The Error in Polynomial Tensor-Product, and Chung-Yao, Interpolation

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Abstract. A formula for the error in Chung-Yao interpolation announced earlier is proved (by induction). In the process, a bivariate divided difference identity of independent interest is proved. Also, an inductive proof of an error formula for polynomial interpolation by tensor-products is given. The main tool is a (convenient notation for a) multivariate divided difference.

In [2], a particular multivariate divided difference is introduced and, as an illustration of its usefulness, error formulæ for three special cases of multivariate polynomial interpolation are stated, but not proved. To be sure, an inductive procedure is indicated which, so it is claimed there, will produce each of these formulæ, but (apparently crucial) detail is missing, for both the error in tensor-product interpolation and in Chung-Yao interpolation. It is the purpose of this note to provide complete, inductive proofs, as outlined in [2], of these formulæ. In the process of proving the formula for Chung-Yao interpolation, an essentially bivariate divided difference identity is proved which may well have a nice multivariate generalization. Short 'direct' proofs (with the induction being hidden in well-known results about univariate divided differences) appear in [1].

This note has the following simple structure. After a quick recall, in Section 1, of the divided difference (notation) introduced in [2], and, in Section 2, of well-known facts about hyperplanes in $\mathbb{R}^d$ in general position, Section 3 brings the inductive proof of the error formula for Chung-Yao interpolation, proving two useful divided difference identities in the process. This is followed by an inductive proof of the error formula for polynomial tensor-product interpolation, in Section 4. The last section points out similarities between these two error formulæ and speculates on the form of a pointwise error formula for polynomial interpolation at an arbitrary pointset, with particular attention to the Sauer-Xu formula for the error in polynomial interpolation at an otherwise arbitrary pointset at which interpolation from the full space Π_k of polynomials of degree $\leq k$ is uniquely possible.

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§1. The Divided Difference Recalled

In [2], the following multivariate divided difference is singled out:

$$[\mathbf{x}_1,\ldots,\mathbf{x}_k,\,\cdot\mid\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_k]g := [\mathbf{x}_1,\,\cdot\mid\boldsymbol{\xi}_1]\,\cdots\,[\mathbf{x}_k,\,\cdot\mid\boldsymbol{\xi}_k]g, \tag{1.1}$$

with

$$[\mathbf{x},\mathbf{y}\mid\boldsymbol{\xi}] : g\mapsto\int_0^1 D_{\boldsymbol{\xi}}g(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))\,\mathrm{d}t, \tag{1.2}$$

hence

$$[\mathbf{x}_1,\ldots,\mathbf{x}_{k+1}\mid\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_k]g = \int_{[\mathbf{x}_1,\ldots,\mathbf{x}_{k+1}]} D_{\boldsymbol{\xi}_1}\cdots D_{\boldsymbol{\xi}_k}g, \tag{1.3}$$

where

$$f\mapsto\int_{[\mathbf{x}_0,\ldots,\mathbf{x}_k]} f := \int_0^1\int_0^{s_1}\cdots\int_0^{s_{k-1}} f(\mathbf{x}_0+s_1\nabla\mathbf{x}_1+\cdots+s_k\nabla\mathbf{x}_k)\,\mathrm{d}s_k\cdots\mathrm{d}s_1 \tag{1.4}$$

(with $\nabla\mathbf{x}_j := \mathbf{x}_j-\mathbf{x}_{j-1}$) is called the *divided difference functional on* $\mathbb{R}^d$ by Micchelli in [8], and is familiar from the Genocchi-Hermite formula

$$[\boldsymbol{\alpha}]g \;=\; \int_{[\boldsymbol{\alpha}]} D^{\#\boldsymbol{\alpha}-1}g \tag{1.5}$$

for the *univariate* divided difference at the (scalar) sequence $\boldsymbol{\alpha}$. See [2] for more detail and for comments concerning the history of this divided difference.

For our purposes here, it is sufficient to know that this divided difference is symmetric in the *points* $\mathbf{x}_i$, and is linear and symmetric in the *directions* $\boldsymbol{\xi}_j$. In particular, by that linearity,

$$\mathbf{x}=\mathbf{y}+\sum_s\alpha_s\mathbf{n}_s \quad\Longrightarrow\quad g(\mathbf{x})=g(\mathbf{y})+\sum_s\alpha_s\,[\mathbf{x},\mathbf{y}\mid\mathbf{n}_s]g \tag{1.6}$$

since $[\mathbf{x}]=[\mathbf{y}]+[\mathbf{x},\mathbf{y}\mid\mathbf{x}-\mathbf{y}]$. Also, for any $\boldsymbol{\vartheta},\mathbf{n}\in\mathbb{R}^d$ and any $\boldsymbol{\alpha}\in\mathbb{R}^k$,

$$[(\boldsymbol{\vartheta}+\alpha_i\mathbf{n}:i=1,\ldots,k)\mid\mathbf{n},\ldots,\mathbf{n}]g \;=\; [\boldsymbol{\alpha}]g(\boldsymbol{\vartheta}+(\,\cdot\,)\mathbf{n}), \tag{1.7}$$

as follows directly from (1.3) and (1.5).

§2. Hyperplanes in General Position

The discussion of Chung-Yao interpolation to follow relies on the following well-known facts concerning hyperplanes in $\mathbb{R}^d$. In their discussion, it is convenient to identify a hyperplane h in $\mathbb{R}^d$ with any one of the linear polynomials whose zero-set is that hyperplane. Denote by $h_\uparrow$ the linear homogeneous part of that polynomial. With this,

$$h(\mathbf{x})=h(\mathbf{y})+h_\uparrow(\mathbf{x}-\mathbf{y}),\quad \mathbf{x},\mathbf{y}\in\mathbb{R}^d\,.$$

In particular,

$$h(\mathbf{x}) = h_\uparrow(\mathbf{x} - \boldsymbol{\vartheta}_h), \quad \mathbf{x} \in \mathbb{R}^d, \boldsymbol{\vartheta}_h \in h,$$

and

$$h_\uparrow(\mathbf{n}) = 0, \quad \text{for } \mathbf{n} \parallel h \ \ (:= \ \mathbf{n} \text{ is parallel to } h).$$

Let H be a collection of d hyperplanes in $\mathbb{R}^d$ *in general position*, meaning that they have exactly one point in common; denote this point by $\boldsymbol{\vartheta}_H$. The hyperplanes are the coordinate planes for any coordinate system with $\boldsymbol{\vartheta}_H$ as its origin and with nontrivial vectors $\mathbf{n}_h$ parallel to

$$\cap(H\backslash h) := \cap_{\tilde{h}\in H\backslash h}\tilde{h},$$

$h \in H$, as its coordinate vectors. To see this, note that $h_\uparrow(\mathbf{n}_{\tilde{h}}) = 0$ for $h \neq \tilde{h}$ (since then $\mathbf{n}_{\tilde{h}} \parallel h$ by assumption), while $h_\uparrow(\mathbf{n}_h) \neq 0$ (since otherwise

$$\boldsymbol{\vartheta}_H \neq \boldsymbol{\vartheta}_H + \mathbf{n}_h \in \cap H = \{\boldsymbol{\vartheta}_H\},$$

a contradiction). Consequently, $(\mathbf{n}_h : h \in H)$ is linearly independent, hence a basis for $\mathbb{R}^d$. Since

$$h(\boldsymbol{\vartheta}_H + \mathbf{n}_{\tilde{h}}) = h(\boldsymbol{\vartheta}_H) + h_\uparrow(\mathbf{n}_{\tilde{h}}) = 0 + 0, \quad h \in H\backslash\tilde{h},$$

h is the affine hull of the points $\boldsymbol{\vartheta}_H, (\boldsymbol{\vartheta}_H + \mathbf{n}_{\tilde{h}} : \tilde{h} \in H\backslash h)$, i.e., the coordinate plane defined by the vanishing of the coordinate associated with $\mathbf{n}_h$.

