Controlled approximation and a characterization of the local approximation order[1]

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Abstract. The local approximation order from a scale (S_h) of approximating functions on $\mathbb{R}^m$ is characterized in terms of the linear spac (and its Fourier transform) of the finitely many compactly supported functions φ whose integer translates $\varphi(\cdot - j)$, $j \in \mathbb{Z}^m$, span the space $S = S_1$ from which the scale is derived. This provides a correction of similar results stated and proved, in part, by Strang and Fix.

The term "controlled approximation" was introduced in 1970 by Strang [**St**]; see also [**FS**]. It concerns approximations of the form $\sum_{\varphi\in\Phi} \varphi * c_\varphi$, with Φ a finite collection of functions on $\mathbb{R}^m$ of compact support and $\varphi * c$ the function obtained from φ by convolution with some "sequence" $c : \mathbb{Z}^m \to \mathbb{R}$; i.e.,

$$\varphi * c := \sum_{j\in\mathbb{Z}^m} \varphi(\cdot - j)c(j).$$

If, more generally, c is some function on $\mathbb{R}^m$, we will still just write $\varphi * c$ instead of the correct but more complicated $\varphi * (c|_{\mathbb{Z}^m})$.

The function u to be approximated lies in the Sobolev space $W_p^k(\mathbb{R}^m)$ with norm

$$\|u\|_{k,p} := \sum_{j\le k} |u|_{j,p},$$

where

$$|u|_{j,p} := \sum_{|\alpha|=j} \|D^\alpha u\|_p$$

and

$$\|u\|_p := \|u\|_{L_p(\mathbb{R}^m)}.$$

We denote by $W_{p,c}^k(\mathbb{R}^m)$ the subspace of $W_p^k(\mathbb{R}^m)$ of compactly supported functions.

The approximations are, more explicitly, of the form

$$\sigma_h \left(\sum_{\varphi\in\Phi} \varphi * c_\varphi^h \right) / h^{m/p} \quad \text{with} \ \ \sigma_h f := f(\cdot/h).$$

Concerning the degree of approximation to $u \in W_p^k(\mathbb{R}^m)$ achievable by proper choice of the weights c_φ^h, Strang and Fix [**SF**, Theorem II] state the following result. In its statement and subsequent analysis, the normalized multivariate monomials appear often enough to deserve an abbreviation of their own. We will use $[\![\ ]\!]^\alpha$ to stand for the function $\mathbb{R}^m \to \mathbb{R} : \ x \mapsto \ x^\alpha/\alpha!$ (and will use standard multi-index notation throughout). In particular, $D^\beta[\![\ ]\!]^\alpha = [\![\ ]\!]^{\alpha-\beta}$, and this holds even when $\beta \not\le \alpha$, since then $[\![\ ]\!]^{\alpha-\beta} = 0$, by convention. Further, Π_j will denote the collection of polynomials on $\mathbb{R}^m$ of total degree $\le j$. Finally, $\widehat{f}$ will denote the Fourier transform of f — i.e., $\widehat{f}(\xi) := \int_{\mathbb{R}^m} e^{-i\xi x} f(x)\,dx$, with ξx the scalar product.

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Theorem SF. *Let Φ be a finite subset of $W_{2,c}^{k-1}(\mathbb{R}^m)$. Then the following are equivalent:*

(i) *There exists a sequence $(\psi_\alpha)_{|\alpha|<k}$ in span Φ which satisfies*

(ia) $\widehat{\psi}_0(0) = 1$, $\widehat{\psi}_0 = 0$ *on* $2\pi\mathbb{Z}^m\backslash 0$;

(ib) $\sum_{\beta\le\alpha} [\![-iD]\!]^\beta \widehat{\psi}_{\alpha-\beta} = 0$ *on* $2\pi\mathbb{Z}^m$ *for* $0 < |\alpha| < k$.

(ii) *There exists a sequence $(\psi_\alpha)_{|\alpha|<k}$ which satisfies*

$$[\![\ ]\!]^\alpha = \sum_{\beta\le\alpha} \psi_{\alpha-\beta} * [\![\ ]\!]^\beta \quad for \quad |\alpha| < k.$$

(iii) *There exist some finitely supported c_φ so that $\psi := \sum_{\varphi\in\Phi} \varphi * c_\varphi$ satisfies $\widehat{\psi}(0) \ne 0$, but $D^\alpha \widehat{\psi} = 0$ on $2\pi\mathbb{Z}^m\backslash 0$ for $|\alpha| < k$.*

(iv) *For each $u \in W_2^k(\mathbb{R}^m)$ there exist weights c_φ^h so that*

(iva) $\|u - \sigma_h(\sum_\varphi \varphi * c_\varphi^h)/h^{m/2}\|_{s,2} \le \mathrm{const}_s h^{k-s} |u|_{k,2}$, $s = 0, \ldots, k-1$;

(ivb) $\sum_\varphi \|c_\varphi^h\|_2^2 \le \mathrm{const}\|u\|_2^2$.

