THE CONDITION OF THE B-SPLINE BASIS FOR POLYNOMIALS*

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Abstract. It is shown that the polynomial basis provided on a knot interval by the kth order B-splines which do not vanish on that interval is well conditioned up to a scaling exactly to the extent that at least $k-1$ of the knots stay finite relative to the length of that knot interval.

Key words. condition, B-splines, polynomials

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Let $\mathbf{t} := (t_i)$ be a knot sequence for kth order splines, i.e., a nondecreasing sequence with $t_i < t_{i+k}$, all i. Assume without loss that the interval

$$I := [t_k, t_{k+1}]$$

is proper. Then it is well known that exactly k B-splines of order k for this knot sequence $\mathbf{t}$ have some support in I, viz., the B-splines $N_{j,k,\mathbf{t}},\ j = 1, \dots, k$. Further, with v_j the polynomial of degree $< k$ which agrees on I with $N_{j,k,\mathbf{t}}$, the sequence $(v_j)_1^k$ forms a basis for the space $\mathbb{P}_k$ of all polynomials of degree $< k$. But the condition of this basis, with respect to the norm

$$\|f\| := \max_{t\in I} |f(t)|,$$

depends strongly on $\mathbf{t}$. In particular, it can become arbitrarily large if I becomes small in comparison to some of its neighboring knot intervals.

In this context, B. Swartz [S] raises the following question: *Is it possible to so scale this basis that its condition with respect to the max-norm on I is bounded independently of $\mathbf{t}$?* Precisely, can one find diagonal matrices $F_\mathbf{t}$ and $E_\mathbf{t}$ so that the matrix

$$A := F_\mathbf{t}(D^{i-1}N_j(\tau))E_\mathbf{t}$$

(with τ some point inside I) has a condition number bounded independently of $\mathbf{t}$? Swartz answers this question in the affirmative in case $k = 3$, and for arbitrary k in case all the knots in $\mathbf{t}$ have at least multiplicity $k/2$, but offers no opinion in the contrary case. It is the purpose of this note to show that these are essentially the only cases in which an affirmative answer is possible.

In terms of the linear map

$$V : \mathbb{R}^k \to \mathbb{P}_k \subseteq C(I) : a \mapsto \sum_{j=1}^k v_j a(j),$$

the matrix A has the form

$$A = F_\mathbf{t} \Delta V E_\mathbf{t},$$

with

$$\Delta : \mathbb{P}_k \to \mathbb{R}^k : p \mapsto \big(D^{i-1}p(\tau)\big)_1^k.$$

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We may assume that $I = [-1,1]$ since we can always arrange that by an appropriate shifting and scaling of $\mathbf{t}$, and the only effect this has on A is to change Δ to

$$\operatorname{diag}\lceil 1, 2/\Delta t_k, \ldots, (2/\Delta t_k)^{k-1}\rfloor\ \Delta,$$

a change that can be incorporated into $F_{\mathbf{t}}$. After such a normalization, we may as well choose $F_{\mathbf{t}} = 1$. In other words, *Swartz' question has an affirmative answer iff $E_{\mathbf{t}}$ can be so chosen that $VE_{\mathbf{t}}$ is bounded and boundedly invertible uniformly in $\mathbf{t}$.* For, the condition of A is unchanged by multiplication by a scalar, hence we may assume that A is bounded, specifically that $\|F_{\mathbf{t}}\| = \|VE_{\mathbf{t}}\| = 1$ (after dividing $F_{\mathbf{t}}$ by $\|F_{\mathbf{t}}\|$ and $E_{\mathbf{t}}$ by $\|VE_{\mathbf{t}}\|$). But then

$$\|Ax\| \le \|\Delta\| \|VE_{\mathbf{t}}x\|,$$

i.e., a lower bound on A provides a lower bound on $VE_{\mathbf{t}}$, while the converse holds with the simple choice $F_{\mathbf{t}} = 1$ since Δ is a fixed invertible map.

A direct attack on the v_i's with the hope of somehow describing precisely the behavior of $\|v_i\|$ as a function of $\mathbf{t}$ leads to seemingly difficult details, but the following end-run seems easy.

