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Tight compactly supported wavelet frames of arbitrarily high smoothness

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ABSTRACT

Based on the method for constructing tight wavelet frames of [RS2], we show that one can construct, for any dilation matrix, and in any spatial dimension, tight wavelet frames generated by compactly supported functions with arbitrarily high smoothness.

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Tight compactly supported wavelet frames of arbitrarily high smoothness

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1. Introduction

In 1988, Ingrid Daubechies constructed univariate compactly supported orthonormal wavelet bases of arbitrarily high smoothness (cf. [D1,2]). Her construction has been widely received, and rightly so, as a milestone in the theory and applications of wavelets, rendering multiresolution analysis great practical importance. However, generalizations of these constructions to higher dimensions and general dilation matrices turned out to be difficult. Aside from the straightforward tensor product construction of Daubechies' wavelet (that is tied to dyadic dilation, and dictates large rectangular support of the wavelet, and an exponential growth in the number of mother wavelets as the spatial dimension increases), only few, specific, constructions of multivariate orthonormal wavelet systems exist presently in the literature. Indeed, even for the simple dilation matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

it is still unknown whether there exists a scaling function in $C^1(\mathbb{R}^2)$ whose integer shifts are orthonormal (cf. Cohen and Daubechies [CD] for results on this topic).

Tight frames generalize orthonormal systems. They preserve the unitary property of the relevant analysis and synthesis operators, while sacrificing the orthonormality and the linear independence of the system (in order to get more flexibility). Precisely, a system $X \subset L_2(\mathbb{R}^d)$ is a **tight frame** if the analysis operator associated with X is unitary, i.e., if, for every $f \in L_2(\mathbb{R}^d)$,

$$\|f\|_2^2 = \sum_{x \in X} |\langle f, x \rangle|^2.$$

This identity implies that

$$\sum_{x \in X} \langle f, x \rangle \, x = f, \quad \forall f \in L_2,$$

hence entitles one to use the same system X for both the decomposition and reconstruction processes; in particular, a tight compactly supported wavelet frame based on a multiresolution analysis employs finite masks during both processes. Albeit, until very recently, tight wavelet frames were hardly considered an issue in wavelet theory; the main reason for this was the lack of a general method for the construction of useful tight wavelet frames.

The paper [RS1] marks a significant change in this state-of-the-art: it laid a foundation to the theory of wavelet frames in general, thereby to tight wavelet frames, via the new notion of *quasi-wavelet frames.* In particular, the paper suggests a new "unitary extension principle" that allows one, at least in theory, to derive tight wavelet frames from any multiresolution analysis, regardless of any conditions of orthonormality or linear independence of the scaling functions. Furthermore, in [RS2] this unitary extension principle is employed to derive various compactly supported spline tight wavelet frames in higher dimensions, i.e., tight wavelet frames based on smooth piecewise-polynomials.

We show in this note that the theory of [RS1] and the construction methods of [RS2], when combined with the existing literature on self-affine tilings for general dilation matrices, lead to the following observation: given any spatial dimension d, any $d \times d$ dilation matrix s, and any smoothness parameter k, there exists a compactly supported tight wavelet system in $C^k(\mathbb{R}^d)$ which is s-invariant. In view of the present inability to construct, for the majority of dilation matrices, compactly supported smooth *orthonormal* wavelet bases, this result might be of practical interest. The details of this observation are the content of the next section.

2. Smooth tight wavelet frames

We first review some of the very basic notions in multiresolution analysis, and then present results from [RS1,2] that are pertinent to the discussion. We then recall some facts concerning self-affine tilings. Finally, all these facts are combined to yield the construction of the compactly supported smooth tight wavelet frames.

A **dilation matrix** s on $\mathbb{R}^d$ is a $d \times d$ *integer* matrix, whose spectrum lies outside the closed unit disc. It induces two closely related dilation operators on functions; on the "space domain" this is the operator

$$D : f \mapsto |\det s|^{-1} f(s^{-1}\cdot),$$

and on the frequency domain, the relevant operator is

$$D_* : f \mapsto f(s^* \cdot);$$

the two operators are connected by the formula $(Df)\widehat{}= D_* \widehat{f}$. A function $\phi \in L_2(\mathbb{R}^d)$ is said to be **a scaling function** (with respect to a given dilation matrix s), if in the Fourier domain it satisfies an equality of the following type:

$$D_* \widehat{\phi} = \tau_\phi \widehat{\phi}, \qquad (2.1)$$

where τ_ϕ is some 2π-periodic function, often referred to as the **mask function**. The **refinement equation** (2.1) is equivalent to having $D\phi$ lie in the closed span V_0 of the integer translates of ϕ, [BDR].

