Time frequency representations of almost-periodic functions

Yeon Hyang Kim
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Dr.
Madison, WI 53706
kim@math.wisc.edu

Amos Ron
Computer Sciences Department
University of Wisconsin-Madison
1210 West Dayton
Madison, WI 53706
amos@cs.wisc.edu

ABSTRACT

In this paper, we characterize the space of almost periodic (AP) functions in one variable using either a *Weyl-Heisenberg* (*WH*) system or an *affine* system. Our observation is that the sought-for characterization of the AP space is valid if and only if the given WH (respectively, *affine*) system is an $L_2(\mathbb{R})$-frame. Moreover, the frame bounds of the system are also the sharpest bounds in our characterization. This draws an intriguing and quite unexpected connection between $L_2(\mathbb{R})$ representations and AP representations.

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1. Introduction

1.1. General

Our paper is devoted to the possible representations and characterization of (univariate) almost periodic (AP) functions via $L_2(\mathbb{R})$-frames. Our analysis deals with the two leading time-frequency representation methods that exist in the L_2-setup: *Weyl-Heisenberg* (*WH*, known also as *Gabor*) representations, and *affine* (aka wavelet) representations. We are not the first to consider this problem. We are particularly aware of the two earlier studies, [PU] and [G], and the results established in those articles. It was the reading of these two papers that motivated us to look into this problem, too, and we will review their main results in the sequel. Our initial reading of [PU] and [G] made it clear that the assumptions made there on the representation system (whether WH or affine) imply that the system is an $L_2(\mathbb{R})$-frame. We wanted to understand whether this is an artifact of the specific approaches that were chosen in those articles or, perhaps, there is a deeper connection between L_2-representations and AP-representations. To this end, we investigated the problem using the fiberization tools that were developed in the context of general shift-invariant systems, [BDR94], [RS95], and the specific results that followed for WH systems, [RS97a] and wavelet systems, [RS97b].

As the two main results of our paper make clear, there is, indeed, a fundamental connection between WH and affine representations in $L_2(\mathbb{R})$ and the corresponding representations for AP functions using such systems. Moreover, the approach we have chosen, viz., the aforementioned fiberization techniques of the underlying operators, was found to be exactly the right tool for revealing this intriguing and unexpected connection. Specifically, we were able to show that the same fiber operators that were employed in the analysis of L_2-representations can be utilized in the analysis of AP representations. The final result is a quantitative equivalence between the notion of WH $L_2(\mathbb{R})$-frame (affine $L_2(\mathbb{R})$-frame, respectively) and the notion of WH AP-frame (affine AP-frame, respectively). We refer to the connection as "quantitative", since the sharpest possible bounds in the L_2-representations are also the sharpest possible bounds in the AP-representations.

In order to state our results, we will need first to introduce the notions of L_2-frames and AP-frames as well as those of WH systems and wavelet systems. We begin with the definition of an AP function.

Definition 1.1. Let f be a complex-valued function defined on $\mathbb{R}$ and let $\epsilon > 0$. An **ϵ-almost period** of f is a number τ such that

$$\sup_t |f(t+\tau) - f(t)| < \epsilon.$$

A function f is **almost periodic** (AP) on $\mathbb{R}$ if it is continuous and if for every ϵ there exists a number $L = L(\epsilon, f)$ such that every interval of length L on $\mathbb{R}$ contains an ϵ-almost period of f. As said, we denote by AP the space of AP functions on $\mathbb{R}$. □

The AP space admits an inner product. The inner product of the AP space is defined by

$$\langle f, g\rangle_{AP} := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)\overline{g}(t)dt. \tag{1.2}$$

Let $\|f\|_{AP}$ denote the AP norm that is induced by this inner product.

Next, we define the notions of a *WH* system and an *affine* system, and start with the former. A WH system is obtained by applying discrete translations and modulations to a subset $\Psi \subset L_2(\mathbb{R})$ of **window functions** defined on $\mathbb{R}$. (The set is usually a singleton, and we assume it to be finite; in principle, a countable Ψ could have been allowed by our results, as well.) We then choose two positive numbers t_0 and w_0, and define the **Weyl-Heisenberg system generated by** Ψ to be the set

$$X := X(\Psi, t_0, w_0) := \{\psi_{k,l} := \psi(\cdot - k)e^{il(\cdot - k)} \,:\, k \in K,\ l \in L,\ \psi \in \Psi\}, \tag{1.3}$$

where

$$K := t_0\mathbb{Z}, \quad L := w_0\mathbb{Z}.$$

We also denote

$$D := 2\pi\mathbb{Z}/t_0.$$

An affine system is obtained when the discrete modulations are replaced by discrete dilations. Our results in this paper treat only the case of *integral* dilations. Thus, given a positive integer $\alpha > 1$ and a finite subset Ψ of $L_2(\mathbb{R})$ of **mother wavelets**, we define

$$\psi_{m,n} := \alpha^{-m}\psi(\alpha^{-m}\cdot -n), \quad \psi \in \Psi,\ m,n \in \mathbb{Z}.$$

The **affine system generated by** Ψ is the set

$$X := X(\Psi,\alpha) := \{\sqrt{\alpha^m}\,\psi_{m,n} \ :\ \psi \in \Psi,\, m,n \in \mathbb{Z}\}. \tag{1.4}$$

(Note that the wavelets $\psi_{m,n}$ are normalized in L_1, not L_2, i.e., $\|\psi_{m,n}\|_{L_1(\mathbb{R})} = \|\psi_{0,0}\|_{L_1(\mathbb{R})}$. This is the right normalization in the AP-setup. The definition of an affine system is geared at an L_2-setup, hence the renormalization of the wavelets.)

1.2. Main results

We first recall the notion of an L_2-frame, and then introduce a corresponding notion of an AP-frame.

