Controlled approximation and a characterization of the local approximation order[1]

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Abstract. The local approximation order from a scale (S_h) of approximating functions on $\mathbb{R}^m$ is characterized in terms of the linear spac (and its Fourier transform) of the finitely many compactly supported functions φ whose integer translates $\varphi(\cdot - j)$, $j \in \mathbb{Z}^m$, span the space $S = S_1$ from which the scale is derived. This provides a correction of similar results stated and proved, in part, by Strang and Fix.

The term "controlled approximation" was introduced in 1970 by Strang [**St**]; see also [**FS**]. It concerns approximations of the form $\sum_{\varphi\in\Phi} \varphi * c_\varphi$, with Φ a finite collection of functions on $\mathbb{R}^m$ of compact support and $\varphi * c$ the function obtained from φ by convolution with some "sequence" $c : \mathbb{Z}^m \to \mathbb{R}$; i.e.,

$$\varphi * c := \sum_{j\in\mathbb{Z}^m} \varphi(\cdot - j)c(j).$$

If, more generally, c is some function on $\mathbb{R}^m$, we will still just write $\varphi * c$ instead of the correct but more complicated $\varphi * (c|_{\mathbb{Z}^m})$.

The function u to be approximated lies in the Sobolev space $W_p^k(\mathbb{R}^m)$ with norm

$$\|u\|_{k,p} := \sum_{j\le k} |u|_{j,p},$$

where

$$|u|_{j,p} := \sum_{|\alpha|=j} \|D^\alpha u\|_p$$

and

$$\|u\|_p := \|u\|_{L_p(\mathbb{R}^m)}.$$

We denote by $W_{p,c}^k(\mathbb{R}^m)$ the subspace of $W_p^k(\mathbb{R}^m)$ of compactly supported functions.

The approximations are, more explicitly, of the form

$$\sigma_h \left(\sum_{\varphi\in\Phi} \varphi * c_\varphi^h \right) / h^{m/p} \quad \text{with } \sigma_h f := f(\cdot/h).$$

Concerning the degree of approximation to $u \in W_p^k(\mathbb{R}^m)$ achievable by proper choice of the weights c_φ^h, Strang and Fix [**SF**, Theorem II] state the following result. In its statement and subsequent analysis, the normalized multivariate monomials appear often enough to deserve an abbreviation of their own. We will use $[\![\ ]\!]^\alpha$ to stand for the function $\mathbb{R}^m \to \mathbb{R} : \ x \mapsto \ x^\alpha/\alpha!$ (and will use standard multi-index notation throughout). In particular, $D^\beta [\![\ ]\!]^\alpha = [\![\ ]\!]^{\alpha-\beta}$, and this holds even when $\beta \not\le \alpha$, since then $[\![\ ]\!]^{\alpha-\beta} = 0$, by convention. Further, Π_j will denote the collection of polynomials on $\mathbb{R}^m$ of total degree $\le j$. Finally, $\widehat{f}$ will denote the Fourier transform of f — i.e., $\widehat{f}(\xi) := \int_{\mathbb{R}^m} e^{-i\xi x} f(x)\, dx$, with ξx the scalar product.

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Theorem SF. *Let Φ be a finite subset of $W_{2,c}^{k-1}(\mathbb{R}^m)$. Then the following are equivalent:*

(i) *There exists a sequence $(\psi_\alpha)_{|\alpha|<k}$ in* span Φ *which satisfies*

(ia) $\widehat{\psi}_0(0) = 1$, $\widehat{\psi}_0 = 0$ *on* $2\pi\mathbb{Z}^m\backslash 0$;

(ib) $\sum_{\beta\le\alpha} [\![-iD]\!]^\beta \widehat{\psi}_{\alpha-\beta} = 0$ *on* $2\pi\mathbb{Z}^m$ *for* $0 < |\alpha| < k$.

