An asymptotic expansion for the error in a linear map that reproduces polynomials of a certain order

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Abstract Han's 'multinode higher-order expansion' in [H] is shown to be a special case of an asymptotic error expansion available for any bounded linear map on $C([a\,..\,b])$ that reproduces polynomials of a certain order. The key is the formula for the divided difference at a sequence containing just two distinct points.

In [H], Han shows that, for linear maps on $C([a\,..\,b])$ of the form $L : f \mapsto \sum_i \varphi_i f(x_i)$ that reproduce polynomials of degree $\le m$, and for a specific choice of coefficients a_j, independent of L and f but depending on m and r, the following asymptotic error expansion

$$f(x) = Lf(x) + \sum_{j=0}^{r} \frac{a_j}{j!} L\left((x-\cdot)^j D^j f\right)(x) + E(f,x)$$

holds, with $E(f,x)$ explicitly given as an integral involving $D^{m+r+1}f$. Since, for his particular choice of L, the sum involves the derivatives of f at the points or nodes x_i associated with L, Han thinks of this as a 'multinode' expansion for f.

It is the purpose of this note to point out that this asymptotic error expansion, properly interpreted, holds for any bounded linear map L on $C([a\,..\,b])$, with the same formula for $E(f,x)$. The key is the formula for the divided difference at a sequence containing just two distinct points.

It is easy to verify, for example by induction on r and m, particularly for the special case $x=0$, $y=1$, that, for any $x \ne y$,

$$(-1)^{m+1}(y-x)^{r+m+1}\,\Delta(x^{[r+1]}, y^{[m+1]}) = \\ \sum_{j=0}^{r} \binom{m+r-j}{r-j}(y-x)^j\,\Delta(x^{[j+1]}) - \sum_{k=0}^{m} \binom{r+m-k}{m-k}(x-y)^k\,\Delta(y^{[k+1]}),$$

with $\Delta(x^{[r+1]}, y^{[m+1]})$ denoting the divided difference at the point sequence that contains x exactly $r+1$ times and y exactly $m+1$ times.

The Peano kernel for the divided difference $\Delta(t_0,\ldots,t_n)$ at the sequence $(t_0,\ldots,t_n)$ is well-known to be the B-spline with knot sequence $(t_0,\ldots,t_n)$ that is normalized to integrate to $1/n!$, hence (cf. (5) below), for arbitrary x and y,

$$(y-x)^{r+m+1}\,\Delta(x^{[r+1]}, y^{[m+1]})f = \int_x^y [\![t-x]\!]^m [\![y-t]\!]^r D^{r+m+1} f(t)\,\mathrm{d}t,$$

with

$$[\![s]\!]^n := s^n/n!$$

a handy notation for the normalized power.

Consequently, for any smooth f and any x and y, and using the fact that $\Delta(z^{[n+1]})f = D^n f(z)/n!$,

$$
\begin{aligned}
(1)\qquad & -\int_x^y [\![x-t]\!]^m [\![y-t]\!]^r D^{r+m+1} f(t)\,\mathrm{d}t \;= \\
& \qquad \sum_{j=0}^{r} \binom{m+r-j}{r-j} [\![y-x]\!]^j D^j f(x) - \sum_{k=0}^{m} \binom{r+m-k}{m-k} [\![x-y]\!]^k D^k f(y).
\end{aligned}
$$

If now L is any bounded linear map on $C([a\,..\,b])$ that reproduces polynomials of degree $\le m$, then, on applying $1-L$ to both sides of (1) as functions of x, we find, for arbitrary y, that

$$
\begin{aligned}
(2)\qquad & \int_a^b (1-L)([\![(\cdot - t)_+]\!]^m)(x)[\![y-t]\!]^r D^{r+m+1} f(t)\,\mathrm{d}t \;= \\
& \qquad \binom{m+r}{m}(f-Lf)(x) + (1-L)\left(\sum_{j=1}^{r} \binom{m+r-j}{r-j} [\![y-\cdot]\!]^j D^j f\right)(x),
\end{aligned}
$$

using the facts that (i) the second sum on the right of (1) is a polynomial of degree $\le m$ in x, hence is annihilated by $1-L$; that (ii) for any (integrable) g and any $x, y \in [a\,..\,b]$,

$$
-\int_x^y g(t)\,\mathrm{d}t = \int_a^b ((x-t)_+^0 - (y-t)_+^0) g(t)\,\mathrm{d}t
$$

(with z_+ equal to z for positive z and 0 otherwise), hence

$$
-\int_x^y [\![x-t]\!]^m [\![y-t]\!]^r g(t)\,\mathrm{d}t = \int_a^b \left([\![(x-t)_+]\!]^m [\![y-t]\!]^r - [\![x-t]\!]^m [\![(y-t)_+]\!]^r\right) g(t)\,\mathrm{d}t,
$$

while (iii) $[\![x-t]\!]^m [\![(y-t)_+]\!]^r$ is of degree $\le m$ in x, hence annihilated by $1-L$. Now notice that $[\![y-x]\!]^j = 0$ for $y = x$ and $j > 0$. So, after setting $y = x$ in (2), we can (and will) replace $(1-L)$ on the right by $-L$, then divide both sides by $\binom{m+r}{m}$ and rearrange to arrive at the sought-for expansion

$$
(3)\qquad f(x) - Lf(x) \;=\; \sum_{j=1}^{r} \frac{\binom{m+r-j}{r-j}}{\binom{m+r}{m}} L\left([\![x-\cdot]\!]^j D^j f\right)(x) \;+\; E(f,x),
$$

with

$$
(4)\qquad E(f,x) \;:=\; \int_a^b (1-L)\left((\cdot - t)_+^m\right)(x)\,(x-t)^r D^{m+r+1} f(t)\,\mathrm{d}t/(m+r)!\,,
$$

in which $\binom{m+r-j}{r-j}/\binom{m+r}{m}$ could be rewritten as $\frac{r!(m+r-j)!}{(m+r)!(r-j)!}$. Thus, when L takes the particular form $Lf := \sum_i \varphi_i f(x_i)$ for some functions φ_i and some points x_i in $[a\,..\,b]$, we now have in hand Theorem 2 of [H].