Definition 1.5. Let $\langle\cdot,\cdot\rangle$ be the usual inner product in $L_2(\mathbb{R})$, and let $X \subset L_2(\mathbb{R})$ be countable. X is called a **fundamental frame** for $L_2(\mathbb{R})$ (L_2-frame for short) if there exist two positive constants A, B such that

$$A\,\|f\|^2_{L_2(\mathbb{R})} \le \sum_{x\in X} |\langle f,x\rangle|^2 \le B\|f\|^2_{L_2(\mathbb{R})}, \quad \forall f \in L_2(\mathbb{R}). \tag{1.6}$$

The sharpest possible constants are known as the **upper frame bound** and the **lower frame bound**. A frame whose upper and lower bounds coincide is a **tight frame** for $L_2(\mathbb{R})$. In particular, an orthonormal basis is a tight frame. If only the right inequality in (1.6) is valid, X is called a **Bessel system**, and the sharpest B in (1.6) is then referred to as the **Bessel bound**. □

The linear map

$$T^* : L_2(\mathbb{R}) \to \ell_2(X) : f \mapsto (\langle f,x\rangle)_{x\in X},$$

which is known as the *analysis map*, underlies the notion of an L_2-frame in the sense that X is a frame if and only if T^* is bounded and has closed range. From this simple point of view, it looks like one has very little hope to use the same system X for analysing AP-functions: it is well known that the AP space is non-separable, and, therefore, no collection of countably many elements of its dual space can be total on it. However, the linear functionals in X are unbounded on the AP-space, and, as we will shortly see, *can* be used to capture the AP-norm. To this end, we follow [PU] and [G] and employ a suitable averaging process that is described separately in the WH case and in the affine case.

Definition 1.7. Let $X := X(\Psi, t_0, w_0)$ be a WH representation system as in (1.3). We say that X is a (WH) **AP-frame** if $\Psi \subset L_1(\mathbb{R})$ and there exist positive constants A, B such that, for every AP-function f,

$$A\,\|f\|^2_{AP} \le \sum_{l\in L} \lim_{N\to\infty} \frac{1}{2Nt_0} \sum_{k\in K(N)} \sum_{\psi\in\Psi} |\langle f,\psi_{k,l}\rangle|^2 \le B\,\|f\|^2_{AP}, \tag{1.8}$$

where $K(N) := \{nt_0 \in K \ :\ -N \le n \le N\}$. The definition includes among its conditions the convergence of the above averaging process. Given a WH AP-frame, the sharpest possible constants in (1.8) are called the **upper AP frame bound** and the **lower AP frame bound**. A frame whose upper and lower bounds coincide is a (WH) **tight frame** for AP. Moreover, if only the right-hand side inequality is valid, then X is a WH **AP Bessel system** and the sharpest constant B in this bound is then the **AP Bessel bound.** □

Definition 1.9. Let $X := X(\Psi, \alpha)$ be an affine representation system as in (1.4). We say that X is an (affine) **AP-frame** if $\Psi \subset L_1(\mathbb{R})$ and there exist positive constants A, B such that every AP-function f that satisfies $\widehat{f}(0) := \lim_{T\to\infty} \int_{-T}^{T} f(t)\, dt = 0$, satisfies also the following inequalities:

$$A\,\|f\|_{AP}^2 \le \sum_{m\in\mathbb{Z}} \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} \sum_{\psi\in\Psi} |\langle f, \psi_{m,n}\rangle|^2 \le B\,\|f\|_{AP}^2. \tag{1.10}$$

The definition includes among its conditions the convergence of the above averaging process. Given an affine AP-frame, the sharpest possible constants in (1.10) are called the **upper AP frame bound** and the **lower AP frame bound**. A frame whose upper and lower bounds coincide is a (affine) **tight frame** for AP. Moreover, if only the right-hand side inequality is valid, then X is an affine **AP Bessel system** and the sharpest constant B in this bound is then the **AP Bessel bound.** □

We note that Definitions 1.7 and 1.9 are new, and are made in anticipation of our results on the matter. One of the main results in this paper says, essentially, that the notions of WH L_2-frame (WH L_2-Bessel system, respectively) and WH AP-frame (WH AP-Bessel system, respectively) are the same. The stronger, quantitative, version of this equivalence is that the L_2 frame bounds coincide with the AP frame bounds (with a similar assertion in the Bessel case). Further, an identical set of statements is established in the affine case, too.

However, as our reader will shortly see, we do impose some *a priori* conditions on the window functions/mother wavelets (i.e., the set Ψ.) We believe (though do not have a formal proof for it) that those restrictions are essential, and are due to the fact that the AP setup is associated with the discrete topology on the frequency domain, while the L_2 setup is associated with the Lebesgue measure on the frequency domain. As a result, while our characterization of L_2 frames (in terms of the fiber operators that we introduce later) is the same as the characterization of AP frames, the former is slightly weaker since it is required to be valid a.e. on the frequency domain while the latter is required to be valid everywhere. By imposing a mild condition on the window functions, we can bridge this small gap and get unconditional equivalence. Here are our two main results.

Theorem 1.11. *Let $X = X(\Psi, t_0, w_0) \subset L_1(\mathbb{R})$ be a WH system. Set $D := 2\pi\mathbb{Z}/t_0$, $L := w_0\mathbb{Z}$. Assume that*

$$\text{for each } \psi \in \Psi \text{ and each } d \in D,\ \sum_{l\in L} \widehat{\psi}(\cdot - l)\overline{\widehat{\psi}}(\cdot - d - l) \text{ is continuous.} \tag{1.12}$$

Then:

- *(a) X is an L_2-frame if and only if it is an AP-frame. Moreover, the L_2 upper frame bound of X is identical to the AP upper frame bound of X. The same holds for the two lower frame bounds.*
- *(b) X is an L_2 Bessel system if and only if it is an AP Bessel system. The two Bessel bounds are then identical as well.*
- *(c) The "if" implications in (a) and (b) are valid even without assumption (1.12).*
- *(d) X is a tight L_2-frame if and only if it is a tight AP-frame.*

In the affine counterpart of the above result, we use the following α-adic valuation function κ:

$$\kappa : \mathbb{R} \to \mathbb{Z} : \lambda \mapsto \inf\{m \in \mathbb{Z} : \alpha^m\lambda \in 2\pi\mathbb{Z}\}.$$

(Thus, $\kappa(0) = -\infty$, and $\kappa(\lambda) = \infty$ unless $\lambda \in \alpha^m 2\pi\mathbb{Z}$ for some integer m, i.e., λ is a 2π-α-adic integer.)