For the very special case when $m = \#\Phi = 1$, such results can already be found in **[Sc]**. In **[SF]** the special case when Φ consists of just one function (but m is arbitrary) is treated first (see **[SF**, Theorem I**]**) and completely. However, for the general case, **[SF]** only gives proof for the implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv). In particular, the validity of (iv) $\Rightarrow$ (i) has recently been questioned. This was finally settled by Jia **[J]**, who shows by a counterexample that (iv) does *not* imply (i) in general.

This raises the question of how to modify (iv) to obtain something equivalent to (i). This is a matter of changing the *control* over the form of the approximation as expressed by (ivb). In this connection it is very useful to recall that Dahmen and Micchelli **[DM]** quote Theorem SF in a modified form. The modification of importance here occurs in condition (iv), which they require to hold locally, as follows:

(iv)$'$ *For each $u \in W_p^k(\mathbb{R}^m)$ there exist weights c_φ^h so that, for any closed domain $G \subset \mathbb{R}^m$,*

(iva)$'$

$$\Big\|u - \sigma_h \sum_\varphi \varphi * c_\varphi^h h^{m/p}\Big\|_p(G) \le \mathrm{const}\, h^k |u|_{k,p}(B_{rh}(G)),$$

(ivb)$'$

$$\Big\{\sum_{G\cap \mathrm{supp}\,\varphi(\,\cdot/h-j)\ne\emptyset} \big|c_\varphi^h(j)\big|^p\Big\}^{1/p} \le \mathrm{const}\|u\|_p(B_{rh}(G))$$

holds for some const *and r independent of h, G, u.*

Here, $B_d(G) := \{x \in \mathbb{R}^m : \mathrm{dist}\,(x, G) < d\}$.

If (iv)$'$ holds, then, with a reference to **[St]**, **[DM]** say that Φ provides "controlled L_p-approximation of order k". They do not comment on the fact that (iv)$'$ is a strengthening of (iv), and they refer to **[SF]** for a proof of the implication (iv)$'$ $\Rightarrow$ (i).

As it turns out, (iv)$'$ does indeed imply (i). But it is the *localness* rather than the *control* that does the job. For this reason we propose here to abandon the notion of "controlled approximation order" in favor of "local approximation order". We say that Φ provides "local L_p-approximation of order k" in case the following condition holds:

(iv)$''$ *For each $u \in W_p^k(\mathbb{R}^m)$ there exist weights c_φ^h so that*

(iva)$''$

$$\Big\|u - \sigma_h \sum_\varphi \varphi * c_\varphi^h\Big\|_p \le \mathrm{const}\, h^k |u|_{k,p};$$

(ivb)$''$

$$c_\varphi^h(j) = 0 \text{ whenever } \mathrm{dist}\,(jh, \mathrm{supp}\, u) > r$$

holds for some const *and some r independent of h and u.*

It is clear that (ivb)$'$ is stronger than both (ivb) and (ivb)$''$, but (ivb) and (ivb)$''$ are not comparable. As we are about to show, (iv)$''$ is the right modification of (iv) to give equivalence with (i). This shows that also (iv)$'$ $\Rightarrow$ (i) and so validates the version of Theorem SF in **[DM]**.

Theorem. *Let Φ be a finite subset of $W^0_{p,c}(\mathbb{R}^m)$. Then the following statements are equivalent:*

(1°) There exists a sequence $(\psi_\alpha)_{|\alpha|<k}$ in $\operatorname{span}\Phi$ which satisfies

(1°a) $\widehat{\psi}_0(0)=1, \widehat{\psi}_0=0$ on $2\pi\mathbb{Z}^m\backslash 0$;

(1°b) $\sum_{\beta\le\alpha} [\![-iD]\!]^\beta \widehat{\psi}_{\alpha-\beta}=0$ on $2\pi\mathbb{Z}^m\backslash 0$ for $0<|\alpha|<k$.