It follows that the affine map

$$\mathbf{x} \mapsto \boldsymbol{\vartheta}_H + \sum_{h\in H} \frac{h(\mathbf{x})}{h_\uparrow(\mathbf{n}_h)}\,\mathbf{n}_h$$

is well-defined on $\mathbb{R}^d$, and that, for every $\tilde{h} \in H$, it carries the point $\mathbf{x} := \boldsymbol{\vartheta}_H + \mathbf{n}_{\tilde{h}}$ to

$$\boldsymbol{\vartheta}_H + \sum_{h\in H} \frac{h_\uparrow(\mathbf{n}_{\tilde{h}})}{h_\uparrow(\mathbf{n}_h)}\,\mathbf{n}_h = \boldsymbol{\vartheta}_H + \mathbf{n}_{\tilde{h}} = \mathbf{x},$$

hence must be the identity (given that $(\mathbf{n}_h : h \in H)$ is a basis). This proves that

$$\mathbf{x} \ = \ \boldsymbol{\vartheta}_H + \sum_{h\in H} \frac{h(\mathbf{x})}{h_\uparrow(\mathbf{n}_h)}\,\mathbf{n}_h, \quad \mathbf{x} \in \mathbb{R}^d. \tag{2.1}$$

Correspondingly, by (1.6),

$$[\mathbf{x}] \ = \ [\boldsymbol{\vartheta}_H] + \sum_{h\in H} \frac{h(\mathbf{x})}{h_\uparrow(\mathbf{n}_h)}\,[\boldsymbol{\vartheta}_H, \mathbf{x} \mid \mathbf{n}_h]. \tag{2.2}$$

§3. Error Formula and Newton Form for Chung-Yao Interpolation

Let $\mathbb{H}$ be a collection of hyperplanes in $\mathbb{R}^d$ *in general position*, meaning not only that the d hyperplanes in any $H \in \binom{\mathbb{H}}{d}$ have exactly one point in common, denoted again by

$$\boldsymbol{\vartheta}_H,$$

but that also $\boldsymbol{\vartheta}_H = \boldsymbol{\vartheta}_{H'}$ only if $H = H'$. As Chung and Yao [4] were the first to observe, this implies that

$$I_{\mathbb{H}} g := \sum_{H \in \binom{\mathbb{H}}{d}} \ell_H \; g(\boldsymbol{\vartheta}_H),$$

with

$$\ell_H \;:=\; \prod_{h \in \mathbb{H} \backslash H} \frac{h}{h(\boldsymbol{\vartheta}_H)},$$

is well-defined, in $\Pi_{\#\mathbb{H}-d}$, and matches g at the $\binom{\#\mathbb{H}}{d} = \dim \Pi_{\#\mathbb{H}-d}$ points in

$$\Theta_{\mathbb{H}} := \{\boldsymbol{\vartheta}_H : H \in \mathbb{H}\},$$

hence is the unique interpolant to g from $\Pi_{\#\mathbb{H}-d}$ at $\Theta_{\mathbb{H}}$.

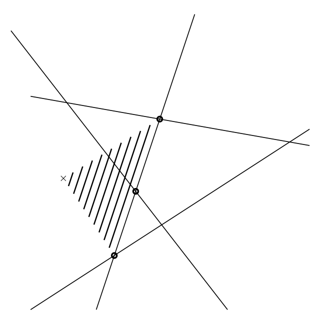

Fig. 1. The typical term in the formula for the error at $\mathbf{x}$ involves the derivative of order $\#\mathbb{H} - d + 1$ in the direction of the line common to $d - 1$ hyperplanes, averaged over a triangle with apex $\mathbf{x}$, multiplied by a polynomial which vanishes on all hyperplanes not containing that line.

Theorem 3.1. *Let $\mathbb{H}$ be a collection of hyperplanes in $\mathbb{R}^d$ in general position. Then,*

$$g \;=\; I_{\mathbb{H}} g \;+ \sum_{K \in \binom{\mathbb{H}}{d-1}} p_K \, [\Theta_{\mathbb{H},K}, \cdot \mid \mathbf{n}_K \ldots, \mathbf{n}_K] g, \tag{3.2}$$

with

$$\Theta_{\mathbb{H},K} \;:=\; \Theta_{\mathbb{H}} \cap (\cap K)$$

the points in $\Theta_{\mathbb{H}}$ on the unique straight line

$$\cap K \;:=\; \bigcap_{h \in K} h$$

common to the $d-1$ hyperplanes in K, and

$$p_K \;:=\; \prod_{h\in \mathbb{H}\backslash K} \frac{h}{h_\uparrow(\mathbf{n}_K)}.$$

Proof: The proof is by induction on $\#\mathbb{H}$, the case $\#\mathbb{H} = d$ being given by (2.2). So, assume (3.2) to hold for a given $\mathbb{H}$ (in general position), and let k be any hyperplane for which

$$\mathbb{H}' := \mathbb{H} \cup k := \mathbb{H} \cup \{k\}$$

is in general position. Then

$$\Theta_{\mathbb{H}'}\backslash\Theta_{\mathbb{H}} \;=\; \{\boldsymbol{\vartheta}_{K\cup k} : K \in \tbinom{\mathbb{H}}{d-1}\} \;=\; \Theta_{\mathbb{H}'} \cap k,$$

and $p_K(\boldsymbol{\vartheta}_{K'\cup k}) = 0$ if (and only if) $K \neq K'$. Hence, by (3.2),

$$g \;=\; I_{\mathbb{H}} g \;+\; \sum_{K\in\binom{\mathbb{H}}{d-1}} p_K\,[\Theta_{\mathbb{H},K}, \boldsymbol{\vartheta}_{K\cup k} \mid \mathbf{n}_K\; \ldots, \mathbf{n}_K]g \quad \text{on} \quad \Theta_{\mathbb{H}'}\backslash\Theta_{\mathbb{H}},$$

while the sum over K on the right side here gives an element of $\Pi_{\#\mathbb{H}+1-d}$ which vanishes on $\Theta_{\mathbb{H}}$. Consequently, the entire right side must equal the unique interpolant from $\Pi_{\#\mathbb{H}'-d}$ to g at $\Theta_{\mathbb{H}'}$, i.e.,

$$I_{\mathbb{H}\cup k} g \;=\; I_{\mathbb{H}} g \;+\; \sum_{K\in\binom{\mathbb{H}}{d-1}} p_K\,[\Theta_{\mathbb{H}\cup k,K} \mid \mathbf{n}_K\; \ldots, \mathbf{n}_K]g, \tag{3.3}$$

thus providing a *Newton form* for Chung-Yao interpolation (also mentioned in [2]).