Theorem 1.13. *Let $X = X(\Psi, \alpha) \subset L_1(\mathbb{R})$ be an affine system. Set $Q := \cup_{m\in\mathbb{Z}} 2\pi\mathbb{Z}/\alpha^m$. Assume that*

$$\text{for each } \psi \in \Psi \text{ and each } \gamma \in Q,\ \sum_{m=\kappa(\gamma)}^{\infty} \widehat{\psi}(\alpha^m\cdot)\overline{\widehat{\psi}}(\alpha^m(\cdot + \gamma)) \text{ is continuous on } \mathbb{R}\backslash\{0, -\gamma\}. \tag{1.14}$$

Then:

- *(a) X is an L_2 affine frame if and only if it is an AP affine frame. Moreover, the L_2 upper frame bound of X is the same as the AP upper frame bound of X. The two lower frame bounds coincide as well.*

(b) X is an L_2 Bessel system if and only if it is an AP Bessel system. The two Bessel bounds are then identical as well.
(c) The "if" implications in (a) and (b) are valid even without assumption (1.14).
(d) X is a tight L_2-frame if and only if it is a tight AP-frame.

Next, we review the main results of [PU] and [G]. We believe that [PU] is the first paper to introduce time frequency representations of almost-periodic functions by WH systems and affine systems. In [G], given a WH system $X(\Psi, t_0, w_0)$, the following inequalities are established:

$$\tilde{A}\,\|f\|_{AP}^2 \le \lim_{N\to\infty} \frac{1}{2Nt_0} \sum_{k\in K(N)} \sum_{l\in L} \sum_{\psi\in\Psi} |\langle f, \psi_{k,l}\rangle|^2 \le \tilde{B}\,\|f\|_{AP}^2,$$

under the following assumptions:
(1) $\Psi = \{\psi\}$ is a singleton;
(2) ψ is a bounded function and $\psi(t) = O(1/t^2)$ as $t \to \pm\infty$;
(3)

$$\tilde{A} := \inf_{\lambda\in\mathbb{R}} \sum_{l\in L} |\widehat{\psi}(\lambda - l)|^2 - \sum_{d\in D\backslash\{0\}} (\Gamma(d)\Gamma(-d))^{1/2} > 0$$

and

$$\tilde{B} := \sup_{\lambda\in\mathbb{R}} \sum_{l\in L} |\widehat{\psi}(\lambda - l)|^2 + \sum_{d\in D\backslash\{0\}} (\Gamma(d)\Gamma(-d))^{1/2} < \infty,$$

where

$$\Gamma(d) := \sup_{\lambda\in\mathbb{R}} \sum_{l\in L} |\widehat{\psi}(\lambda - l)\overline{\widehat{\psi}}(\lambda - d - l)|.$$

One finds that the conditions $\tilde{A} > 0$ and $\tilde{B} < 0$ are the hypotheses used by [D] in the construction of L_2-frames. These assumptions are sufficient conditions for X to be an L_2-frame. However, they are not necessary.

In the affine case, [PU] established the following identity, with ψ the Haar wavelet:

$$\lim_{k\to\infty} \sum_{m=-\infty}^{k} \sum_{n=-2^{k-m}}^{2^{k-m}-1} 2^{-m-k-1} |\langle f, \psi_{m,n}\rangle|^2 = \|f\|_{AP}^2.$$

The reference [G] extended the aforementioned result to a wider family of affine systems. It proved that $X := X(\Psi, \alpha)$ is an (affine) AP-frame under the following assumptions:
(1) $\alpha > 1$ and $\beta > 0$;
(2) $\Psi = \{\psi\}$ is a singleton and $\psi_{m,n} := \alpha^{-m}\psi(\alpha^{-m}\cdot - \beta n)$;
(3) $\psi \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ and $\int_{\mathbb{R}} |\psi(\lambda)|^2/|\lambda|\, d\lambda < \infty$;
(4)

$$\tilde{A} := \inf_{\lambda\in\mathbb{R}} \sum_{m\in\mathbb{Z}} |\widehat{\psi}(\lambda\alpha^m)|^2 - \sum_{n\in\mathbb{Z}\backslash\{0\}} (\Gamma(n)\Gamma(-n))^{1/2} > 0$$

and

$$\tilde{B} := \sup_{\lambda\in\mathbb{R}} \sum_{m\in\mathbb{Z}} |\widehat{\psi}(\lambda\alpha^m)|^2 + \sum_{n\in\mathbb{Z}\backslash\{0\}} (\Gamma(n)\Gamma(-n))^{1/2} < \infty,$$

where

$$\Gamma(n) := \sup_{\lambda\in\mathbb{R}} \sum_{m\in\mathbb{Z}} |\widehat{\psi}(\lambda\alpha^m)\overline{\widehat{\psi}}(\lambda\alpha^m - n/\beta)|.$$

Again, the assumptions made in [G] on the affine system are sufficient conditions for the system to be an L_2-frame, [D]. Once again, those conditions are not necessary. Moreover, the constants $\widetilde{A}$, $\widetilde{B}$ (either those from the WH case or those from the affine case) are known to be valid frame bounds; however they are generally (and generically) different from the sharpest bounds, i.e., *the* frame bounds.