(2°) There exists a sequence $(\psi_\alpha)_{|\alpha|<k}$ in $\operatorname{span}\Phi$ such that

$$[\![\ ]\!]^\alpha - \sum_{\beta\le\alpha}\psi_{\alpha-\beta} * [\![\ ]\!]^\beta \in \Pi_{|\alpha|-1} \quad for \quad |\alpha|<k.$$

(3°) There exist some finitely supported c_φ so that $\psi := \sum_{\varphi\in\Phi}\varphi * c_\varphi$ satisfies

$$[\![\ ]\!]^\alpha - \psi * [\![\ ]\!]^\alpha \in \Pi_{|\alpha|-1} \quad for\ |\alpha|<k.$$

(4°) For all $p\in[1,\infty]$, Φ provides local L_p–approximation order k.

(5°) For some $p\in[1,\infty]$, Φ provides local L_p–approximation order k.

Remarks. Condition (1°) differs from (i) in that the latter requires, additionally, that $\sum_{\beta\le\alpha}[\![-iD]\!]^\beta\widehat{\psi}_{\alpha-\beta}=0$ at 0. Already Dahmen and Micchelli **[DM]** prove that it is possible to get away with the weaker condition (1°).

Condition (2°) is, off-hand, weaker than (ii), but an inductive argument leads from (2°) to (ii).

Condition (3°) seems more useful to us in applications than the (equivalent) condition (iii). We note that induction gives the seemingly stronger statement

(3°)′ *There exist some finitely supported c_φ so that $\psi := \sum_{\varphi\in\Phi}\varphi * c_\varphi$ satisfies $\psi * [\![\ ]\!]^\alpha = [\![\ ]\!]^\alpha$ for $|\alpha|<k$.*

The asserted equivalence between (4°) and (5°) shows that one might as well drop the qualifier "L_p-" and just speak of the local approximation order provided by Φ.

Proof of the Theorem. While the main point of this note is the implication (5°) $\Rightarrow$ (1°), we give a proof of the entire implication cycle (1°) $\Rightarrow$ (2°) $\Rightarrow\cdots\Rightarrow$ (1°). The arguments for (1°) $\Rightarrow$ (2°) $\Rightarrow$ (3°) are adaptations of those in **[SF]**.

(1°) $\Rightarrow$ (2°). Observe that the Fourier transform of $\psi(x-\cdot)[\![\ ]\!]^\beta$ is $[\![iD]\!]^\beta(e^{-i(\cdot)x}\widehat{\psi}(-\cdot))$. Hence, by Poisson's summation formula, we have

$$\begin{aligned}\psi * [\![\ ]\!]^\beta(x) &= \sum_j [\![iD]\!]^\beta(e^{-i\xi x}\widehat{\psi}(-\xi))\big|_{\xi=2\pi j}\\ &= \sum_j\sum_{\gamma\le\beta}[\![x]\!]^\gamma e^{-2\pi i j x}[\![-iD]\!]^{\beta-\gamma}\widehat{\psi}(-2\pi j).\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{\beta\le\alpha}\psi_{\alpha-\beta} * [\![\ ]\!]^\beta &= \sum_j\sum_{\beta\le\alpha}\sum_{\gamma\le\beta}[\![\ ]\!]^\gamma e^{-2\pi i j()}[\![-iD]\!]^{\beta-\gamma}\widehat{\psi}_{\alpha-\beta}(-2\pi j)\\ &= \sum_j e^{-2\pi i j()}\sum_{\gamma\le\alpha}[\![\ ]\!]^\gamma\sum_{\beta-\gamma\le\alpha-\gamma}[\![-iD]\!]^{\beta-\gamma}\widehat{\psi}_{(\alpha-\gamma)-(\beta-\gamma)}(-2\pi j)\\ &= [\![\ ]\!]^\alpha + \sum_{\gamma<\alpha}[\![\ ]\!]^\gamma\sum_{\delta\le\alpha-\gamma}[\![-iD]\!]^\delta\widehat{\psi}_{\alpha-\gamma-\delta}(0),\end{aligned}$$

the last equation by (1°).