It follows that, for any $\mathbf{x} \in \mathbb{R}^d$,

$$g(\mathbf{x}) \;=\; \bigl(I_{\mathbb{H}'} g\bigr)(\mathbf{x}) \;+\; \sum_{K\in\binom{\mathbb{H}}{d-1}} p_K(\mathbf{x})\, E_K,$$

with

$$\begin{aligned} E_K \;&:= [\Theta_{\mathbb{H},K}, \mathbf{x} \mid \mathbf{n}_K\; \ldots, \mathbf{n}_K]g - [\Theta_{\mathbb{H},K}, \boldsymbol{\vartheta}_{K\cup k} \mid \mathbf{n}_K\; \ldots, \mathbf{n}_K]g \\ &= [\Theta_{\mathbb{H}',K}, \mathbf{x} \mid \mathbf{n}_K\; \ldots, \mathbf{n}_K, \mathbf{x} - \boldsymbol{\vartheta}_{K\cup k}]g. \end{aligned}$$

Use of (2.2) with $H = K \cup k$ gives

$$E_K \;=\; \sum_{h\in K\cup k} \frac{h(\mathbf{x})}{h_\uparrow(\mathbf{n}_{K\cup k\backslash h})} [\Theta_{\mathbb{H}',K}, \mathbf{x} \mid \mathbf{n}_K\; \ldots, \mathbf{n}_K, \mathbf{n}_{K\cup k\backslash h}]g.$$

Consequently,

$$\big(g - I_{\mathbb{H}'} g\big)(\mathbf{x}) \;=\; \sum_{K\in\binom{\mathbb{H}}{d-1}} p_K(\mathbf{x}) \frac{k(\mathbf{x})}{k_\uparrow(\mathbf{n}_K)} [\Theta_{\mathbb{H}',K}, \mathbf{x} \mid \mathbf{n}_K \ldots, \mathbf{n}_K] g \;+\; \sum_{L\in\binom{\mathbb{H}}{d-2}} F_L, \tag{3.4}$$

with

$$F_L \;:=\; \sum_{h\in\mathbb{H}\backslash L} p_{L\cup h}(\mathbf{x}) \frac{h(\mathbf{x})}{h_\uparrow(\mathbf{n}_{L\cup k})} [\Theta_{\mathbb{H}',L\cup h}, \mathbf{x} \mid \mathbf{n}_{L\cup h} \ldots, \mathbf{n}_{L\cup h}, \mathbf{n}_{L\cup k}] g.$$

In particular, for $K \in \binom{\mathbb{H}}{d-1}$, we obtain the correct form of the error term, therefore, by symmetry and the independence of the polynomials

$$q_K \;:=\; \begin{cases} p_K \frac{k}{k_\uparrow(\mathbf{n}_K)}, & K \in \binom{\mathbb{H}}{d-1}; \\ \prod_{h\in\mathbb{H}\backslash L} \frac{h}{h_\uparrow(\mathbf{n}_{L\cup k})}, & K = L\cup k, L \in \binom{\mathbb{H}}{d-2}, \end{cases}$$

necessarily

$$F_L \;=\; q_{L\cup k}(\mathbf{x})\, [\Theta_{\mathbb{H}',L\cup k}, \mathbf{x} \mid \mathbf{n}_{L\cup k}, \ldots, \mathbf{n}_{L\cup k}] g, \quad L \in \binom{\mathbb{H}}{d-2}. \tag{3.5}$$

However, a rigorous version of this proof would have to deal with the fact that, in (3.4), the coefficients of the polynomials q_K are not just constants, unless g is in $\Pi_{\#\mathbb{H}+1-d}$, hence would have to prove the claim that the identity

$$\sum_{\tilde h\in\mathbb{H}\backslash L} \left(\prod_{h\in\mathbb{H}\backslash(L\cup\tilde h)} \frac{h_\uparrow(\mathbf{n}_{L\cup k})}{h_\uparrow(\mathbf{n}_{L\cup\tilde h})} \right) [\Theta_{\mathbb{H}\cup k, L\cup\tilde h}, \mathbf{x} \mid \mathbf{n}_{L\cup\tilde h} \ldots, \mathbf{n}_{L\cup\tilde h}, \mathbf{n}_{L\cup k}] g$$
$$= [\Theta_{\mathbb{H}\cup k, L\cup k}, \mathbf{x} \mid \mathbf{n}_{L\cup k}, \ldots, \mathbf{n}_{L\cup k}] g, \quad L \in \binom{\mathbb{H}}{d-2}, \tag{3.6}$$

needed here to prove (3.5), holds for all smooth g if it holds for $g \in \Pi_{\#\mathbb{H}+1-d}$. But the details of such reasoning, while certainly of interest for a better understanding of this multivariate divided difference, become much more involved than the following direct proof of (3.6).

Since $\cap L$ is a two-dimensional plane, (3.6) can be obtained by applying to

$$[\,\cdot\,, \mathbf{x} \mid \mathbf{n}_{L\cup k}] g$$

the *bivariate* divided difference identity established in the Lemma 3.9 below. This proves (3.5), hence gives, with (3.4), that (3.2) holds with $\mathbb{H}$ replaced by $\mathbb{H}'$, thus finishing the induction step, hence the proof of the theorem. ■

In the proof of Lemma 3.9, the following lemma is used.

Lemma 3.7. *Let $\ell = \boldsymbol{\vartheta} + \mathrm{span}(\mathbf{n})$ be a straight line in $\mathbb{R}^d$ and let $\mathbb{H}$ be a collection of hyperplanes in $\mathbb{R}^d$* in general position with respect to ℓ, *meaning that each $h \in \mathbb{H}$ has exactly one point in common with ℓ, denoted by $\boldsymbol{\vartheta}_h$, and that $\boldsymbol{\vartheta}_h = \boldsymbol{\vartheta}_{\tilde{h}}$ only if $h = \tilde{h}$. Then*

$$[(\boldsymbol{\vartheta}_h : h \in \mathbb{H}) \mid \mathbf{n}, \ldots, \mathbf{n}] \;=\; \sum_{k\in\mathbb{H}} \left(\prod_{h\in\mathbb{H}\backslash k} \frac{h_\uparrow(\mathbf{n})}{h(\boldsymbol{\vartheta}_k)} \right) [\boldsymbol{\vartheta}_k]. \tag{3.8}$$

Proof: Let

$$\boldsymbol{\vartheta}_h =: \boldsymbol{\vartheta} + \alpha_h \mathbf{n}, \quad h \in \mathbb{H}.$$