We conclude the introductory section with some notations as well as background material on AP functions. That latter material can be found, for example, in [B] and [K].

1.3. Notations and background material

For $\lambda \in \mathbb{R}$, the exponential e_λ is the function

$$e_\lambda : \mathbb{R} \to \mathbb{C} : t \mapsto e^{i\lambda t}.$$

A **trigonometric polynomial** in this paper is not restricted to periodic functions, i.e., it is a finite combination of *arbitrary* bounded exponentials:

$$f = \sum_{k=1}^{n} a_k e_{\lambda_k}, \quad a_k \in \mathbb{C},\ \lambda_k \in \mathbb{R},\ n \in \mathbb{N}.$$

The exponentials $(e_\lambda)_{\lambda \in \mathbb{R}}$ are AP functions and form an orthonormal system in the AP space. Thus, every trigonometric polynomial is an AP function and satisfies

$$\| \sum_{k=1}^{n} a_k e_{\lambda_k} \|_{AP}^2 = \sum_{k=1}^{n} |a_k|^2.$$

The exponentials actually form an orthonormal *basis* for the AP space, i.e., the above Parseval's identity extends to the entire AP space:

$$\|f\|_{AP}^2 = \sum_{\lambda \in \mathbb{R}} |a_f(\lambda)|^2,$$

where

$$a_f(\lambda) := \widehat{f}(\lambda) := \langle f, e_\lambda \rangle_{AP}. \tag{1.15}$$

In particular, the trigonometric polynomials are dense in the AP space. They are even dense in it in the stronger uniform norm; as a matter of fact, the AP space is the uniform closure of the trigonometric polynomials.

The **norm spectrum** of $f \in AP$ is defined as

$$\sigma := \sigma(f) := \{\lambda : \langle f, e_\lambda \rangle_{AP} \neq 0\}.$$

It follows from the above that σ is (at most) countable.

Next, we define in this paper the Fourier transform on $L_1(\mathbb{R})$ by

$$\widehat{f}(\lambda) := \int_{\mathbb{R}} f(t) e^{-i\lambda t} dt,$$

and extend it in the usual way to an isometry from $L_2(\mathbb{R})$ onto itself.

Finally, we use the following shorthand notations for some of the quantities that appear in Theorems 1.11 and 1.13, respectively:

$$H(f, \Psi) := \sum_{l \in L} \lim_{N \to \infty} \frac{1}{2Nt_0} \sum_{k \in K(N)} \sum_{\psi \in \Psi} |\langle f, \psi_{k,l} \rangle|^2,$$

$$F(f, \Psi) := \sum_{m \in \mathbb{Z}} \lim_{N \to \infty} \frac{1}{2N} \sum_{n=-N}^{N} \sum_{\psi \in \Psi} |\langle f, \psi_{m,n} \rangle|^2.$$

2. WH representations of almost periodic functions

Our goal in this section is to prove Theorem 1.11. Our tool to this end is the fiberization of the analysis operator that is associated with the given WH system, [RS97a]. We discuss this topic in the first subsection, and present the aforementioned proof in the second and last subsection.

2.1. Fiberization of WH systems

Let Ψ be a finite subset of $L_2(\mathrm{IR})$, and let $X := X(\Psi, t_0, w_0)$ be a WH system. For each $\lambda \in \mathrm{IR}$, the fiber $\widetilde{G}(\lambda)$ of the dual Gramian $\widetilde{G}$ of X is defined as

$$\tilde{G}(\lambda) := \Big(\widetilde{G}(\lambda)(d,d')\Big)_{(d,d')\in D^2}, \quad \widetilde{G}(\lambda)(d,d') := \frac{1}{t_0}\sum_{\psi\in\Psi}\sum_{l\in L}\widehat{\psi}\,(\lambda-d-l)\,\overline{\widehat{\psi}}\,(\lambda-d'-l)\,,$$

where $D := 2\pi\mathbb{Z}/t_0$ and $L := w_0\mathbb{Z}$. Each fiber $\tilde{G}(\lambda)$ is a non-negative definite self-adjoint operator and is considered as an endomorphism of $\ell_2(D)$ with norm denoted by $\mathcal{G}^*(\lambda)$ and inverse norm $\mathcal{G}^{*-}(\lambda)$. It is tacitly assumed, hence understood, that $\mathcal{G}^*(\lambda) := \infty$ whenever $\tilde{G}(\lambda)$ does not represent a bounded operator; for example, since we only know that $\Psi \subset L_2$, the sum that defines $\widetilde{G}(\lambda)(d,d')$ converges merely locally in L_1, hence is defined only a.e. Thus, there exists a nullset of fibers with entries that are not even well-defined. The convention $\mathcal{G}^*(\lambda) := \infty$ automatically applies to each of these matrices. A similar remark applies to $\mathcal{G}^{*-}(\lambda)$; moreover, in this case we automatically have $\mathcal{G}^{*-}(\lambda) = \infty$ whenever $\mathcal{G}^*(\lambda) = \infty$. A more detailed discussion of the dual Gramian fibers of X is provided in [RS95] (for a general shift-invariant X), and in [RS97a] (for the current WH case). The following result is quoted from [RS97a]:

Result 2.1. *Let $X = X(\Psi, t_0, w_0)$, $\Psi \subset L_2(\mathrm{IR})$, be a WH system. Let $\tilde{G}$ be the associated dual Gramian, and let $\mathcal{G}^*$ and $\mathcal{G}^{*-}$ be the dual Gramian norm functions that are defined as above. Then:*

(a) The following conditions are equivalent:
- *(a1) X is a Bessel system;*
- *(a2) $\mathcal{G}^* \in L_\infty(\mathrm{IR})$.*

Furthermore, $\|\mathcal{G}^*\|_\infty$ is the Bessel bound of X.