(2°) $\Rightarrow$ (3°). With $\gamma := (k,\ldots,k)$, the monomial $[\![\ ]\!]^\gamma$ and its integer shifts span the space $\Pi_k(\mathbb{R})\times\cdots\times\Pi_k(\mathbb{R})$. This implies that there exist finitely supported sequences c_β so that

$$[\![\ ]\!]^\gamma * c_\beta = [\![\ ]\!]^{\gamma-\beta} \quad \text{for } \beta\le\gamma.$$

On applying $D^{\gamma-\alpha}$ to both sides, we find that $[\![\ ]\!]^\alpha * c_\beta = [\![\ ]\!]^{\alpha-\beta}$; hence,

$$c_\beta * [\![\ ]\!]^\alpha = [\![\ ]\!]^{\alpha-\beta} \quad \text{on } \mathbb{Z}^m \quad \text{for } \alpha,\beta\le\gamma.$$

Now set

$$\psi := \sum_{|\beta|<k} \psi_\beta * c_\beta.$$

Then

$$\psi * [\![\;]\!]^\alpha = \sum_{|\beta|<k} \psi_\beta * c_\beta * [\![\;]\!]^\alpha = \sum_{|\beta|<k} \psi_\beta * [\![\;]\!]^{\alpha-\beta} = \sum_{\beta\le\alpha} \psi_\beta * [\![\;]\!]^{\alpha-\beta} \in [\![\;]\!]^\alpha + \Pi_{|\alpha|-1},$$

the inclusion by (2°).

The proof of (3°) ⇒ (4°) follows the argument for [**BH**, Corollary to Theorem 6].

This leaves (5°) ⇒ (1°). We approximate a tensor product of univariate B-splines—namely, the function

$$u(x) := \prod_{\nu=1}^{m} M_{k+1}(x(\nu)),$$

with

$$M_{k+1}(t) := \big[-\frac{k+1}{2}, -\frac{k-1}{2}, \ldots, \frac{k+1}{2}\big](\,\cdot - t)_+^k, \qquad t \in \mathbb{R}$$

(see [**Sc**]). Since $u \in W_p^k(\mathbb{R}^m)$ for any $p \in [1,\infty]$, we can find weights c_φ^h so that (iva)″ and (ivb)″ hold (whatever the p might be). Set

$$u_h := \sigma_h\Big(\sum_\varphi \varphi * c_\varphi^h\Big)$$

and consider the Fourier-Laplace transform

$$\widehat{f}(z) := \int e^{-izx} f(x)\,dx, \qquad z \in \mathbb{C}^m,$$

of the error $f := u - u_h$. Since u has compact support, so does u_h by (ivb)″; hence, so does f uniformly in h. This means that $\operatorname{supp} f$ lies in some ball B_a of finite radius a independently of h. Consequently,

$$|\widehat{f}(z)| = \Big|\int_{B_a} e^{-izx} f(x)\,dx\Big| \le e^{a\|\operatorname{Im} z\|}\mathrm{const}_a \|f\|_p,$$

and, by (iva)″,

$$\|f\|_p \le \mathrm{const}\; h^k |u|_{k,p}.$$

This implies that

$$|\widehat{f}(z)| \le \mathrm{const}\; h^k \quad \text{for} \;\; \|\operatorname{Im} z\| \le \mathrm{const}.$$

We can therefore invoke Cauchy's formula (see, e.g., [**R**]) to get the estimate

$$\text{(1)} \qquad \|D^\alpha \widehat{f}\|_\infty(\mathbb{R}) \;\le\; \mathrm{const}\; h^k.$$

The Fourier transform of u is well known (see [**Sc**]); it is

$$\widehat{u}(z) = \prod_{\nu=1}^{m} C(z(\nu))^{k+1}$$

where $C(t) := (\sin t/2)/(t/2)$ is Whittaker's cardinal function. From this we deduce that

$$\text{(2)} \qquad \widehat{u}(0) = 1$$

and

$$\text{(3)} \qquad \lim_{h\to 0} \frac{(D^\alpha \widehat{u})\,(x/h)}{h^k} = 0 \quad \text{for } x \in \mathbb{R}^m\backslash 0 \text{ and } |\alpha| < k;$$

hence, by (1), $\widehat{u}_h$ must satisfy corresponding conditions.