Then

$$h(\boldsymbol{\vartheta}_{\tilde{h}}) = h(\boldsymbol{\vartheta}_h) + h_\uparrow(\boldsymbol{\vartheta}_{\tilde{h}} - \boldsymbol{\vartheta}_h) = 0 + (\alpha_{\tilde{h}} - \alpha_h) h_\uparrow(\mathbf{n}),$$

hence the right side of (3.8) equals the linear functional

$$f \mapsto [(\alpha_h : h \in \mathbb{H})] f(\boldsymbol{\vartheta} + (\cdot)\mathbf{n}) \;=\; [(\boldsymbol{\vartheta}_h : h \in \mathbb{H}) \mid \mathbf{n}, \ldots, \mathbf{n}] f,$$

the last equation by (1.7). ■

Lemma 3.9. *Let $\mathbb{H}' := \mathbb{H} \cup k$ be a collection of straight lines in $\mathbb{R}^2$ in general position, and, for each $h \in \mathbb{H}'$, let $\mathbf{n}_h$ be a nonzero vector parallel to h, and set*

$$\Theta_h := \Theta_{\mathbb{H}',h} \;=\; \Theta_{\mathbb{H}'} \cap h.$$

Then

$$\sum_{\tilde{h}\in\mathbb{H}} \left(\prod_{h\in\mathbb{H}\backslash\tilde{h}} \frac{h_\uparrow(\mathbf{n}_k)}{h_\uparrow(\mathbf{n}_{\tilde{h}})} \right) [\Theta_{\tilde{h}} \mid \mathbf{n}_{\tilde{h}}, \ldots, \mathbf{n}_{\tilde{h}}] \;=\; [\Theta_k \mid \mathbf{n}_k, \ldots, \mathbf{n}_k]. \tag{3.10}$$

Proof: By Lemma 3.7 and for every $\tilde{h} \in \mathbb{H}'$,

$$[\Theta_{\tilde{h}} \mid \mathbf{n}_{\tilde{h}}, \ldots, \mathbf{n}_{\tilde{h}}] \;=\; \sum_{\widehat{h}\in\mathbb{H}'\backslash\tilde{h}} \left(\prod_{h\in\mathbb{H}'\backslash\{\tilde{h},\widehat{h}\}} \frac{h_\uparrow(\mathbf{n}_{\tilde{h}})}{h(\boldsymbol{\vartheta}_{\tilde{h},\widehat{h}})} \right) [\boldsymbol{\vartheta}_{\tilde{h},\widehat{h}}], \tag{3.11}$$

with

$$\boldsymbol{\vartheta}_{\tilde{h},\widehat{h}}$$

the unique point common to the two straight lines $\tilde{h}$ and $\widehat{h}$. Substitution of this into the left side of (3.10) produces the double sum

$$\sum_{\tilde{h}\in\mathbb{H}} \sum_{\widehat{h}\in\mathbb{H}'\backslash\tilde{h}} w(\tilde{h}, \widehat{h})\, [\boldsymbol{\vartheta}_{\tilde{h},\widehat{h}}], \tag{3.12}$$

with

$$w(\tilde h, \widehat h) \;:=\; \left(\prod_{h\in\mathbb{H}\backslash\tilde h} \frac{h_\uparrow(\mathbf{n}_k)}{h_\uparrow(\mathbf{n}_{\tilde h})}\right)\left(\prod_{h\in\mathbb{H}'\backslash\{\tilde h,\widehat h\}} \frac{h_\uparrow(\mathbf{n}_{\tilde h})}{h(\boldsymbol{\vartheta}_{\tilde h,\widehat h})}\right).$$

Each $[\boldsymbol{\vartheta}]$ with $\boldsymbol{\vartheta}\in\Theta_{\mathbb{H}}$ occurs exactly twice in the sum (3.12). For any such $\boldsymbol{\vartheta}$, set

$$\boldsymbol{\vartheta} =: \boldsymbol{\vartheta}_{\tilde h,\widehat h}.$$

Then the coefficient of $[\boldsymbol{\vartheta}]$ in (3.12) is

$$w(\tilde h,\widehat h)+w(\widehat h,\tilde h) \;=\; \left(\prod_{h\in\mathbb{H}\backslash\{\tilde h,\widehat h\}} \frac{h_\uparrow(\mathbf{n}_k)}{h(\boldsymbol{\vartheta})}\right) F\,,$$

with

$$F \;:=\; \frac{\widehat h_\uparrow(\mathbf{n}_k)\,k_\uparrow(\mathbf{n}_{\tilde h})}{\widehat h_\uparrow(\mathbf{n}_{\tilde h})\,k(\boldsymbol{\vartheta})} + \frac{\tilde h_\uparrow(\mathbf{n}_k)\,k_\uparrow(\mathbf{n}_{\widehat h})}{\tilde h_\uparrow(\mathbf{n}_{\widehat h})\,k(\boldsymbol{\vartheta})}.$$

Now note that F depends linearly on $\mathbf{n}_k$ and is constant as a function of $\mathbf{n}_{\tilde h}$ and $\mathbf{n}_{\widehat h}$, hence must be zero for all choices of $\mathbf{n}_k$, $\mathbf{n}_{\tilde h}$, $\mathbf{n}_{\widehat h}$ if it can be shown to be zero for a particular (nontrivial) choice. Choose, in particular,

$$\begin{aligned}\mathbf{n}_k &= \boldsymbol{\vartheta}_{\tilde h,k} - \boldsymbol{\vartheta}_{\widehat h,k},\\ \mathbf{n}_{\tilde h} &= \boldsymbol{\vartheta} - \boldsymbol{\vartheta}_{\tilde h,k},\\ \mathbf{n}_{\widehat h} &= \boldsymbol{\vartheta} - \boldsymbol{\vartheta}_{\widehat h,k}.\end{aligned}$$

Then (see Figure 2 below)

$$k_\uparrow(\mathbf{n}_{\tilde h}) = k(\boldsymbol{\vartheta}) = k_\uparrow(\mathbf{n}_{\widehat h}),$$

while

$$\widehat h_\uparrow(\mathbf{n}_k) = -\widehat h_\uparrow(\mathbf{n}_{\tilde h})$$

and

$$\tilde h_\uparrow(\mathbf{n}_k) = \tilde h_\uparrow(\mathbf{n}_{\widehat h}),$$

and this implies the vanishing of F, for this choice of the $\mathbf{n}_h$, hence for any (nontrivial) choice.

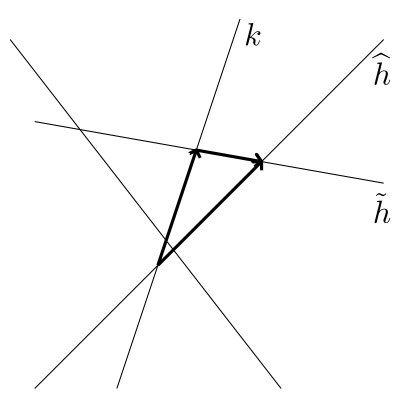

Fig. 2. The vectors $\mathbf{n}_k, \mathbf{n}_{\tilde h}, \mathbf{n}_{\widehat h}$.