Now, let Ψ be any finite subset of V_0. The assumption $\psi \in V_0$ is equivalent to the existence of a 2π-periodic function, τ_ψ, that satisfies the **wavelet equation**

$$D_* \widehat{\psi} = \tau_\psi \widehat{\phi}.$$

Let Γ be the quotient group $\mathbb{Z}^d/(s\mathbb{Z}^d)$, and let $\widetilde{\Gamma}$ be its dual group i.e., the quotient group $2\pi(s^{*-1}\mathbb{Z}^d/\mathbb{Z}^d)$ (both of order $|\det s|$). We may always identify any set of representers of Γ ($\widetilde{\Gamma}$) with the group itself. We then consider the following matrix-valued function Δ on $\mathbb{R}^d$, whose rows are indexed by $F := \Psi \cup \{\phi\}$, whose columns are indexed by $\widetilde{\Gamma}$, and whose $(f,\gamma) \in F \times \widetilde{\Gamma}$ entry is defined as

$$\Delta_{f,\gamma}(\omega) = E^\gamma \tau_f(\omega) = \tau_f(\omega + \gamma),$$

where E^t, $t \in \mathbb{R}^d$, is the translation operator

$$E^t : f \mapsto f(\cdot + t).$$

The *unitary extension principle* of [RS1] then reads as follows:

Result 2.2. [RS1: Corollary 6.7]. *In the above notations, if $\widehat{\phi}(0) = 1$, and if the columns of matrix $\Delta(\omega)$ are orthonormal for almost all $\omega \in \mathbb{R}^d$, then the system defined by*

$$X := \{D^k E^j \psi : \ \psi \in \Psi, \ j \in \mathbb{Z}^d, \ k \in \mathbb{Z}\}$$

is a tight (wavelet) frame.

Remark. The original proof in [RS1] requires a very mild smoothness assumption (which does not even imply continuity); but it follows from a result in [CSS] that Result 2.2 is valid without any smoothness condition. In any event, since we aim at very smooth scaling functions ϕ, we need not worry about this technical detail.

In principle, one may be able to construct a tight wavelet frame from any scaling function ϕ with mask τ_0 by solving the following matrix extension problem: given the mask function τ_0, find a (possibly large) number of periodic functions $\tau_i, i = 1, \ldots, r$ such that the column vectors $u_\gamma(\omega) = (\tau_0(\gamma + \omega), \ldots, \tau_r(\gamma + \omega))^T$, $\gamma \in \widetilde{\Gamma}$, are orthonormal for almost all ω. While that seems still to be a hard problem, a partial solution was provided by an inductive method in [RS2]. *The objective of this note is to show that the class of refinable functions to which the [RS2]-method can be applied to, is broad enough to include compactly supported ϕ's of arbitrarily high smoothness.* But, first, we need to provide the necessary details of the [RS2]-construction.

For this, let ϕ_1, ϕ_2 be refinable compactly supported distributions with respect to the same dilation matrix s, viz.,

$$D_* \widehat{\phi}_j = \tau_j \widehat{\phi}_j, \quad j = 1, 2,$$

with τ_j, $j = 1, 2$ being 2π-periodic. Roughly speaking, the goal is to derive a wavelet system from the scaling function $\phi_1 * \phi_2$, based on the ability to do this separately with each of the scaling functions ϕ_j, $j = 1, 2$. Suppose, indeed, that we are able to use the unitary extension principle with respect to ϕ_1, and to obtain a collection $\mathrm{T}_1 = (\tau)_{\tau \in \mathrm{T}_1}$ of 2π-periodic functions that satisfy the extension principle, in other words the columns of the matrix $\Delta(\omega)$ associated with $\tau_1 \cup \mathrm{T}_1$ are orthonormal for almost all $\omega \in \mathbb{R}^d$. As with respect to ϕ_2, the assumption here is even milder, viz., that we only need to find a collection $\mathrm{T}_2 = (\tau)_{\tau \in \mathrm{T}_2}$ so that