As a check, for $L : f \mapsto f(a)$, hence $m = 0$, we obtain

$$f(x) - f(a) = \sum_{j=1}^{r} [\![x-a]\!]^j D^j f(a) + \int_a^b (x-t)_+^r D^{r+1} f(t)\, \mathrm{d}t / r!\,,$$

i.e., the truncated Taylor series with integral remainder.

Consider now the error $E(f, x)$ in the asymptotic error expansion (3) for general L.

To be sure, (4) is correct offhand only for $m > 0$. Even when $m = 0$, it is correct in Han's context, i.e., when L is of the form $f \mapsto \sum_i \varphi_i f(x_i)$. For more general L, $t \mapsto (L(\cdot - t)_+^0)(x)$ is not defined (since $L(\cdot - t)_+^0$ is not defined) and so must be interpreted properly, namely as the function $k(x, \cdot)$ of bounded variation that vanishes at b and represents the linear functional $\lambda : g \mapsto -(L \int_a^{\cdot} g(t)\, \mathrm{d}t)(x)$ in the sense that $\lambda f = \int f\, \mathrm{d}k(x, \cdot)$ for all $f \in C([a \,..\, b])$, with the existence of such $k(x, \cdot)$ guaranteed by the Riesz Representation Theorem.

With that concern set to rest, assume that $f \in C^{(r+m+1)}([a \,..\, b])$ and that, for a given $x \in [a \,..\, b]$,

$$[a \,..\, b] : t \mapsto (1 - L)\left((\cdot - t)_+^m\right)(x)$$

is of one sign (as it is, for any $x \in [a \,..\, b]$, when Lf is the Bernstein polynomial for f, or the Lagrange polynomial interpolant). Then (see (4)) the Peano kernel for $E(\cdot, x)$ is of one sign on $[a \,..\, x]$ and on $[x \,..\, b]$. Correspondingly,

$$E(f, x) = c_1(x) D^{m+r+1} f(\xi_1) + c_2(x) D^{m+r+1} f(\xi_2), \qquad \text{some } \ \xi_1 \in [a \,..\, x],\ \xi_2 \in [x \,..\, b],$$

with

$$c_1(x) := E((-1)^{m+r+1} [\![(x - \cdot)_+]\!]^{m+r+1}, x) \quad \text{and} \quad c_2(x) := E([\![(\cdot - x)_+]\!]^{m+r+1}, x)$$

readily computable by retracing the steps that brought us to (3) but choosing, specifically, $f = (-1)^{m+r+1} [\![(x - \cdot)_+]\!]^{m+r+1}$, i.e., $D^{m+r+1} f = (x - \cdot)_+^0$, to get $c_1(x)$ and choosing $f = [\![(\cdot - x)_+]\!]^{m+r+1}$, i.e., $D^{m+r+1} f = (\cdot - x)_+^0$, to get $c_2(x)$. For this, we note that

$$-\int_x^y [\![x - t]\!]^m [\![y - t]\!]^r \,\mathrm{d}t = (-1)^{m+1} [\![y - x]\!]^{m+r+1}, \tag{5}$$

for arbitrary x and y, hence, e.g.,

$$-\int_x^y [\![x - t]\!]^m [\![y - t]\!]^r (x - t)_+^0 \,\mathrm{d}t = (-1)^{m+1} (x - y)_+^0 [\![y - x]\!]^{m+r+1}.$$

Recalling that we obtained from this the corresponding error term by applying $1 - L$ to it as a function of x, then setting $y = x$ and dividing by $\binom{m+r}{m}$, we get

$$\begin{aligned} c_1(x) &= (-1)^{m+1} (1 - L)([\![(x - \cdot)_+]\!]^{m+r+1})(x) / \tbinom{m+r}{m} \\ &= (-1)^m L([\![(x - \cdot)_+]\!]^{m+r+1})(x) / \tbinom{m+r}{m}. \end{aligned}$$

In the same way, we find that

$$c_2(x) = (-1)^m L([\![(x - \cdot)_-]\!]^{m+r+1})(x)/\binom{m+r}{m}.$$

If now r is even, then $c_1(x)$ and $c_2(x)$ are of the same sign and, in that case,

$$E(f, x) = c(x) D^{m+r+1} f(\xi), \quad \text{some} \ \ \xi \in [a \,..\, b],$$

with

$$c(x) := c_1(x) + c_2(x) = E([\![\,\cdot\,]\!]^{m+r+1}, x) = (-1)^m L([\![x - \cdot]\!]^{m+r+1})(x)/\binom{m+r}{m}.$$

Thus, when L takes the particular form $Lf := \sum_i \varphi_i f(x_i)$ for some functions φ_i and some points x_i in $[a \,..\, b]$, we now have in hand Theorem 3 of [H].

References

[H] Xuli Han (2003), "Multinode higher order expansions of a function", *J. Approx. Theory* **124(2)**, 242–253.