(b) Assume X to be a Bessel system. Then the following conditions are equivalent:
- *(b1) X is an L_2-frame;*
- *(b2) $\mathcal{G}^{*-} \in L_\infty(\mathrm{IR})$.*

Furthermore, $\|\mathcal{G}^{*-}\|_\infty^{-1}$ is the lower L_2 frame bound of X.

Next, we define, for each $l \in L$ and each $\lambda \in \mathrm{IR}$,

$$\tilde{G}_l(\lambda) := \Big(\tilde{G}_l(\lambda)(d,d')\Big)_{(d,d')\in D^2}, \quad \tilde{G}_l(\lambda)(d,d') := \frac{1}{t_0}\sum_{\psi\in\Psi}\widehat{\psi}\,(\lambda-d-l)\,\overline{\widehat{\psi}}\,(\lambda-d'-l)\,. \tag{2.2}$$

Then we have the following lemma.

Lemma 2.3. *If $X := X(\Psi, t_0, w_0)$ is a WH Bessel system, then, for each $a \in \ell_2(D)$ and a.e. $\lambda \in \mathrm{IR}$,*

$$\sum_{l\in L} a^*\tilde{G}_l(\lambda)a = a^*\tilde{G}(\lambda)a.$$

Proof. First, note that, in view of (a) of Result 2.1, the claim trivially holds for any *finitely* supported sequence $a_0 \in \ell_2(D)$. Since X is a Bessel system for $L_2(\mathrm{IR})$, then, [RS95], for each $l \in L$ and a.e. $\lambda \in \mathrm{IR}$, $\tilde{G}_l(\lambda) : \ell_2(D) \to \ell_2(D)$ is a non-negative definite self-adjoint operator and $\|\sum_{l\in I}\tilde{G}_l(\lambda)\| \le \|\mathcal{G}^*\|_\infty$, for any finite $I \subset L$. So, for each $M \in \mathrm{IN}$ and a.e. $\lambda \in \mathrm{IR}$, $E_M(\lambda) := \tilde{G}(\lambda) - \sum_{|l|\le M}\tilde{G}_l(\lambda)$ is also a non-negative definite self-adjoint operator and $\|E_M(\lambda)\| \le 2\|\mathcal{G}^*\|_\infty$. Therefore, given a finitely supported $a_0 \in \ell_2(D)$ such that $\|a_0\|_{\ell_2} \le \|a\|_{\ell_2}$,

$$a^*E_M(\lambda)a = (a_0 + (a-a_0))^*E_M(\lambda)(a_0 + (a-a_0)) \le a_0^*E_M(\lambda)a_0 + 6\|a\|_{\ell_2}\|a-a_0\|_{\ell_2}\|\mathcal{G}^*\|_\infty.$$

This, together with the fact that $\lim_{M\to\infty} a_0^*E_M(\lambda)a_0 = 0$ easily implies that, given $\varepsilon > 0$, we have that $a^*E_M(\lambda)a < \varepsilon$ for all sufficiently large M. □

We also need the following corollary of Result 2.1:

Corollary 2.4. *If $X := X(\Psi, t_0, w_0)$ is a WH Bessel system, then the functions*

$$\sum_{l\in L}\sum_{\psi\in\Psi}\left|\widehat{\psi}(\cdot - l)\overline{\widehat{\psi}}(\cdot - t - l)\right|, \quad t\in L, \tag{2.5}$$

lie, each, in $L_\infty(\mathbb{R})$, and form a bounded set there.

Proof. Since X is a Bessel system, Result 2.1 implies that $\mathcal{G}^*$ is essentially bounded, say by C/t_0. Thus, for a.e. $\lambda\in\mathbb{R}$,

$$\sum_{l\in L}\sum_{\psi\in\Psi}|\widehat{\psi}(\lambda-l)|^2 = t_0\widetilde{G}(\lambda)(0,0)\le t_0\mathcal{G}^*(\lambda)\le C.$$

The requisite boundedness follows then by Schwarz' inequality. □

Remark. Note that the last corollary clarifies the context of assumption (1.12) in Theorem 1.11: that condition is used in the proofs of the "only if" implications in (a) and (b) of the theorem. Thus, whenever it is needed, X is always known to be a Bessel system. In this event, Corollary 2.4 guarantees that the sum in (2.5) converges absolutely (a.e.). Assumption (1.12) in Theorem 1.11 then merely asserts that this sum is well-defined everywhere and is also continuous. Similar remark can be applied to the affine case too.

2.2. WH-based Characterization of AP functions

We are now ready to prove Theorem 1.11.

Proof of Theorem 1.11. First, we prove the "only if" implication in (a). The proof of the "only if" implication of (b) is omitted, since that proof uses a subset of the arguments that we use in the proof of (a). We begin with the analysis of the case when $f\in AP$ is a trigonometric polynomial of the following specific form:

$$f := \sum_{d\in D} a(d)e_{(\lambda-d)}.$$

Here, $\lambda\in\mathbb{R}$ is arbitrary. Since $\Psi\subset L_1(\mathbb{R})$, for each $\psi\in\Psi$, each $k\in K$, and each $l\in L$,

$$\langle f,\psi_{k,l}\rangle = \sum_{d\in D} a(d)\overline{\widehat{\psi}}(\lambda-d-l)e^{ik(\lambda-d)}$$

is finite and hence, since $t_0(D-D)\subset 2\pi\mathbb{Z}$,

$$|\langle f,\psi_{k,l}\rangle|^2 = \sum_{(d,d')\in D^2} a(d)\overline{a}(d')\overline{\widehat{\psi}}(\lambda-d-l)\widehat{\psi}(\lambda-d'-l).$$