(ii) *There exists a sequence $(\psi_\alpha)_{|\alpha|<k}$ which satisfies*

$$[\![\,]\!]^\alpha = \sum_{\beta\le\alpha} \psi_{\alpha-\beta} * [\![\,]\!]^\beta \quad \textit{for} \quad |\alpha| < k.$$

(iii) *There exist some finitely supported c_φ so that $\psi := \sum_{\varphi\in\Phi} \varphi * c_\varphi$ satisfies $\widehat{\psi}(0) \neq 0$, but $D^\alpha\widehat{\psi} = 0$ on $2\pi\mathbb{Z}^m\backslash 0$ for $|\alpha| < k$.*

(iv) *For each $u \in W_2^k(\mathbb{R}^m)$ there exist weights c_φ^h so that*

(iva) $\|u - \sigma_h(\sum_\varphi \varphi * c_\varphi^h)/h^{m/2}\|_{s,2} \le \text{const}_s h^{k-s}|u|_{k,2}$, $s = 0, \dots, k-1$;

(ivb) $\sum_\varphi \|c_\varphi^h\|_2^2 \le \text{const}\|u\|_2^2$.

For the very special case when $m = \#\Phi = 1$, such results can already be found in [**Sc**]. In [**SF**] the special case when Φ consists of just one function (but m is arbitrary) is treated first (see [**SF**, Theorem I]) and completely. However, for the general case, [**SF**] only gives proof for the implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv). In particular, the validity of (iv) $\Rightarrow$ (i) has recently been questioned. This was finally settled by Jia [**J**], who shows by a counterexample that (iv) does *not* imply (i) in general.

This raises the question of how to modify (iv) to obtain something equivalent to (i). This is a matter of changing the *control* over the form of the approximation as expressed by (ivb). In this connection it is very useful to recall that Dahmen and Micchelli [**DM**] quote Theorem SF in a modified form. The modification of importance here occurs in condition (iv), which they require to hold locally, as follows:

(iv)′ *For each $u \in W_p^k(\mathbb{R}^m)$ there exist weights c_φ^h so that, for any closed domain $G \subset \mathbb{R}^m$,*

(iva)′

$$\left\| u - \sigma_h \sum_\varphi \varphi * c_\varphi^h h^{m/p} \right\|_p (G) \le \text{const}\, h^k |u|_{k,p}(B_{rh}(G)),$$

(ivb)′

$$\left\{ \sum_{G\cap \text{supp}\,\varphi(\,\cdot/h - j)\neq\emptyset} |c_\varphi^h(j)|^p \right\}^{1/p} \le \text{const}\|u\|_p(B_{rh}(G))$$

holds for some const *and r independent of h, G, u.*

Here, $B_d(G) := \{x \in \mathbb{R}^m : \text{dist}\,(x, G) < d\}$.

If (iv)′ holds, then, with a reference to [**St**], [**DM**] say that Φ provides "controlled L_p-approximation of order k". They do not comment on the fact that (iv)′ is a strengthening of (iv), and they refer to [**SF**] for a proof of the implication (iv)′ $\Rightarrow$ (i).

As it turns out, (iv)′ does indeed imply (i). But it is the *localness* rather than the *control* that does the job. For this reason we propose here to abandon the notion of "controlled approximation order" in favor of "local approximation order". We say that Φ provides "local L_p-approximation of order k" in case the following condition holds:

(iv)″ *For each $u \in W_p^k(\mathbb{R}^m)$ there exist weights c_φ^h so that*

(iva)″

$$\left\| u - \sigma_h \sum_\varphi \varphi * c_\varphi^h \right\|_p \le \text{const}\ h^k |u|_{k,p};$$

(ivb)″

$$c_\varphi^h(j) = 0 \ \textit{whenever}\ \text{dist}\,(jh, \text{supp}\,u) > r$$

holds for some const *and some r independent of h and u.*

It is clear that (ivb)′ is stronger than both (ivb) and (ivb)″, but (ivb) and (ivb)″ are not comparable. As we are about to show, (iv)″ is the right modification of (iv) to give equivalence with (i). This shows that also (iv)′ $\Rightarrow$ (i) and so validates the version of Theorem SF in [**DM**].

Theorem. *Let Φ be a finite subset of $W^0_{p,c}(\mathbb{R}^m)$. Then the following statements are equivalent:*
(1°) There exists a sequence $(\psi_\alpha)_{|\alpha|<k}$ in $\operatorname{span}\Phi$ which satisfies
(1°a) $\widehat{\psi}_0(0) = 1, \widehat{\psi}_0 = 0$ on $2\pi\mathbb{Z}^m\backslash 0$;
(1°b) $\sum_{\beta\le\alpha} [\![-iD]\!]^\beta \widehat{\psi}_{\alpha-\beta} = 0$ on $2\pi\mathbb{Z}^m\backslash 0$ for $0 < |\alpha| < k$.
(2°) There exists a sequence $(\psi_\alpha)_{|\alpha|<k}$ in $\operatorname{span}\Phi$ such that

$$[\![\,]\!]^\alpha - \sum_{\beta\le\alpha} \psi_{\alpha-\beta} * [\![\,]\!]^\beta \in \Pi_{|\alpha|-1} \quad \textit{for} \quad |\alpha| < k.$$