To be sure, the vanishing of F is nothing more than the fact that

$$[\boldsymbol{\vartheta}_{\tilde{h},k}] - [\boldsymbol{\vartheta}_{\widehat{h},k}] = ([\boldsymbol{\vartheta}] - [\boldsymbol{\vartheta}_{\widehat{h},k}]) \; - \; ([\boldsymbol{\vartheta}] - [\boldsymbol{\vartheta}_{\tilde{h},k}]).$$

In any event, (3.12) reduces to the sum of terms involving $\boldsymbol{\vartheta} \in \Theta_k$, i.e., to the sum

$$\sum_{\tilde{h}\in\mathbb{H}} w(\tilde{h},k)[\boldsymbol{\vartheta}_{\tilde{h},k}],$$

with

$$\begin{aligned} w(\tilde{h},k) \; &= \left(\prod_{h\in\mathbb{H}\backslash\tilde{h}} \frac{h_\uparrow(\mathbf{n}_k)}{h_\uparrow(\mathbf{n}_{\tilde{h}})}\right) \left(\prod_{h\in\mathbb{H}\backslash\tilde{h}} \frac{h_\uparrow(\mathbf{n}_{\tilde{h}})}{h(\boldsymbol{\vartheta}_{\tilde{h},k})}\right) \\ &= \prod_{h\in\mathbb{H}\backslash\tilde{h}} \frac{h_\uparrow(\mathbf{n}_k)}{h(\boldsymbol{\vartheta}_{\tilde{h},k})}, \end{aligned}$$

and this sum equals the right side of (3.10), by Lemma 3.7. ■

It seems certain that (3.10) is the bivariate case of a *multi*variate divided difference identity.

§4. An Error Formula for Tensor-product Interpolation

Let $\mathbf{t}_i =: (t_{i,1},\ldots,t_{i,d})$, $i = 0,1,2,\ldots$, be a sequence of points in $\mathbb{R}^d$, with $t_{i,s} = t_{j,s}$ for some s only if $i = j$. For any particular $\mathbf{k} = (k_1,\ldots,k_d)$ with nonnegative integer entries, consider the rectangular mesh

$$\Theta_{\mathbf{k}} := \mathop{\times}_{s=1}^{d} \{t_{i,s} : i = 0,\ldots,k_s\}.$$

It is well known that the tensor-product polynomial space

$$\Pi_{\mathbf{k}} \; := \; \Pi_{k_1} \otimes \cdots \otimes \Pi_{k_d}$$

contains, for given g defined at least on $\Theta_{\mathbf{k}}$, exactly one element,

$$I_{\mathbf{k}} g,$$

that matches g on $\Theta_{\mathbf{k}}$.

Error formulæ for this polynomial interpolant can be found in the standard literature as early as [7] (and probably earlier). However, all these formulæ are variants on the following simple tensor-product construct. With $I_{s,\mathbf{k}}$ denoting univariate polynomial interpolation at $t_{0,s},\ldots,t_{k_s,s}$, one has

$$\mathrm{id} \; = \; \bigotimes_{s=1}^{d} \left(I_{s,\mathbf{k}} + (\mathrm{id}_s - I_{s,\mathbf{k}})\right),$$

hence the resulting error formula

$$\mathrm{id} - I_{\mathbf{k}} \;=\; \sum_{\boldsymbol{\alpha}\in\{0,1\}^d\backslash\mathbf{0}} \bigotimes_{s=1}^{d} I_{s,\mathbf{k}}^{\alpha_s}(\mathrm{id}_s - I_{s,\mathbf{k}})^{1-\alpha_s}$$

contains explicitly the term

$$\bigotimes_{s=1}^{d}(\mathrm{id}_s - I_{s,\mathbf{k}}),$$

and, therefore, when applied to the interpoland g and written in standard integral form, involves the high-order mixed derivative $D^{\mathbf{k+1}}g = D_1^{k_1+1}\cdots D_d^{k_d+1}g$ (along with various other mixed derivatives).

On the other hand, it has been known at least since [5] that the distance of a smooth function g from $\Pi_{\mathbf{k}}$ can be bounded entirely in terms of the low-order pure derivatives $D^{k_s+1}g$, $s=1,\ldots,d$, of the interpoland, g.

It is the purpose of this section to show that, correspondingly, tensor-product polynomial interpolation admits an error formula that involves only these pure derivatives $D^{k_s+1}g$, $s=1,\ldots,d$, of the interpoland.

The formula to be proved, for the error at $\mathbf{x}=(x_s : s=1,\ldots,d)$, involves d terms, with the sth term the product, of

$$\psi_{s,\mathbf{k}}(\mathbf{x}) := (x_s - t_{0,s})\cdots(x_s - t_{k_s,s})$$

with the value at

$$\mathbf{x}_{\backslash s} \;:=\; (x_i : i \neq s)$$

of the interpolant, on the mesh

$$\Theta_{\mathbf{k},\backslash s} := \mathop{\times}_{\sigma\neq s}\{t_{0,\sigma},\ldots,t_{k_\sigma,\sigma}\}$$

from the polynomial space

$$\Pi_{\mathbf{k},\backslash s} \;:=\; \bigotimes_{\sigma\neq s}\Pi_{k_\sigma}$$

to the function

$$\mathbb{R}^{d-1} \to \mathbb{R} : \mathbf{y} \mapsto [((t_{i,s}|_s\mathbf{y}) : i=0,\ldots,k_s)\,,\mathbf{x} \mid \mathbf{i}_s,\ldots,\mathbf{i}_s]g.$$

Here,

$$\mathbf{i}_s \;:=\; (\underbrace{0,\ldots,0}_{s-1},1,0,\ldots),$$

and

$$(t|_s\mathbf{y}) := (y_1,\ldots,y_{s-1},t,y_s,\ldots,y_{d-1})$$

is an attempt to avoid more cumbersome notation. In the same spirit, I will denote by

$$I_{\mathbf{k},\backslash s}$$

the operator of polynomial interpolation on $\Theta_{\mathbf{k},\backslash s}$ from $\Pi_{\mathbf{k},\backslash s}$.

Theorem 4.1. *The error at $\mathbf{x} \in \mathbb{R}^d$ of the polynomial interpolant $I_{\mathbf{k}}g$ from $\Pi_{\mathbf{k}}$ at $\Theta_{\mathbf{k}}$ to g can be written as the sum*

$$\begin{aligned}(g - I_{\mathbf{k}}g)(\mathbf{x}) \;=\;& \\ \sum_{s=1}^{d} \psi_{s,\mathbf{k}}(\mathbf{x}) \, \big(I_{\mathbf{k},\backslash s}\, [((t_{i,s}|_s\, \cdot\,) : i = 0,\ldots,k_s)\,, \mathbf{x} \mid \mathbf{i}_s,\ldots,\mathbf{i}_s]g\big)(\mathbf{x}_{\backslash s}).&\end{aligned} \tag{4.2}$$

Proof: The proof is by induction on $\mathbf{k}$, the simplest case, $\mathbf{k} = \mathbf{0}$, being covered by (2.2). Assume that (4.2) holds for $\mathbf{k} = \mathbf{h}$ and consider, without loss of generality, just the case $\mathbf{k} = \mathbf{h}' := \mathbf{h} + \mathbf{i}_1$.