$$|\tau_2|^2 + \sum_{\tau \in \mathrm{T}_2} |\tau|^2 = 1.$$

The method in [RS2] provides then a unitary extension to $\tau_0 := \tau_1 \, D_* \tau_2$, which is the mask of the scaling function $\phi := \phi_1 * D\phi_2$ (instead of the expected extension to the mask $\tau_1 \tau_2$ of $\phi_1 * \phi_2$). The wavelets masks that corresponds to ϕ are

$$\mathrm{T} := \mathrm{T}_1 \cup (\tau_1 D_* \mathrm{T}_2);$$

i.e., the masks of T_1 are augmented by the modified masks of T_2, where the modification is done by dilating each of these masks and then multiplying each by τ_1. It is shown in [RS2: §4], that $\tau_0 \cup \mathrm{T}$ satisfies the unitary extension principle, and hence by Result 2.2 the wavelet system generated by $\Psi := (\psi_\tau)_{\tau \in \mathrm{T}}$, where

$$D_* \widehat{\psi}_\tau := \tau \widehat{\phi},$$

is a tight wavelet frame, provided that ϕ is in $L_2(\mathbb{R}^d)$, and not merely a distribution.

In order to construct smooth tight wavelet frames with respect to s, we first find tight frames that are not smooth at all, something that is relatively easy; we then use the outlined method to improve on the smoothness. For that, we first invoke the following simple lemma:

Lemma 2.3. *For a given dilation matrix s with determinant $n = |\det s|$, fix a set $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ of representers for the group $\mathbb{Z}^d / s\mathbb{Z}^d$, and a set $\widetilde{\Gamma} = \{g_1, \ldots, g_n\}$ of representers for the dual group. For any unitary $n \times n$ matrix $U := (u_{jk})$, set*

$$\tau_j(\omega) = n^{-1/2} \sum_{k=1}^{n} u_{jk} e^{i\gamma_k \cdot \omega}.$$

Then the $n \times n$ matrix Δ with entries $E^{g_k}\tau_j, j, k = 1, \ldots, n$ is unitary for all ω.

Proof: Let $R(\omega)$ be the $n \times n$-matrix whose (j,k)-entry is $n^{-1/2}e^{i\gamma_j\cdot(\omega+g_k)}$. This matrix is unitary due to the well-known orthogonality relations of group characters (cf. for instance [GH], Lemma 5.1). This implies that $\Delta(\omega)$ is unitary, too, as the product $UR(\omega)$ of two unitary matrices. □

The lemma, when combined with Result 2.2, can be used for the construction of tight wavelet frame in the following special case. Suppose Γ is a set of representers for the group $\mathbb{Z}^d/(s\mathbb{Z}^d)$, and suppose that ϕ is a scaling function whose refinement mask τ_ϕ has the special form

$$\tau_\phi(\omega) = \sum_{\gamma\in\Gamma} a_\gamma e^{i\gamma\cdot\omega}, \tag{2.4}$$

so that (with $n := |\det s|$) $a := n^{1/2}(a_\gamma)_{\gamma\in\gamma}$ is a unit vector. Then, there exists a unitary matrix U whose first row is a, and the lemma, then, when combined with Result 2.2, provides us with a tight compactly supported wavelet system generated by $n-1$ functions. The next two results assert now that: (a) there exist L_2-scaling functions with mask of the form (2.4), and (b) these functions, though are not really smooth, have a minimal amount of smoothness.

The first result is due to [GM]:

Result 2.5. [GM: Theorem 2]. *Let s be a dilation matrix, and Γ a representer set of $\mathbb{Z}^d/(s\mathbb{Z}^d)$. Set, with $n := |\det s|$,*

$$\tau := n^{-1}\sum_{\gamma\in\Gamma} e^{i\gamma\cdot\omega}.$$

Then the dilation equation

$$D_*\widehat{\phi} = \tau\widehat{\phi}$$

has an L_2-compactly supported solution, with non-zero mean value. Furthermore, up to a multiplicative constant, ϕ is the support function of a uniquely determined (compact) set Ω which satisfies $s\Omega = \Gamma + \Omega$. □

The wavelets that can be constructed by combining Result 2.2, Result 2.5, and Lemma 2.3, generate a tight frame, but lack smoothness: each wavelet is piecewise constant. On the positive side, they inherit their smoothness properties from the corresponding scaling function, and that scaling function is known to be "minimally smooth".

Result 2.6. [S: Lemma 3.1]. *For any s-self-affine tile Ω, (i.e., a non-null measurable set Ω satisfying $s\Omega = \Gamma + \Omega$ for some set of representers Γ of $\mathbb{Z}^d/s\mathbb{Z}^d$), there exists $\epsilon > 0$ such that*

$$|\widehat{\chi}(\omega)| = O(|\omega|^{-\epsilon}).$$

Here, χ is the support function of Ω.