(3°) There exist some finitely supported c_φ so that $\psi := \sum_{\varphi\in\Phi} \varphi * c_\varphi$ satisfies

$$[\![\,]\!]^\alpha - \psi * [\![\,]\!]^\alpha \in \Pi_{|\alpha|-1} \quad \textit{for } |\alpha| < k.$$

(4°) For all $p \in [1,\infty]$, Φ provides local L_p–approximation order k.
(5°) For some $p \in [1,\infty]$, Φ provides local L_p–approximation order k.

Remarks. Condition (1°) differs from (i) in that the latter requires, additionally, that $\sum_{\beta\le\alpha} [\![-iD]\!]^\beta \widehat{\psi}_{\alpha-\beta} = 0$ at 0. Already Dahmen and Micchelli **[DM]** prove that it is possible to get away with the weaker condition (1°).

Condition (2°) is, off-hand, weaker than (ii), but an inductive argument leads from (2°) to (ii).

Condition (3°) seems more useful to us in applications than the (equivalent) condition (iii). We note that induction gives the seemingly stronger statement

(3°)′ *There exist some finitely supported c_φ so that $\psi := \sum_{\varphi\in\Phi} \varphi * c_\varphi$ satisfies $\psi * [\![\,]\!]^\alpha = [\![\,]\!]^\alpha$ for $|\alpha| < k$.*

The asserted equivalence between (4°) and (5°) shows that one might as well drop the qualifier "L_p-" and just speak of the local approximation order provided by Φ.

Proof of the Theorem. While the main point of this note is the implication (5°) ⇒ (1°), we give a proof of the entire implication cycle (1°) ⇒ (2°) ⇒ ⋯ ⇒ (1°). The arguments for (1°) ⇒ (2°) ⇒ (3°) are adaptations of those in **[SF]**.

(1°) ⇒ (2°). Observe that the Fourier transform of $\psi(x - \cdot)[\![\,]\!]^\beta$ is $[\![iD]\!]^\beta(e^{-i(\cdot)x}\widehat{\psi}(-\cdot))$. Hence, by Poisson's summation formula, we have

$$\begin{aligned}\psi * [\![\,]\!]^\beta(x) &= \sum_j [\![iD]\!]^\beta (e^{-i\xi x}\widehat{\psi}(-\xi))\big|_{\xi=2\pi j} \\ &= \sum_j \sum_{\gamma\le\beta} [\![x]\!]^\gamma e^{-2\pi i j x} [\![-iD]\!]^{\beta-\gamma}\widehat{\psi}(-2\pi j).\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{\beta\le\alpha} \psi_{\alpha-\beta} * [\![\,]\!]^\beta &= \sum_j \sum_{\beta\le\alpha} \sum_{\gamma\le\beta} [\![\,]\!]^\gamma e^{-2\pi i j()} [\![-iD]\!]^{\beta-\gamma}\widehat{\psi}_{\alpha-\beta}(-2\pi j) \\ &= \sum_j e^{-2\pi i j()} \sum_{\gamma\le\alpha} [\![\,]\!]^\gamma \sum_{\beta-\gamma\le\alpha-\gamma} [\![-iD]\!]^{\beta-\gamma}\widehat{\psi}_{(\alpha-\gamma)-(\beta-\gamma)}(-2\pi j) \\ &= [\![\,]\!]^\alpha + \sum_{\gamma<\alpha} [\![\,]\!]^\gamma \sum_{\delta\le\alpha-\gamma} [\![-iD]\!]^\delta \widehat{\psi}_{\alpha-\gamma-\delta}(0),\end{aligned}$$

the last equation by (1°).