Abbreviate the summands in (4.2), for $\mathbf{k} = \mathbf{h}$,

$$F_s(\mathbf{x}) \;:=\; \psi_{s,\mathbf{h}}(\mathbf{x}) \, \big(I_{\mathbf{h},\backslash s}\, [((t_{i,s}|_s\, \cdot\,) : i = 0,\ldots,h_s)\,, \mathbf{x} \mid \mathbf{i}_s,\ldots,\mathbf{i}_s]g\big)(\mathbf{x}_{\backslash s}),$$

write the first summand in Lagrange form:

$$F_1(\mathbf{x}) \;=\; \psi_{1,\mathbf{h}}(\mathbf{x}) \sum_{\boldsymbol{\vartheta}\in\Theta_{\mathbf{h},\backslash 1}} \ell_{\boldsymbol{\vartheta}}(\mathbf{x}_{\backslash 1})\; [((t_{i,1}|_1\boldsymbol{\vartheta}) : i = 0,\ldots,h_1)\,, \mathbf{x} \mid \mathbf{i}_1,\ldots,\mathbf{i}_1]g$$

and note that, by the assumed validity of (4.2) for $\mathbf{k} = \mathbf{h}$, on replacing here the $\mathbf{x}$ in the divided difference by $(t_{h_1',1}|_1\boldsymbol{\vartheta})$, the resulting function

$$p : \mathbf{x} \mapsto \; \psi_{1,\mathbf{h}}(\mathbf{x}) \sum_{\boldsymbol{\vartheta}\in\Theta_{\mathbf{h},\backslash 1}} \ell_{\boldsymbol{\vartheta}}(\mathbf{x}_{\backslash 1})\; [((t_{i,1}|_1\boldsymbol{\vartheta}) : i = 0,\ldots,h_1') \mid \mathbf{i}_1,\ldots,\mathbf{i}_1]g$$

is a polynomial in $\Pi_{\mathbf{h}'}$ which agrees with $g - I_{\mathbf{h}}g$ at $(t_{h_1',1}|_1\boldsymbol{\vartheta})$ for all $\boldsymbol{\vartheta} \in \Theta_{\mathbf{h},\backslash 1}$; it also vanishes on $\Theta_{\mathbf{h}}$, because of the factor $\psi_{1,\mathbf{h}}$. Hence $I_{\mathbf{h}}g + p = I_{\mathbf{h}'}g$, and

$$\begin{aligned}(g - I_{\mathbf{h}'}g)(\mathbf{x}) \;=\;& \psi_{1,\mathbf{h}}(\mathbf{x}) \times \\ &\sum_{\boldsymbol{\vartheta}\in\Theta_{\mathbf{h},\backslash 1}} \ell_{\boldsymbol{\vartheta}}(\mathbf{x}_{\backslash 1}) \; \sum_{s=1}^{d} (x_s - \vartheta_s)[((t_{i,1}|_1\boldsymbol{\vartheta}) : i = 0,\ldots,h_1')\,, \mathbf{x} \mid \mathbf{i}_1,\ldots,\mathbf{i}_1, \mathbf{i}_s]g \\ &+ \sum_{s=2}^{d} F_s(\mathbf{x}),\end{aligned}$$

making use of the fact that, by (1.1) and (1.6),

$$\begin{aligned}&[((t_{i,1}|_1\boldsymbol{\vartheta}) : i = 0,\ldots,h_1)\,, \mathbf{x} \mid \mathbf{i}_1,\ldots,\mathbf{i}_1]g \\ &\quad - [((t_{i,1}|_1\boldsymbol{\vartheta}) : i = 0,\ldots,h_1') \mid \mathbf{i}_1,\ldots,\mathbf{i}_1]g \\ &\qquad = \sum_{s=1}^{d} (x_s - \vartheta_s)[((t_{i,1}|_1\boldsymbol{\vartheta}) : i = 0,\ldots,h_1')\,, \mathbf{x} \mid \mathbf{i}_1,\ldots,\mathbf{i}_1, \mathbf{i}_s]g.\end{aligned}$$

It follows that

$$\begin{aligned}(g - I_{\mathbf{h}'}g)(\mathbf{x}) \ =&\\ \psi_{1,\mathbf{h}'}(\mathbf{x})\big(I_{\mathbf{h}',\backslash 1}\,[((t_{i,1}|_1\,\cdot\,) : i = 0, \ldots, h'_1)\,, \mathbf{x} \mid \mathbf{i}_1, \ldots, \mathbf{i}_1]g\big)(\mathbf{x}_{\backslash 1})&\\ + \sum_{s=2}^{d} \big(E_s(\mathbf{x}) + F_s(\mathbf{x})\big),&\end{aligned} \tag{4.3}$$

with

$$\begin{aligned}E_s(\mathbf{x}) \ :=& \ \psi_{1,\mathbf{h}}(\mathbf{x}) \cdot\\ &\sum_{\boldsymbol{\vartheta}\in\Theta_{\mathbf{h},\backslash 1}} \ell_{\boldsymbol{\vartheta}}(\mathbf{x}_{\backslash 1})(x_s - \vartheta_s)[((t_{i,1}|_1\boldsymbol{\vartheta}) : i = 0, \ldots, h'_1)\,, \mathbf{x} \mid \mathbf{i}_1, \ldots, \mathbf{i}_1, \mathbf{i}_s]g\ .\end{aligned}$$

Now, as for the sum in $E_s(\mathbf{x})$, observe that

$$\begin{aligned}\sum_{\boldsymbol{\vartheta}\in\Theta_{\mathbf{h},\backslash 1}} \ell_{\boldsymbol{\vartheta}}(\mathbf{x}_{\backslash 1})(x_s - \vartheta_s)[((t_{i,1}|_1\boldsymbol{\vartheta}) : i = 0, \ldots, h'_1)\,, \mathbf{x} \mid \mathbf{i}_1, \ldots, \mathbf{i}_1, \mathbf{i}_s]g&\\ = \big((I_{\mathbf{h}'\backslash\{1,s\}} \otimes I_{s,\mathbf{h}'})f\big)(\mathbf{x}_{\backslash 1}),&\end{aligned}$$

with

$$f(\boldsymbol{\vartheta}) \ := \ (x_s - \vartheta_s)[(t_{i,1} : i = 0, \ldots, h'_1)][(\,\cdot\,|_1\boldsymbol{\vartheta}), \mathbf{x} \mid \mathbf{i}_s]g.$$

Further, for any univariate φ (defined at least on $(t_{i,1} : i = 0, \ldots, h'_1)$),

$$\begin{aligned}\big(I_{s,\mathbf{h}'}\,(\mathbf{x} -\ \cdot\,)_s\,\varphi\big)(x_s) \ =& \ \sum_{j=0}^{h'_s} \big(\prod_{i\neq j} \frac{x_s - t_{i,s}}{t_{j,s} - t_{i,s}}\big)\,(x_s - t_{j,s})\varphi(t_{j,s})\\ =& \ \psi_{s,\mathbf{h}'}(\mathbf{x})\,[(t_{i,s} : i = 0, \ldots, h'_s)]\varphi.\end{aligned}$$