We now compute $\widehat{u}_h$. Since

$$\int e^{-izx}\varphi(x/h-j)\,dx = h^m\widehat{\varphi}(hz)e^{-ihzj},$$

we find that

$$\widehat{u}_h(z) = \sum_\varphi \widehat{\varphi}(hz)v^h_{\varphi,0}(z)$$

with

$$v^h_{\varphi,0}(z) := h^m\sum_j c^h_\varphi(j)e^{-ihzj}.$$

Thus, from (1) and (2),

$$\lim_{h\to 0}\sum_\varphi \widehat{\varphi}(0)v^h_{\varphi,0}(0) = 1. \tag{4}$$

Further,

$$[\![D]\!]^\alpha\big(\widehat{\varphi}(hz)v^h_{\varphi,0}(z)\big) = \sum_{\beta\le\alpha} h^{|\beta|}[\![D]\!]^\beta\widehat{\varphi}(hz)\,(-ih)^{|\alpha-\beta|}v^h_{\varphi,\alpha-\beta}(z)$$

with

$$v^h_{\varphi,\gamma}(z) := h^m\sum_j c^h_\varphi(j)[\![j]\!]^\gamma e^{-ihzj}$$

each a $2\pi/h$-periodic function. (Note that, for $\gamma=0$, this agrees with the earlier definition, as it should.) This implies that

$$v_{\varphi,\gamma}(2\pi j/h) = v^h_{\varphi,\gamma}(0),$$

and therefore (1) and (3) give

$$\sum_\varphi\sum_{\beta\le\alpha} h^{|\alpha|}(-i)^{|\alpha-\beta|}[\![D]\!]^\beta\widehat{\varphi}(2\pi j)v^h_{\varphi,\alpha-\beta}(0) = O(h^k);$$

hence, for $|\alpha|<k$,

$$\lim_{h\to 0}\sum_\varphi\sum_{\beta\le\alpha}[\![-iD]\!]^\beta\widehat{\varphi}(2\pi j)v^h_{\varphi,\alpha-\beta}(0) = 0 \quad\text{for } j\in\mathbb{Z}^m\backslash 0. \tag{5}$$

From (4) and (5) we deduce (1°) as follows. By (4) we cannot have $\widehat{\varphi}(0)=0$ for every $\varphi\in\Phi$. Without loss of generality we can therefore assume that, for some $\chi\in\Phi$, $\widehat{\chi}(0)=1$, and $\widehat{\varphi}(0)=0$ for all $\varphi\in\Phi\backslash\chi$. In particular, this implies with (4) that $\lim_{h\to0} v^h_{\chi,0}(0)=1$. Now consider the space S of all vectors $w=(w_{\varphi,\gamma})$ for which

$$\lim_{h\to 0}\sum_\varphi\sum_{|\gamma|<k} w_{\varphi,\gamma}v^h_{\varphi,\gamma}(0) = 0.$$

We claim that $S\perp$ contains a vector w' with $w'_{\chi,0}=1$. Indeed, if $w'_{\chi,0}=0$ for all $w'\in S\perp$, then $(S\perp)\perp = S$ would contain the unit vector $(\delta_{\varphi\chi}\delta_{\gamma0})$; hence, $\lim_{h\to0} v^h_{\chi,0}(0)=0$ would follow.

With this, define

$$\psi_\gamma := \sum_\varphi w'_{\varphi,\gamma}\varphi.$$

Then

$$\widehat{\psi}_0(0) = \sum_\varphi w'_{\varphi,0}\widehat{\varphi}(0) = w'_{\chi,0}\widehat{\chi}(0) = 1.$$

Further, by (5), the vector $([\![-iD]\!]^{\alpha-\gamma}\widehat{\varphi}(2\pi j))_{\varphi,\gamma}$ is in S for $|\alpha|<k$ and $j\in\mathbb{Z}^m\backslash 0$; therefore

$$\sum_{\varphi}\sum_{\beta\le\alpha} w'_{\varphi,\alpha-\beta}[\![-iD]\!]^{\beta}\widehat{\varphi}(2\pi j)=0 \qquad \text{for } |\alpha|<k \text{ and } j\in\mathbb{Z}^m\backslash 0,$$

and this implies that

$$\sum_{\beta\le\alpha}[\![-iD]\!]^{\beta}\widehat{\psi}_{\alpha-\beta}=0 \qquad \text{on } 2\pi\mathbb{Z}^m\backslash 0 \text{ for } |\alpha|<k,$$

as we wanted to show.

Remark. In contrast to **[SF]**, we assume Φ only to lie in L_p. For this reason we do not get (iva); i.e., we do not get simultaneous approximation to derivatives. But this is easily obtained under the assumption that Φ lie in an appropriately smoother space, using the quasi-interpolant constructed in the proof of $(3°) \Rightarrow (4°)$. In this connection we note that, using $(3°)$ in the equivalent formulation $(3°)'$, such a quasi-interpolant for u takes the particularly simple form

$$\psi * u.$$

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