(2°) ⇒ (3°). With $\gamma := (k,\dots,k)$, the monomial $[\![\,]\!]^\gamma$ and its integer shifts span the space $\Pi_k(\mathbb{R}) \times \cdots \times \Pi_k(\mathbb{R})$. This implies that there exist finitely supported sequences c_β so that

$$[\![\,]\!]^\gamma * c_\beta = [\![\,]\!]^{\gamma-\beta} \quad \text{for } \beta \le \gamma.$$

On applying $D^{\gamma-\alpha}$ to both sides, we find that $[\![\,]\!]^\alpha * c_\beta = [\![\,]\!]^{\alpha-\beta}$; hence,

$$c_\beta * [\![\,]\!]^\alpha = [\![\,]\!]^{\alpha-\beta} \quad \text{on } \mathbb{Z}^m \quad \text{for } \alpha,\beta \le \gamma.$$

Now set

$$\psi := \sum_{|\beta|<k} \psi_\beta * c_\beta.$$

Then

$$\psi * [\![\ ]\!]^\alpha = \sum_{|\beta|<k} \psi_\beta * c_\beta * [\![\ ]\!]^\alpha = \sum_{|\beta|<k} \psi_\beta * [\![\ ]\!]^{\alpha-\beta} = \sum_{\beta\le\alpha} \psi_\beta * [\![\ ]\!]^{\alpha-\beta} \in [\![\ ]\!]^\alpha + \Pi_{|\alpha|-1},$$

the inclusion by (2°).

The proof of (3°) ⇒ (4°) follows the argument for [**BH**, Corollary to Theorem 6].

This leaves (5°) ⇒ (1°). We approximate a tensor product of univariate B-splines—namely, the function

$$u(x) := \prod_{\nu=1}^{m} M_{k+1}(x(\nu)),$$

with

$$M_{k+1}(t) := \big[-\frac{k+1}{2}, -\frac{k-1}{2}, \dots, \frac{k+1}{2}\big](\cdot - t)_+^k, \qquad t \in \mathbb{R}$$

(see [**Sc**]). Since $u \in W_p^k(\mathbb{R}^m)$ for any $p \in [1,\infty]$, we can find weights c_φ^h so that (iva)″ and (ivb)″ hold (whatever the p might be). Set

$$u_h := \sigma_h\Big(\sum_\varphi \varphi * c_\varphi^h\Big)$$

and consider the Fourier-Laplace transform

$$\widehat{f}(z) := \int e^{-izx} f(x)\, dx, \qquad z \in \mathbb{C}^m,$$

of the error $f := u - u_h$. Since u has compact support, so does u_h by (ivb)″; hence, so does f uniformly in h. This means that supp f lies in some ball B_a of finite radius a independently of h. Consequently,

$$|\widehat{f}(z)| = \Big|\int_{B_a} e^{-izx} f(x)\, dx\Big| \le e^{a\|\operatorname{Im} z\|}\text{const}_a \|f\|_p,$$

and, by (iva)″,

$$\|f\|_p \le \text{const}\ h^k |u|_{k,p}.$$

This implies that

$$|\widehat{f}(z)| \le \text{const}\ h^k \quad \text{for } \|\operatorname{Im} z\| \le \text{const}.$$

We can therefore invoke Cauchy's formula (see, e.g., [**R**]) to get the estimate

$$\|D^\alpha \widehat{f}\|_\infty(\mathbb{R}) \le \text{const}\ h^k. \tag{1}$$

The Fourier transform of u is well known (see [**Sc**]); it is

$$\widehat{u}(z) = \prod_{\nu=1}^{m} C(z(\nu))^{k+1}$$

where $C(t) := (\sin t/2)/(t/2)$ is Whittaker's cardinal function. From this we deduce that

$$\widehat{u}(0) = 1 \tag{2}$$

and

$$\lim_{h\to 0} \frac{(D^\alpha \widehat{u})(x/h)}{h^k} = 0 \quad \text{for } x \in \mathbb{R}^m\backslash 0 \text{ and } |\alpha| < k; \tag{3}$$

hence, by (1), $\widehat{u}_h$ must satisfy corresponding conditions.

We now compute $\widehat{u}_h$. Since

$$\int e^{-izx}\varphi(x/h-j)\,dx = h^m\widehat{\varphi}(hz)e^{-ihzj},$$

we find that

$$\widehat{u}_h(z) = \sum_\varphi \widehat{\varphi}(hz)v^h_{\varphi,0}(z)$$

with

$$v^h_{\varphi,0}(z) := h^m\sum_j c^h_\varphi(j)e^{-ihzj}.$$

Thus, from (1) and (2),

$$\lim_{h\to 0}\sum_\varphi \widehat{\varphi}(0)v^h_{\varphi,0}(0) = 1. \tag{4}$$