Hence, altogether,

$$E_s(\mathbf{x}) \ = \ \psi_{1,\mathbf{h}}(\mathbf{x})\psi_{s,\mathbf{h}'}(\mathbf{x}) \sum_{\boldsymbol{\vartheta}\in\Theta_{\mathbf{h}',\backslash\{1,s\}}} \ell_{\boldsymbol{\vartheta}}(\mathbf{x}_{\backslash\{1,s\}})\,G_s(\mathbf{x}, \boldsymbol{\vartheta}),$$

with

$$\begin{aligned}G_s(\mathbf{x}, \boldsymbol{\vartheta}) \ :=& \ [(t_{i,1} : i = 0, \ldots, h'_1)] \otimes [(t_{i,s} : i = 0, \ldots, h'_s)][(\,\cdot\,|_{\{1,s\}}\boldsymbol{\vartheta}), \mathbf{x} \mid \mathbf{i}_s]g\\ =& \ [(t_{i,1} : i = 0, \ldots, h'_1)]\,[((\,\cdot\,|_1(t_{i,s}|_s\boldsymbol{\vartheta})) : i = 0, \ldots, h'_s)\,, \mathbf{x} \mid \mathbf{i}_s, \ldots, \mathbf{i}_s]g,\end{aligned}$$

the second equality by (1.7). Further, for any univariate φ (defined at least on $(t_{i,1} : i = 0, \ldots, h'_1)$),

$$\psi_{1,\mathbf{h}}(\mathbf{x})([(t_{i,1} : i = 0, \ldots, h'_1)]\varphi)(x_1) = \big((I_{1,\mathbf{h}'} - I_{1,\mathbf{h}})\varphi\big)(x_1).$$

Hence, altogether,

$$\begin{aligned}E_s(\mathbf{x}) \ =& \ \psi_{s,\mathbf{h}'}(\mathbf{x})\times\\ &\big((I_{\mathbf{h}',\backslash s} - I_{\mathbf{h},\backslash s})[((t_{i,s}|_s\,\cdot\,) : i = 0, \ldots, h_s = h'_s)\,, \mathbf{x} \mid \mathbf{i}_s, \ldots, \mathbf{i}_s]g\big)(\mathbf{x}_{\backslash s}),\end{aligned}$$

from which, with $\psi_{s,\mathbf{h}} = \psi_{s,\mathbf{h}'}$ for $s > 1$, we conclude that

$$\begin{aligned}E_s(\mathbf{x}) + F_s(\mathbf{x}) \ =&\\ &\psi_{s,\mathbf{h}'}(\mathbf{x})\big(I_{\mathbf{h}',\backslash s}\,[((t_{i,s}|_s\,\cdot\,) : i = 0, \ldots, h'_s)\,, \mathbf{x} \mid \mathbf{i}_s, \ldots, \mathbf{i}_s]g\big)(\mathbf{x}_{\backslash s}).\end{aligned}$$

Substitution of this into (4.3) gives (4.2) for $\mathbf{k} = \mathbf{h}'$, and so finishes the induction step. ∎

§5. More General Error Formulæ

Both error formulæ involve weighted integrals, of certain derivatives of the interpoland, over *triangles*, even when $d > 2$. Both formulæ reduce, on certain straight lines, to the standard error formula for univariate polynomial interpolation. It would be interesting to explore the error formula for Chung-Yao interpolation in the limiting case, as some of the hyperplanes become parallel and/or coincident, in which case the 'finite part' of the interpolant becomes the Dyn-Ron interpolant introduced in [6]. Since tensor-product interpolation is a special case of the latter, it should be possible to connect the two error formulæ in this way and, in the process, provide a corresponding error formula for Dyn-Ron interpolation.

Both error formulæ are of the form

$$(g - Ig)(\mathbf{x}) \;=\; \sum_{\varphi\in\Phi} \varphi(\mathbf{x}) M_{\mathbf{x},\varphi}(q_\varphi(D)g), \tag{5.1}$$

with Φ a minimal generating set for ideal(Θ), the ideal of all polynomials which vanish on Θ, and $(q_\varphi : \varphi \in \Phi)$ a sequence of *homogeneous* polynomials *dual* to Φ in the sense that

$$q_\varphi(D)(\tilde{\varphi}_\uparrow) = 0 \quad\Longleftrightarrow\quad \varphi \neq \tilde{\varphi},$$

and $M_{\mathbf{x},\varphi}$ certain distributions. (Here, $p_\uparrow$ denotes the *leading term* of the polynomial p, i.e., the homogeneous polynomial characterized by the fact that the degree of $p - p_\uparrow$ is less than that of p (with the zero polynomial being its own leading term, by definition).) One would hope that a formula of the form (5.1) would be available for more general polynomial interpolation schemes, at least for the least interpolation introduced in [3]. The least interpolation scheme for a given pointset Θ chooses a particular polynomial space from which to interpolate, and this is just Π_k whenever interpolation from Π_k to data on Θ is uniquely possible, i.e., whenever Θ is minimally total for Π_k. It is therefore instructive to compare the form (5.1) with the only error formula presently available in this case, namely the remarkable formula of Sauer and Xu, in [9].

The Sauer-Xu formula relies on the fact that, since Θ is miminally total for Π_k, it must be possible to partition Θ into subsets $\Theta_0, \ldots, \Theta_k$ so that, for each $j = 0, \ldots, k$, the pointset

$$\Theta_{\le j} \;:=\; \bigcup_{i\le j} \Theta_i$$

is minimally total for Π_j. For any such partition of Θ and for each $\boldsymbol{\vartheta} \in \Theta_j$, let $\ell_{\boldsymbol{\vartheta}}$ be the unique element in Π_j for which

$$\ell_{\boldsymbol{\vartheta}}(\boldsymbol{\vartheta}') = \delta_{\boldsymbol{\vartheta},\boldsymbol{\vartheta}'} \qquad \forall \boldsymbol{\vartheta}' \in \Theta_{\le j}.$$

With this, the Sauer-Xu formula, for the error in the interpolant $I_k g$ from Π_k to g at Θ, can be written

$$(g - I_k g)(\mathbf{x}) = \sum_{\boldsymbol{\vartheta}\in\Theta_k} \sum_{\boldsymbol{\mu}\in \mathrm{M}^k_{\boldsymbol{\vartheta}}(\mathbf{x})} \left(\prod_{i=0}^{k} \ell_{\boldsymbol{\mu}_i}(\boldsymbol{\mu}_{i+1})\right) [\boldsymbol{\mu} \mid \Delta\boldsymbol{\mu}]g, \tag{5.2}$$

with

$$\mathrm{M}^k_{\boldsymbol{\vartheta}}(\mathbf{x}) \ := \ (\mathop{\times}_{i=0}^{k-1} \Theta_i) \times \{\boldsymbol{\vartheta}\} \times \{\mathbf{x}\}$$

and

$$\Delta\boldsymbol{\mu} := (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0, \boldsymbol{\mu}_2 - \boldsymbol{\mu}_1, \ldots).$$

For the sake of comparison with (5.1), here is the formula with its dependence on $\mathbf{x}$ made more explicit:

$$(g - I_k g)(\mathbf{x}) = \sum_{\boldsymbol{\vartheta}\in\Theta_k} \ell_{\boldsymbol{\vartheta}}(\mathbf{x}) \sum_{\boldsymbol{\mu}\in \mathrm{M}^k_{\boldsymbol{\vartheta}}} \left(\prod_{i=0}^{k-1} \ell_{\boldsymbol{\mu}_i}(\boldsymbol{\mu}_{i+1})\right) [\boldsymbol{\mu}, \mathbf{x} \mid \Delta\boldsymbol{\mu}, \mathbf{x} - \boldsymbol{\vartheta}]g, \tag{5.3}$$

with

$$\mathrm{M}^k_{\boldsymbol{\vartheta}} \ := \ (\mathop{\times}_{i=0}^{k-1} \Theta_i) \times \{\boldsymbol{\vartheta}\}.$$

It is, of course, possible to apply (1.6) in many ways to bring this formula into the form

$$(g - I_k g)(\mathbf{x}) = \sum_{\varphi\in\Phi} \varphi(\mathbf{x}) M_{\mathbf{x},\varphi}(g).$$

However, careless application of (1.6) may well result in a Φ which, while generating ideal(Θ), fails to be minimally generating. Also, at present, I have no idea just how to use (1.6) here so as to ensure that the coefficient of φ here is the result of a distribution applied to some appropriate derivative of g (rather than the sum of distributions applied to various derivatives of g).