Thus $|\langle f,\psi_{k,l}\rangle|^2$ is independent of k, and we conclude that

$$t_0H(f,\Psi) = \sum_{l\in L}\sum_{(d,d')\in D^2}\sum_{\psi\in\Psi} a(d)\overline{a}(d')\overline{\widehat{\psi}}(\lambda-d-l)\widehat{\psi}(\lambda-d'-l).$$

By assumption (1.12) of the present theorem,

$$\sum_{\psi\in\Psi}\sum_{l\in L}\widehat{\psi}(\cdot - l)\overline{\widehat{\psi}}(\cdot - d - l)$$

converges everywhere for each $d\in D$. This justifies the change in the summation order in the first equality of the following derivation:

$$\begin{aligned}
&\sum_{l\in L}\sum_{(d,d')\in D^2}\sum_{\psi\in\Psi} a(d)\overline{a}(d')\overline{\widehat{\psi}}(\lambda-d-l)\widehat{\psi}(\lambda-d'-l)\\
&= \sum_{(d,d')\in D^2} a(d)\overline{a}(d')\sum_{l\in L}\sum_{\psi\in\Psi}\overline{\widehat{\psi}}(\lambda-d-l)\widehat{\psi}(\lambda-d'-l)\\
&= t_0\, a^*\tilde{G}(\lambda)a.
\end{aligned}$$

Thus, for this special type of AP functions, the averaging process $H(f,\Psi)$ coincides with the action of the quadratic form $\widetilde{G}(\lambda)$ on the coefficient vector a.

Now, let f be a general trigonometric polynomial, say, $f=\sum_{\lambda\in\sigma}a(\lambda)e_\lambda$, with $\sigma\subset\mathbb{R}$ finite. We define an equivalence relation $\sim$ on σ by $\lambda\sim\lambda'\iff\lambda-\lambda'\in D$. Let $\Lambda\subset\sigma$ be a set of representers of the equivalence classes. For $\lambda\in\Lambda$, denote by $\sigma_\lambda\subset\sigma$ the corresponding equivalence class. Further, set

$$\Sigma_1:=\cup_{\lambda\in\Lambda}\sigma_\lambda\times\sigma_\lambda,\quad \Sigma_2:=(\sigma\times\sigma)\setminus\Sigma_1.$$

Now, for each $\lambda\in\Lambda$, we define a sequence

$$a_\lambda:D\to\mathbb{C},\ d\mapsto\begin{cases}a(\lambda-d), & \text{if } \lambda-d\in\sigma_\lambda,\\ 0, & \text{otherwise.}\end{cases}\tag{2.6}$$

The argument used in the first part of the proof can be repeated now to yield that

$$H(f,\Psi)-\sum_{\lambda\in\Lambda}a_\lambda^*\widetilde{G}(\lambda)a_\lambda=\sum_{l\in L}\lim_{N\to\infty}\frac{1}{2Nt_0}\sum_{k\in K(N)}\sum_{\psi\in\Psi}\sum_{(\lambda,\lambda')\in\Sigma_2}a(\lambda)\overline{a}(\lambda')\overline{\widehat{\psi}}(\lambda-l)\widehat{\psi}(\lambda'-l)e^{ik(\lambda-\lambda')},$$

provided that we can show that the limit in the right-hand-side above exists for every l and ψ. Here, we used again Corollary 2.4 (as well as assumption (1.12) in the current theorem), this time for the choice $t:=\lambda-\lambda'$. We will show that the right-hand-side in the last display equals 0. To this end, we fix $l\in L$ and $\psi\in\Psi$, and examine the expression

$$\frac{1}{2N}\sum_{k\in K(N)}\sum_{(\lambda,\lambda')\in\Sigma_2}a(\lambda)\overline{a}(\lambda')\overline{\widehat{\psi}}(\lambda-l)\widehat{\psi}(\lambda'-l)e^{ik(\lambda-\lambda')}.$$

The above sum is actually finite, hence equals

$$\sum_{(\lambda,\lambda')\in\Sigma_2}a(\lambda)\overline{a}(\lambda')\overline{\widehat{\psi}}(\lambda-l)\widehat{\psi}(\lambda'-l)\frac{1}{2N}\sum_{k\in K(N)}e^{ik(\lambda-\lambda')}.$$

Since, by our definition of Σ_2, $\lambda-\lambda'\notin D$, we have that

$$\lim_{N\to\infty}\frac{1}{2N}\sum_{k\in K(N)}e^{ik(\lambda-\lambda')}=0.$$

Consequently,

$$H(f,\Psi)=\sum_{\lambda\in\Lambda}a_\lambda^*\widetilde{G}(\lambda)a_\lambda.\tag{2.7}$$

Since we assume that X is an L_2-frame, we know from Result 2.1 that, for a.e. $\lambda\in\mathbb{R}$, and for any $a_\lambda\in\ell_2(D)$,

$$\frac{\|a_\lambda\|_{\ell_2}^2}{\|\mathcal{G}^{*-}\|_\infty}\le a_\lambda^*\,\tilde{G}(\lambda)\,a_\lambda\le\|\mathcal{G}^*\|_\infty\|a_\lambda\|_{\ell_2}^2.\tag{2.8}$$

However, assumption (1.12) in the current theorem guarantees the entries of the dual Gramian to be pointwise continuous, and hence (2.8) is valid *everywhere*, hence also at every $\lambda\in\Lambda$. Thus,

$$\frac{\|a\|_{\ell_2}^2}{\|\mathcal{G}^{*-}\|_\infty}=\sum_{\lambda\in\Lambda}\frac{\|a_\lambda\|_{\ell_2}^2}{\|\mathcal{G}^{*-}\|_\infty}\le\sum_{\lambda\in\Lambda}a_\lambda^*\,\tilde{G}(\lambda)\,a_\lambda\le\sum_{\lambda\in\Lambda}\|\mathcal{G}^*\|_\infty\|a_\lambda\|_{\ell_2}^2=\|\mathcal{G}^*\|_\infty\|a\|_{\ell_2}^2.\tag{2.9}$$

