha|!/\prod_v \alpha(v)!$, $\ell^\alpha := \prod_{v\in V}(\ell_v)^{\alpha(v)}$, and with $\ell_v := \ell_{v,\tau}$ the affine polynomial that vanishes on $V\backslash v$ and takes the value 1 at v, i.e., the ℓ_v are the Lagrange polynomials for linear interpolation to data given at V, hence $(\ell_{v,\tau}(x) : v \in V)$ are the socalled barycentric coordinates of x with respect to the vertex set V of τ. Further, it turns out to be very helpful to associate the coefficient $c(\alpha) = c(\alpha,\tau)$ with the "domain point"

$$\xi_{\alpha,\tau} := \sum_{v\in V}\alpha(v)v/k$$

(which happens to be the location of the unique maximum of ℓ_v^α (in τ)). For example, $v \in V$ is a domain point, namely $\xi_{k\delta_v,\tau}$, with δ_v the vector whose only nonzero entry is a 1 in position v, and all ℓ_w with $w \ne v$ vanish at that point, hence the corresponding coefficient, $c(k\delta_v)$, equals $p(v)$. More generally, on the edge of τ not containing v, i.e., on the zero set of ℓ_v, the only terms in (6) not obviously zero are those with $\alpha(v) = 0$, i.e., whose domain point lies on that edge. Hence continuity across that edge of a piecewise polynomial function is guaranteed by having the B-form coefficients of the two polynomial pieces abutting along that edge agree in the sense that coefficients associated with the same domain point coincide. Tbis sets up a 1-1 linear correspondence between the elements of $S_k^{(0)}(\Delta)$ and their "B-net", i.e., the scalar-valued map $\xi_{\alpha,\tau} \mapsto c(\alpha,\tau)$ on $\{\xi_{\alpha,\tau} : \alpha \in \mathbb{Z}_+^V, |\alpha| = k; \tau \in \Delta\}$.

Further, for any vector y, $D_y\ell_v = \ell_{v\uparrow}(y)$, with $\ell_{v\uparrow}$ the homogeneous linear part of the affine map ℓ_v, hence

$$D_y \sum_{|\alpha|=k} c(\alpha)\binom{|\alpha|}{\alpha}\ell^\alpha = k\sum_{|\beta|=k-1}\left(\sum_{v\in V} c(\beta+\delta_v)\ell_{v\uparrow}(y)\right)\binom{|\beta|}{\beta}\ell^\beta. \tag{7}$$

Hence, as Gerald Farin, in [Fa], was the first to stress, C^1-continuity across the edge of τ not containing v is guaranteed by the equalities

$$\sum_{w\in V} c(\beta+\delta_w)\ell_{w,\tau\uparrow}(y) = \sum_{w\in V'} c(\beta+\delta_w)\ell_{w,\sigma\uparrow}(y), \qquad |\beta| = k-1, \beta \in \mathbb{Z}_+^{V\cap V'},$$

with V' the vertex set of the triangle σ sharing that edge with τ. Note that the coefficients in these homogeneous equations are independent of the index β.

It is clear how ρ-fold iteration of this process produces the homogeneous linear equations that the B-net coefficients of an element of $S_k^{(0)}(\Delta)$ must satisfy for C^ρ continuity across the edge of τ not containing v. Each such equation involves the "quadrilateral" of coefficients $c(\beta+\gamma)$ and $c(\beta+\gamma')$, with $\beta \in \mathbb{Z}_+^{V \cap V'}$, $|\beta| = k - \rho$, and, $\gamma \in \mathbb{Z}_+^V$, $\gamma' \in \mathbb{Z}_+^{V'}$, $|\gamma| = \rho = |\gamma'|$.

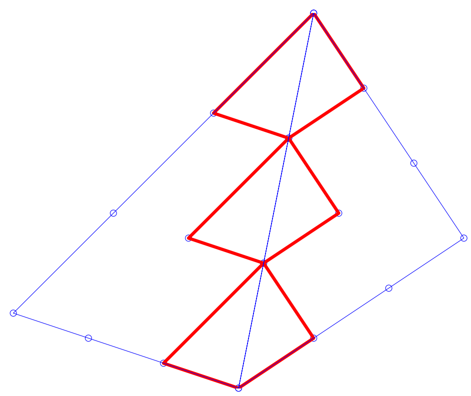


Fig. 3. C^1-conditions across an edge in the cubic case.