Further,

$$[\![D]\!]^\alpha\big(\widehat{\varphi}(hz)v^h_{\varphi,0}(z)\big) = \sum_{\beta\le\alpha} h^{|\beta|}[\![D]\!]^\beta\widehat{\varphi}(hz)\,(-ih)^{|\alpha-\beta|}v^h_{\varphi,\alpha-\beta}(z)$$

with

$$v^h_{\varphi,\gamma}(z) := h^m\sum_j c^h_\varphi(j)[\![j]\!]^\gamma e^{-ihzj}$$

each a $2\pi/h$-periodic function. (Note that, for $\gamma = 0$, this agrees with the earlier definition, as it should.) This implies that

$$v_{\varphi,\gamma}(2\pi j/h) = v^h_{\varphi,\gamma}(0),$$

and therefore (1) and (3) give

$$\sum_\varphi\sum_{\beta\le\alpha} h^{|\alpha|}(-i)^{|\alpha-\beta|}[\![D]\!]^\beta\widehat{\varphi}(2\pi j)v^h_{\varphi,\alpha-\beta}(0) = O(h^k);$$

hence, for $|\alpha| < k$,

$$\lim_{h\to 0}\sum_\varphi\sum_{\beta\le\alpha}[\![-iD]\!]^\beta\widehat{\varphi}(2\pi j)v^h_{\varphi,\alpha-\beta}(0) = 0 \quad \text{for } j\in\mathbb{Z}^m\backslash 0. \tag{5}$$

From (4) and (5) we deduce (1°) as follows. By (4) we cannot have $\widehat{\varphi}(0) = 0$ for every $\varphi\in\Phi$. Without loss of generality we can therefore assume that, for some $\chi\in\Phi$, $\widehat{\chi}(0) = 1$, and $\widehat{\varphi}(0) = 0$ for all $\varphi\in\Phi\backslash\chi$. In particular, this implies with (4) that $\lim_{h\to 0} v^h_{\chi,0}(0) = 1$. Now consider the space S of all vectors $w = (w_{\varphi,\gamma})$ for which

$$\lim_{h\to 0}\sum_\varphi\sum_{|\gamma|<k} w_{\varphi,\gamma}v^h_{\varphi,\gamma}(0) = 0.$$

We claim that $S\perp$ contains a vector w' with $w'_{\chi,0} = 1$. Indeed, if $w'_{\chi,0} = 0$ for all $w'\in S\perp$, then $(S\perp)\perp = S$ would contain the unit vector $(\delta_{\varphi\chi}\delta_{\gamma 0})$; hence, $\lim_{h\to 0} v^h_{\chi,0}(0) = 0$ would follow.

With this, define

$$\psi_\gamma := \sum_\varphi w'_{\varphi,\gamma}\varphi.$$

Then

$$\widehat{\psi}_0(0) = \sum_\varphi w'_{\varphi,0}\widehat{\varphi}(0) = w'_{\chi,0}\widehat{\chi}(0) = 1.$$

Further, by (5), the vector $(\llbracket -iD\rrbracket^{\alpha-\gamma}\widehat{\varphi}(2\pi j))_{\varphi,\gamma}$ is in S for $|\alpha| < k$ and $j \in \mathbb{Z}^m\backslash 0$; therefore

$$\sum_{\varphi}\sum_{\beta\le\alpha} w'_{\varphi,\alpha-\beta}\llbracket -iD\rrbracket^{\beta}\widehat{\varphi}(2\pi j) = 0 \qquad \text{for } |\alpha| < k \text{ and } j \in \mathbb{Z}^m\backslash 0,$$

and this implies that

$$\sum_{\beta\le\alpha} \llbracket -iD\rrbracket^{\beta}\widehat{\psi}_{\alpha-\beta} = 0 \qquad \text{on } 2\pi\mathbb{Z}^m\backslash 0 \text{ for } |\alpha| < k,$$

as we wanted to show.

Remark. In contrast to **[SF]**, we assume Φ only to lie in L_p. For this reason we do not get (iva); i.e., we do not get simultaneous approximation to derivatives. But this is easily obtained under the assumption that Φ lie in an appropriately smoother space, using the quasi-interpolant constructed in the proof of $(3^\circ) \Rightarrow (4^\circ)$. In this connection we note that, using (3°) in the equivalent formulation $(3^\circ)'$, such a quasi-interpolant for u takes the particularly simple form

$$\psi * u.$$

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