Therefore, for a trigonometric polynomial f, we obtain the requisite inequalities:

$$\frac{\|f\|_{AP}^2}{\|\mathcal{G}^{*-}\|_\infty} \le H(f,\Psi) \le \|\mathcal{G}^*\|_\infty \|f\|_{AP}^2.$$

Next, let f be a general almost periodic function and let σ be the norm spectrum of f. Since σ is countable, we can enumerate its elements: $\sigma = \{\lambda_1, \lambda_2, \cdots\}$. We set $\sigma^s := \{\lambda_1, \cdots, \lambda_s\}$ for each $s \in \mathbb{N}$. Then, [K], there is a trigonometric polynomial sequence $\{f_s\}_{s=1}^\infty$ which converges uniformly to f such that for each $s \in \mathbb{N}$ the norm spectrum of f_s is σ^s. We denote, for each $s \in \mathbb{N}$,

$$f_s = \sum_{\lambda \in \sigma^s} a^s(\lambda) e_\lambda.$$

Since $\lim_{s\to\infty} f_s = f$ in $L_\infty(\mathbb{R})$, $\langle f_s, \psi_{k,l}\rangle$ converges to $\langle f, \psi_{k,l}\rangle$ uniformly in k, l as $s \to \infty$. So, it is easy to check that

$$\lim_{s\to\infty} \lim_{N\to\infty} \frac{1}{2Nt_0} \sum_{k\in K(N)} \sum_{\psi\in\Psi} |\langle f_s, \psi_{k,l}\rangle|^2 = \lim_{N\to\infty} \frac{1}{2Nt_0} \sum_{k\in K(N)} \sum_{\psi\in\Psi} |\langle f, \psi_{k,l}\rangle|^2. \tag{2.10}$$

Consequently,

$$H(f,\Psi) = \sum_{l\in L} \lim_{s\to\infty} \lim_{N\to\infty} \frac{1}{2Nt_0} \sum_{k\in K(N)} \sum_{\psi\in\Psi} |\langle f_s, \psi_{k,l}\rangle|^2. \tag{2.11}$$

This, together with Fatou's lemma implies that

$$\begin{aligned} H(f,\Psi) &\le \limsup_{s\to\infty} \sum_{l\in L} \lim_{N\to\infty} \frac{1}{2Nt_0} \sum_{k\in K(N)} \sum_{\psi\in\Psi} |\langle f_s, \psi_{k,l}\rangle|^2 \\ &= \limsup_{s\to\infty} H(f_s, \Psi) \\ &\le \lim_{s\to\infty} \|\mathcal{G}^*\|_\infty \|a^s\|_{\ell_2}^2 \\ &= \|\mathcal{G}^*\|_\infty \|a\|_{\ell_2}^2. \end{aligned} \tag{2.12}$$

Now, with the equivalence relation $\sim$, and the set of representers Λ as before, we define, for each $s \in \mathbb{N}$,

$$\Lambda_s := \{\lambda \in \Lambda : \ \lambda - D \cap \sigma^s \neq \emptyset\}.$$

Given $\lambda \in \Lambda_s$, we also define a sequence

$$a_\lambda^s : D \to \mathbb{C}, \ d \mapsto \begin{cases} a^s(\lambda - d), & \lambda - d \in \sigma^s, \\ 0, & \text{otherwise}. \end{cases}$$

Note that $\sum_{\lambda\in\Lambda_s} \|a_\lambda^s\|_{\ell_2}^2 = \|a^s\|_{\ell_2}^2$. Since $\Psi \subset L_1(\mathbb{R})$, each entry of $\tilde{G}_l(\lambda)$ is also continuous for every $\lambda \in \mathbb{R}$ and each $l \in L$. Hence the argument used in the first part of the proof can be repeated here to yield that

$$\begin{aligned} H(f,\Psi) &= \sum_{l\in L} \lim_{s\to\infty} \lim_{N\to\infty} \frac{1}{2Nt_0} \sum_{k\in K(N)} \sum_{\psi\in\Psi} |\langle f_s, \psi_{k,l}\rangle|^2 \\ &= \sum_{l\in L} \lim_{s\to\infty} \sum_{\lambda\in\Lambda_s} (a_\lambda^s)^* \tilde{G}_l(\lambda) a_\lambda^s. \end{aligned}$$

Now, for each $\lambda \in \Lambda$, a_λ^s converges to a_λ in $\ell_2(D)$ as $s \to \infty$, where a_λ is defined in (2.6) with infinite set σ at this time. Thus,

$$\lim_{s\to\infty} (a_\lambda^s)^* \tilde{G}_l(\lambda) a_\lambda^s = a_\lambda^* \tilde{G}_l(\lambda) a_\lambda \le \|a_\lambda\|_{\ell_2}^2 \|\mathcal{G}^*\|_\infty,$$

and $\sum_{\lambda\in\Lambda}\|a_\lambda\|_{\ell_2}^2 = \|a\|_{\ell_2}^2$. So, by, e.g., dominated convergence argument,

$$H(f,\Psi) = \sum_{l\in L}\sum_{\lambda\in\Lambda} a_\lambda^* \tilde{G}_l(\lambda) a_\lambda.$$

Since each term of this series is non-negative and $H(f,\Psi) < \infty$ by (2.12), this series converges absolutely so that

$$H(f,\Psi) = \sum_{\lambda\in\Lambda}\sum_{l\in L} a_\lambda^* \tilde{G}_l(\lambda) a_\lambda.$$

By Lemma 2.3, we finally have

$$\sum_{\lambda\in\Lambda}\sum_{l\in L} a_\lambda^* \tilde{G}_l(\lambda) a_\lambda = \sum_{\lambda\in\Lambda} a_\lambda^* \tilde{G}(\lambda) a_\lambda \geq \sum_{\lambda\in\Lambda} \frac{\|a_\lambda\|_{\ell_2}^2}{\|\mathcal{G}^{*-}\|_\infty} = \frac{\|a\|_{\ell_2}^2}{\|\mathcal{G}^{*-}\|_\infty}.$$