Consider instead the inverse of V. It has the same condition as V and, by [BF], is known to have the form

$$\Lambda : \mathbb{P}_k \to \mathbb{R}^k : p \mapsto \left(\lambda_{\psi_i} p\right)_1^k$$

with

$$\lambda_\psi p := \sum_{j<k} (-D)^{k-1-j}\psi(\tau)\ D^j p(\tau)$$

and

$$\psi_i := (t_{i+1} - \cdot) \cdots (t_{i+k-1} - \cdot)/(k-1)!.$$

In other words,

$$V^{-1} = \Lambda = (\hat\Delta\Psi)'\Delta = \Psi'\hat\Delta'\Delta,$$

with

$$\Psi : \mathbb{R}^k \to \mathbb{P}_k : a \mapsto \sum_j \psi_j a(j),$$

and

$$\hat\Delta : \mathbb{P}_k \to \mathbb{R}^k : p \mapsto \left((-D)^{k-j}p(\tau)\right)_1^k.$$

Hence, Δ and $\hat\Delta$ being fixed, Swartz' question has a positive answer if and only if it is possible to construct the diagonal matrix $E_{\mathbf{t}}$ so that

$$\Psi E_{\mathbf{t}}^{-1}$$

is bounded above and below uniformly in $\mathbf{t}$. Further, if there is such a matrix

$$E_{\mathbf{t}} =: \operatorname{diag}\lceil e_{11}, \ldots, e_{kk}\rfloor,$$

then the norm of the resulting scaled polynomials

$$\tilde\psi_j := \psi_j/e_{jj}, \quad j = 1, \ldots, k,$$

must be bounded and bounded away from 0 uniformly in $\mathbf{t}$, i.e., we must have

$$\inf_{\mathbf{t}} \min_j \|\tilde\psi_j\| > 0, \quad \sup_{\mathbf{t}} \max_j \|\tilde\psi_j\| < \infty. \tag{1}$$

For, with the max-norm on $\mathbb{R}^k$ and

$$\Psi E_{\mathbf{t}}^{-1} =: \tilde{\Psi} : a \mapsto \sum_j \tilde{\psi}_j a(j),$$

we have

$$\max_j \|\tilde{\psi}_j\| \le \|\tilde{\Psi}\| \le \sum_j \|\tilde{\psi}_j\|;$$

hence $\tilde{\Psi}$ is bounded uniformly in $\mathbf{t}$ iff $\max_j \|\tilde{\psi}\|$ is. Also,

$$\|\tilde{\Psi}^{-1}\| \ge \max_j 1/\|\tilde{\psi}_j\|;$$

hence $\tilde{\Psi}$ is bounded below uniformly in $\mathbf{t}$ only if $\min_j \|\tilde{\psi}_j\|$ is.

On the other hand, there is essentially only one scaling that achieves (1), i.e., if also

$$\inf_{\mathbf{t}} \min_j \|\psi_j/\tilde{e}_{jj}\| > 0, \quad \sup_{\mathbf{t}} \max_j \|\psi_j/\tilde{e}_{jj}\| < \infty,$$

then $|e_{jj}/\tilde{e}_{jj}|$ is bounded and bounded below uniformly in $\mathbf{t}$. Since

$$1 \le \| \prod_{i=j+1}^{j+k-1} \frac{t_i - \cdot}{t_i - \tau} \| \le (2/(1-|\tau|))^{k-1},$$

we conclude that the answer to Swartz' question is positive iff

$$\Psi E_{\mathbf{t}}^{-1} := \hat{\Psi} : a \mapsto \sum_j \hat{\psi}_j a(j)$$

is bounded below uniformly in $\mathbf{t}$, with

$$\hat{f} := f/f(\tau),$$

i.e.,

$$\hat{\psi}_j := \psi_j/\psi_j(\tau) = \prod_{r=j+1}^{j+k-1} \frac{t_r - \cdot}{t_r - \tau} \ .$$