In Figure 3, the situation is illustrated for the cubic case, $k = 3$. It shows the relevant domain points in the two triangles τ and σ sharing an edge, as well as the quadruples of domain points whose corresponding B-net coefficients must satisfy the *same* homogeneous linear equation for C^1-continuity across that edge.

This figure makes it immediate why the question of the dimension and approximation order of the space of bivariate C^1-cubics on a given triangulation might be difficult: there is only one domain point in the interior of each triangle, and its coefficient is involved in three homogeneous equations. Hence, the determination of an element of $S_3^{(1)}(\Delta)$ involves a global linear system. Correspondingly, it is not even clear whether there is an element of $S_3^{(1)}(\Delta)$ with prescribed values at the vertices of all the triangles, i.e., with the B-net coefficients corresponding to the vertices prescribed.

On the other hand, it has been known for some time that there is a local quasi-interpolant onto $S_5^{(1)}(\Delta)$ reproducing $\Pi_{\le 5}$ for any triangulation Δ (though its stability will depend on the smallest angle in the triangulation). Checking the geometry of the smoothness conditions, one realizes (see Figure 4) that 5 is the smallest value of k for which there is on each edge a "free" C^1-smoothness condition, i.e., one not touching a smoothness condition for any other edge. This led to the guess that, in the general case, $S_k^{(\rho)}(\Delta)$ has a local quasi-interpolant reproducing $\Pi_{\le k}$ if there is a "free" C^ρ-smoothness condition on each edge, i.e., one not belonging to the "ring" of C^ρ-smoothness conditions associated with some vertex v by virtue of the fact that its edge and the edge of a smoothness condition it touches both contain v. For, one could hope to use such "free" conditions to "disentangle" or separate neighboring

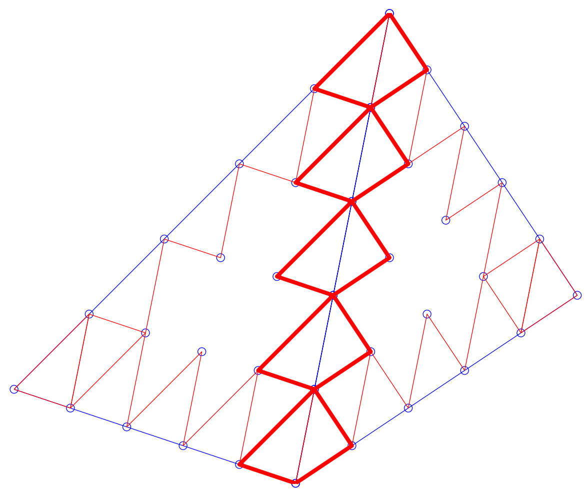


Fig. 4. C^1-conditions across an edge in the quintic case.

vertex rings. If ξ_α is the apex of such a “free” $C^{(\rho)}$-condition, it would have $\alpha(v) = \rho$ for some v, and would have $\alpha(w) > \rho$ for all $w \in V\backslash v$, hence $k = |\alpha| \geq 3\rho + 2$. For that case, [dBH6] contains a “proof” that, for a triangulation in which all the angles are bounded below by a constant, the approximation order is full, i.e., of the order h^{k+1}, where h is the mesh size. Unfortunately, the “proof” fails to take into account the possibility that the quadrangles corresponding to smoothness conditions across an edge can become nearly, but not exactly, flat which spoils a certain estimate on which the “proof” relies. This is explained in more detail in [dB6] which also contains a detailed account of the construction of a local basis for such spaces. A satisfactory proof of the main claim of [dBH6] was first given in [CHJ].

The above description of B-form and B-net readily applies to d dimensions (with the role of triangles played by d-simplices and the role of edges played by faces). However, in d dimension, existence of “free” $C^{(\rho)}$-smoothness conditions requires $k \geq (d + 1)\rho + d$ for a generic partition into simplices. In particular, already for $d = 3$ one would need $k \geq 7$ for $C^{(1)}$, which discouraged me from pursuing the study of all smooth piecewise polynomials on a “triangulation” in higher dimension.

Another result using B-nets in an essential way was the discovery in [dBH3] that, even on a certain regular triangulation, namely the 3-direction mesh Δ_3, $S_3^{(1)}(\Delta_3)$ does not have full approximation order, even though the space contains $\Pi_{\leq 3}$ locally. This has been reproved in more generality and with very different methods in [dBDR].

Altogether, the appearance of the B-net revolutionized the analysis of smooth piecewise polynomials even (and particularly) in the bivariate case, as is illustrated by its prominence in [LS].

3aug11 10//6: subset `-->` k-subset

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