The statement in [S] assumes that the smallest s-invariant lattice containing Γ equals $\mathbb{Z}^d$, but can be modified to yield our stronger statement. Since this strengthening is critical for the development here, we outline below the modified proof.

Proof of Result 2.6. With Ω and Γ as in the result, one immediately obtains that

$$D_* \widehat{\chi} = \tau \widehat{\chi}$$

with $\tau(\omega) = |\det s|^{-1} \sum_{\gamma \in \Gamma} e^{i\gamma \cdot \omega}$. Let Λ be the smallest s-invariant lattice that contains Γ. Since Λ, as any lattice in $\mathbb{R}^d$, is finitely generated, it is easy to see that, for a sufficiently large N, the set

$$\Gamma_N := \sum_{j=0}^{N-1} s^j \Gamma$$

generates Λ. Since we still have $s^N \Omega = \Gamma_N + \Omega$, it follows that χ is refinable also with respect to s^N, and with mask $|\det s|^{-N} \sum_{\gamma \in \Gamma_N} e^{i\gamma \cdot \omega}$. This allows one to assume (by replacing Γ by Γ_N if necessary) that the original Γ generates Λ.

Let $\Lambda^\perp$ be the lattice dual to Λ:

$$\Lambda^\perp := \{\omega \in \mathbb{R}^d : \ \lambda \cdot \omega \in 2\pi\mathbb{Z}, \ \forall \lambda \in \Lambda\} = \{\omega \in \mathbb{R}^d : \ \gamma \cdot \omega \in 2\pi\mathbb{Z}, \ \forall \gamma \in \Gamma\},$$

with the second equality following from the fact that Γ generates Λ. Then,

$$|\tau(\omega)| = 1 \iff (e^{i\gamma \cdot \omega} = 1, \ \forall \gamma \in \Gamma) \iff \omega \in \Lambda^\perp. \tag{2.7}$$

Also, if $\omega \notin \Lambda^\perp$, but $s^*\omega \in \Lambda^\perp$, then ω is a non-trivial character of the (well-defined) group $\Lambda/(s\Lambda)$, and hence

$$\tau(\omega) = \sum_{\gamma \in \Gamma} e^{i\gamma \cdot \omega} = 0, \quad s^*\omega \in \Lambda^\perp \backslash (s^* \Lambda^\perp) \tag{2.8}$$

(say, because every non-trivial character is orthogonal to the trivial one).

The proof now proceeds in the same manner as in [S], with the lattice $2\pi\mathbb{Z}^d$ there being replaced by $\Lambda^\perp$. The essence of the proof is the observation that maximal chains of the form $\prod_{j=1}^k D_*^j \widehat{\chi}(\omega)$, where each $|D_*^j \widehat{\chi}(\omega)|$ is "near 1", must be preceded by a "near 0" term $\widehat{\chi}(\omega)$. That observation requires only (2.7) and (2.8) above. $\square$

We can now easily combine Result 2.6 with the inductive method of [RS2]: we define

$$\phi_k := \phi * \underbrace{D\phi * D\phi * \ldots * D\phi}_{k-\text{times}},$$

with ϕ being the scaling function of Result 2.5. Then

$$|\widehat{\phi}_k(\omega)| = O(|\omega|^{-k\epsilon}),$$

and choosing k large enough, ϕ_k has any desired smoothness.

The function ϕ_k is refinable with mask

$$\tau^{(k)} := \tau_1 \, (D_* \tau_1)^k,$$

and the wavelets derived from ϕ_k are defined inductively as follows: for $k = 0$, we use the construction that is detailed after Lemma 2.3. Next, we assume that T_{k-1} are the wavelet masks used in the derivation of a tight wavelet frame from ϕ_{k-1}. Since we can write

$$\phi_k = \phi_{k-1} * D\phi,$$

and since the refinement mask of ϕ_{k-1} is $\tau^{(k-1)} = \tau_1 \, (D_* \tau_1)^{k-1}$, the [RS2]-method provides us with the new wavelets masks

$$\mathrm{T}_k := \mathrm{T}_{k-1} \cup (\tau^{(k-1)} \, D_* \mathrm{T}_0), \tag{2.9}$$

and the wavelets are then defined in the Fourier domain as

$$D_* \widehat{\psi}_\tau := \tau \widehat{\phi}_k, \quad \tau \in \mathrm{T}_k.$$

Obviously, they have the same smoothness as ϕ_k. Note that each inductive step increases the number of mother wavelets by $|\det s| - 1$, hence $(k+1)(|\det s| - 1)$ mother wavelets are derived from ϕ_k.