Note that $\hat{\psi}_j$ depends continuously on $\mathbf{t}$ and converges to

$$\prod_{\max\{j,m\}<r<\min\{j+k,n\}} \frac{t_r - \cdot}{t_r - \tau} =: \hat{\psi}_{\max\{j,m\},\min\{j+k,n\}}$$

as $t_m \to -\infty$ and $t_n \to \infty$. Consequently, $\hat{\Psi}$ is bounded below uniformly in $\mathbf{t}$ provided it is invertible for every $\mathbf{t}$ including those with some knots infinite. To look into this invertibility, consider the situation that

$$-\infty = t_m < t_{m+1} \le -1 = t_k, \qquad t_{k+1} = 1 \le t_{n-1} < t_n = \infty.$$

Then we have to show that the polynomials

$$\psi_{\max\{i,m\},\min\{i+k,n\}}, \qquad i = 1, \ldots, k \tag{2}$$

are linearly independent. Since there are k polynomials, all of degree $< k$, this is equivalent to having them be spanning for $\mathbb{P}_k$. Hence a necessary condition is that *the sequence* (2) *contain a polynomial of degree* $k-1$. This condition is also sufficient, as can be seen as follows: Let $\tilde{\mathbf{t}} = (\tilde{t}_i)$ be a finite knot sequence with $\tilde{t}_j = t_j$ for $m < j < n$ but which is otherwise strictly increasing, and let $\tilde{\psi}_{i,j} := \prod_{i<r<j}(\tilde{t}_r - \cdot)$ be the corresponding polynomials. Then (2) coincides with the sequence

$$(2)' \qquad \tilde{\psi}_{\max\{i,m\},\min\{i+k,n\}}, \qquad i = 1,\dots,k.$$

On the other hand, if one of these is of exact degree $k-1$, i.e., if $n - m \ge k$, then the identity

$$\tilde{\psi}_{i,j} - \tilde{\psi}_{i-1,j-1} = (\tilde{t}_{j-1} - \tilde{t}_i)\tilde{\psi}_{i,j-1}$$

(applied for $i \le m$, $j > k$; or $i \le k$, $j > n$; and $j - i \le k$) shows that this sequence contains in its span the polynomials

$$\tilde{\psi}_j = \tilde{\psi}_{j,j+k}, \qquad j = 1,\dots,k,$$

hence must span $\mathbb{P}_k$ since the latter are linearly independent, e.g., by the invertibility of Λ.

The condition $n - m \ge k$ can be guaranteed for $k \le 3$ and also when each knot has multiplicity $\ge (k-1)/2$, and in any other case in which the assumptions on $\mathbf{t}$ guarantee that at least $k-1$ knots stay finite, but in no other case.

I am indebted to Blair Swartz for various constructive comments and for the formulation of the following proposition which summarizes the result of this note in terms of an appropriate mesh-ratio.

Proposition. *Let T be a collection of knot sequences $\mathbf{t}$ appropriate to B-splines of order $k > 1$ on $[0,1]$; that is to say, each $\mathbf{t} =: (t_i)_{i=-k+1}^{N(\mathbf{t})+k-1}$ is a monotone nondecreasing sequence to $[0,1]$ satisfying $t_i < t_{i+k}$. Then, the local B-spline basis for each nontrivial local polynomial space $\mathbb{P}_k[t_{i-1}, t_i]$, $1 \le i \le N(\mathbf{t})$, can be locally scaled so that the change of basis to a locally scaled local monomial basis is uniformly bounded and boundedly invertible if and only if the "kth-order knot-interval ratio"*

$$R_{\mathbf{t},k} := \max_{1 \le i \le N(\mathbf{t})} \min_{0 \le j < k} (t_{i+j-1} - t_{i+j-k+1})/(t_i - t_{i-1})$$

is uniformly bounded on T.

REFERENCES

[BF] C. de Boor and G. Fix (1973), *Spline approximation by quasi-interpolants*, J. Approx. Theory 8, pp. 19–45.

[S] B. Swartz, *Conditioning collocation*, Los Alamos National Laboratory Report LA-UR-85-49, January 1987; This Journal, 25 (1988), pp. 124–147.