This difficulty exists even in the case of Chung-Yao interpolation, in which case we know what end result we have in mind. To elaborate, in that case, any ordering

$$\mathbb{H} =: \{h_1, \ldots, h_m\}$$

of the set $\mathbb{H}$ produces a suitable partition of $\Theta_{\mathbb{H}}$ as follows:

$$\Theta_{j-d} \ := \ \{\boldsymbol{\vartheta}_{K\cup h_j} : K \in \binom{\mathbb{H}_{<j}}{d-1}\},$$

with

$$\mathbb{H}_{<j} \ := \ \{h_i : i < j\},$$

$j = d, \ldots, m =: d + k$. Correspondingly,

$$\ell_{\boldsymbol{\vartheta}_{K\cup h_j}} = \ell_{K,j} := \prod_{h\in\mathbb{H}_{<j}\backslash K} \frac{h}{h(\boldsymbol{\vartheta}_{K\cup h_j})}.$$

Therefore, with

$$\boldsymbol{\mu}_{j-d} := \boldsymbol{\vartheta}_{K_j \cup h_j}, \quad j = d, \ldots, m = d + k,$$

we have

$$\ell_{\boldsymbol{\mu}_{i-d}}(\boldsymbol{\mu}_{i+1-d}) = 0 \quad \text{unless} \quad (K_{i+1} \cup h_{i+1}) \cap (\mathbb{H}_{<i} \backslash K_i) = \emptyset,$$

i.e., unless $(K_{i+1} \backslash h_i) \subset K_i$. In other words,

$$\ell_{\boldsymbol{\mu}_{i-d}}(\boldsymbol{\mu}_{i+1-d}) \neq 0 \quad \Longrightarrow \quad K_i = \begin{cases} (K_{i+1} \backslash h_i) \cup h, h \in \mathbb{H}_{<i}, & \text{if } h_i \in K_{i+1}; \\ K_{i+1} & \text{otherwise.} \end{cases} \tag{5.4}$$

This greatly reduces the inner sum in (5.3) since, with

$$\boldsymbol{\vartheta} =: \boldsymbol{\vartheta}_{K \cup h_m},$$

(5.4) implies that we only have to sum over $\boldsymbol{\mu}$ in

$$\begin{aligned} M_K^k \ := \{ (\boldsymbol{\vartheta}_{K_d \cup h_d}, \ldots, \boldsymbol{\vartheta}_{K_m \cup h_m}) : \\ K_{i+1} \backslash h_i \subset K_i \subset \mathbb{H}_{<i}, i = d, \ldots, m-1; K_m = K \}. \end{aligned}$$

In other words, for Chung-Yao interpolation, (5.3) reduces to

$$\begin{aligned} &(g - I_{\mathbb{H}} g)(\mathbf{x}) \ = \\ &\sum_{K \in \binom{\mathbb{H}_{<m}}{d-1}} \ell_{K,m}(\mathbf{x}) \sum_{\boldsymbol{\mu} \in \mathrm{M}_K^k} \left(\prod_{i=0}^{k-1} \ell_{\boldsymbol{\mu}_i}(\boldsymbol{\mu}_{i+1}) \right) [\boldsymbol{\mu}, \mathbf{x} \mid \Delta \boldsymbol{\mu}, \mathbf{x} - \boldsymbol{\vartheta}_{K \cup h_m}] g. \end{aligned} \tag{5.5}$$

In particular, by (5.4), for $K = \mathbb{H}_{<d}$, this leaves just one $\boldsymbol{\mu}$, as was first noticed by Vladimir Yegorov [10], during an attempt to derive (3.2) from (5.2). Specifically, in the notation used here,

$$M_K^k = \{ (\boldsymbol{\vartheta}_{K \cup h_i} : i = d, \ldots, m) \}, \quad K = \mathbb{H}_{<d}.$$

Correspondingly,

$$\ell_{K,m}(\mathbf{x}) \sum_{\boldsymbol{\mu} \in \mathrm{M}_K^k} \left(\prod_{i=0}^{k-1} \ell_{\boldsymbol{\mu}_i}(\boldsymbol{\mu}_{i+1}) \right) \ = \ \prod_{i=d}^{m-1} \frac{h_i(\mathbf{x})}{h_i(\boldsymbol{\vartheta}_{K \cup h_{i+1}})}.$$

Since $h_i(\boldsymbol{\vartheta}_{K \cup h_{i+1}}) = h_{i\uparrow}(\boldsymbol{\vartheta}_{K \cup h_{i+1}} - \boldsymbol{\vartheta}_{K \cup h_i})$, hence

$$(\boldsymbol{\vartheta}_{K \cup h_{i+1}} - \boldsymbol{\vartheta}_{K \cup h_i}) / h_i(\boldsymbol{\vartheta}_{K \cup h_{i+1}}) \ = \ \mathbf{n}_K / h_{i\uparrow}(\mathbf{n}_K),$$

this reduces the term in (5.5) corresponding to $K = \mathbb{H}_{<d}$ to

$$\left(\prod_{i=d}^{m-1} \frac{h_i(\mathbf{x})}{h_{i\uparrow}(\mathbf{n}_K)} \right) [(\boldsymbol{\vartheta}_{K \cup h_i} : i = d, \ldots, m), \mathbf{x} \mid \mathbf{n}_K, \ldots, \mathbf{n}_K, \mathbf{x} - \boldsymbol{\vartheta}_{K \cup h_m}] g.$$

Thus, as Yegorov also noted, after using (2.2) with $H = K \cup h_m$ to remove the dependence on $\mathbf{x}$ in the direction set of this last divided difference, one obtains, among other terms, exactly the term in (3.2) corresponding to this K. Symmetry considerations then would suggest that, after rewriting (5.5) as a weighted sum of p_K over $K \in \binom{\mathbb{H}}{d-1}$, the resulting, quite complicated, expression multiplying such a p_K and involving various divided differences of g must equal the corresponding simple coefficient of p_K in (3.2), thus supplying a remarkable set of conjectured multivariate divided difference identities.

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6apr09: The case $\#\mathbb{H} = d+1$ of Theorem 3.1 is due to Shayne Waldron; see [S. Waldron; The error in linear interpolation at the vertices of a simplex; SIAM J. Numer. Anal.; 35(3); 1998; 1191–1200].

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