We have thus proved:

Theorem 2.10. *Given a dilation matrix s on $\mathbb{R}^d$ and a smoothness parameter n, there exists a collection Ψ of wavelet functions with compact support in $C^n(\mathbb{R}^d)$, such that*

$$X := \{D^k E^j \psi : \ \psi \in \Psi, \ j \in \mathbb{Z}^d, \ k \in \mathbb{Z}\}$$

is a tight frame.

Remark. In practice, one would like the smoothness of the sequence $(\phi_k)_k$ to increase as fast as possible. Using k-fold convolution of $D\phi$ is certainly far from being an optimal strategy for a speedy increase in the smoothness of ϕ_k, as $k \to \infty$. An obvious modification is to take at each inductive step a different tile constructed as in Result 2.5 (by changing the set of representers Γ that is used there). The construction details remain essentially unchanged: one only needs to modify the set T_0, that is used in (2.10) according to the nature of the fractal ϕ used. We refer to [DDL] for a related construction.

Remark. A significant improvement in the above construction is available in case the matrix s satisfies a relation

$$s^m = (\det s)^m I,$$

for some integer m. In that event, one can smooth the scaling function ϕ of Result 2.5, by convolving it with a smooth box spline ϕ_ξ instead of $D\phi$. For a given vector $\xi \in \mathbb{Z}^d \backslash 0$ the box spline ϕ_ξ is defined as

$$\widehat{\phi_\xi}(\omega) := \prod_{j=1}^{m} \int_0^1 e^{-i(s^j \xi \cdot \omega)t}\, dt.$$

The details of that construction are essentially in [RS2: §4] where the resulting wavelets were dubbed *boxlets* for the case of dyadic dilation. In that case each additional convolution increases the number of resulting wavelets by 1 only, and the smoothness of ϕ_k can be increased by as much as m degrees per each inductive step.

References

[BDR] C. de Boor, R. DeVore and A. Ron, The structure of shift invariant spaces and applications to approximation theory, J. Functional Anal. **119** (1994), 37–78. Ftp site: `anonymous@ftp.cs.wisc.edu/Approx` file name `several.ps`

[CD] A. Cohen and I. Daubechies, Non-separable bidimensional wavelets bases, Rev. Math. Iberoamericana, Vol **9** (1993), 51–137.

[CSS] C.K. Chui, X.L. Shi and J. Stöckler, Affine frames, quasi-affine frames and their duals, CAT Report 372, Texas A&M University, College Station, TX, 77843, June 1996.

[DDL] S. Dahlke, W. Dahmen, V. Latour, Smooth refinable functions and wavelets obtained from convolution products. Appl. Comp. Harm. Anal. **2** (1995), 68 – 84.

[D1] I. Daubechies, Orthonormal bases of compactly supported wavelets, Comm. Pure and Applied Math., **41** (1988), 909–996.

[D2] I. Daubechies, Ten lectures on wavelets, CBMS-NSF conference series in applied mathematics, Vol **61**, SIAM, Philadelphia, 1992.

[GH] K. Gröchenig and A. Haas, Self-similar lattice tilings. J. Fourier Anal. Appl. **1** (1994), 131–170.

[GM] K. Gröchenig and W. Madych, Multiresolution analysis, Haar bases, and self-similar tilings. IEEE Trans. Inform. Th. **38(2)**, (1992), 556–568.

[RS1] A. Ron and Z. Shen, Affine systems in $L_2(\mathbb{R}^d)$, the analysis of the analysis operator, CMS-TSR # 96–02 , University of Wisconsin - Madison, Madison WI 53706, December 1995. Ftp site: `anonymous@ftp.cs.wisc.edu/Approx` file `affine.ps`

[RS2] A. Ron and Z. Shen, Compactly supported tight affine spline frames in $L_2(\mathbb{R}^d)$, Math. Comp., to appear. Ftp site: `anonymous@ftp.cs.wisc.edu/Approx` file `tight.ps`

[S] R. S. Strichartz, Wavelets and self-affine tilings. Constr. Approx. **9** (1993), 327 – 346.