We therefore conclude that for any $f \in AP$,

$$\frac{\|f\|_{AP}^2}{\|\mathcal{G}^{*-}\|_\infty} \leq H(f,\Psi) \leq \|\mathcal{G}^*\|_\infty \|f\|_{AP}^2.$$

Now, we prove the "if" assertion in (a). We first choose f to be an exponential e_λ, $\lambda \in \mathbb{R}$. The argument in the first part of the proof applies to yield that

$$\frac{1}{t_0}\sum_{\psi\in\Psi}\sum_{m\in\mathbb{Z}} |\widehat{\psi}(\lambda - l)|^2 = H(f,\Psi).$$

By our current assumption, $H(f,\Psi) < \infty$. Thus, all the diagonal entries of each dual Gramian $\widetilde{G}(\lambda)$ converge absolutely to a finite limit, and hence, by Schwartz' inequality, all the entries of all the dual Gramians are finite. Next, for each $\lambda \in \mathbb{R}$, let

$$f := \sum_{d\in D} a(d) e_{(\lambda-d)},$$

with a finitely supported (hence in ℓ_2). We proved at the beginning of the proof that, for such a

$$H(f,\Psi) = a^* \tilde{G}(\lambda) a. \tag{2.13}$$

Since we assume here that $B\|f\|_{AP}^2 \leq H(f,\Psi) \leq A\|f\|_{AP}^2$, since we have that $\|f\|_{AP} = \|a\|_{\ell_2}$, and since the finitely supported sequence a is arbitrary, we conclude from (2.13) that the self-adjoint operator $\widetilde{G}(\lambda)$ is bounded above by A and bounded below by B, which is exactly what we needed to prove. Result 2.1 can be invoked now to yield that X is an L_2-frame with upper frame bound $\leq A$ and lower frame bound $\geq B$.

The proof of the "if" assertion in (b) is entirely analogous. That is, we proved (c). The statement in (d) follows from the other statements. □

3. Affine representations of AP functions

We move now to the proof of Theorem 1.13. Once again, we employ the fiberization of the analysis operator that is associated with the given affine system, [RS97b]. We present the relevant details on this fiberization in the first subsection, and prove the theorem in the second one.

3.1. The dual Gramian of affine systems

Let $X := X(\Psi, \alpha)$ be an affine system. The dual Gramian $\widetilde{G}$ of the system is based on the notion of the affine product $\Psi[\cdot, \cdot]$, [RS97b], of X, which is defined as

$$\Psi[\lambda, \lambda'] := \sum_{m=\kappa(\lambda-\lambda')}^{\infty} \sum_{\psi \in \Psi} \widehat{\psi}(\alpha^m \lambda)\overline{\widehat{\psi}}(\alpha^m \lambda'), \quad \lambda, \lambda' \in \mathbb{R},$$

where κ is the α-adic valuation

$$\kappa : \mathbb{R} \to \mathbb{Z} : \lambda \mapsto \inf\{m \in \mathbb{Z} \,:\, \alpha^m \lambda \in 2\pi\mathbb{Z}\}.$$

Then, for $(j, l) \in (2\pi\mathbb{Z})^2$ and $\lambda \in \mathbb{R}$, the (j, l)-entry of the dual Gramian fiber $\tilde{G}(\lambda)$, associated with the affine system X, is defined, [RS97b],

$$\tilde{G}(\lambda)(j, l) := \Psi[\lambda - j, \lambda - l].$$

In an analogous way to the WH case, we consider each fiber $\widetilde{G}(\lambda)$ as an endomorphism of $\ell_2(2\pi\mathbb{Z})$. This gives rise to the associated norm functions:

$$\begin{aligned} \mathcal{G}^* &: \mathbb{R} \to \mathbb{R} : \lambda \mapsto \|\tilde{G}(\lambda)\|, \\ \mathcal{G}^{*-} &: \mathbb{R} \to \mathbb{R} : \lambda \mapsto \|\tilde{G}^{-1}(\lambda)\|. \end{aligned}$$

As in the WH case, these norm functions are defined conservatively, with an automatic definition $\mathcal{G}^*(\lambda) := \infty$, whenever the fiber operator fails to represent a bounded endomorphism, for whatever reason; similarly, for $\mathcal{G}^{*-}$.

The following result concerning the fiberization of affine systems is taken from [RS97b].$^{\aleph}$

Result 3.1. *Let $X = X(\Psi, \alpha)$ be an affine system, and let $\tilde{G}$, $\mathcal{G}^*$ and $\mathcal{G}^{*-}$ be the associated dual Gramian and the resulting norm functions, defined as above. Then:*

(a) The following conditions are equivalent:
(a1) X is a Bessel system;
(a2) $\mathcal{G}^* \in L_\infty(\mathbb{R})$;
Furthermore, $\|\mathcal{G}^*\|_\infty$ is then the Bessel bound of X.

(b) Assume X to be a Bessel system. Then the following conditions are equivalent:
(b1) X is an L_2-frame;
(b2) $\mathcal{G}^{*-} \in L_\infty(\mathbb{R})$;
Furthermore, $\|\mathcal{G}^{*-}\|_\infty^{-1}$ is the lower frame bound of X.

The fibers $\widetilde{G}(\lambda)$ in the affine case are inter-related: $\widetilde{G}(\lambda)$ is a submatrix of $\widetilde{G}(\alpha\lambda)$. It is more convenient to us to eliminate this redundancy in the fiber operators. To this end, we define, for each $\lambda \in \mathbb{R}$, an augmented non-negative self-adjoint matrix $\tilde{\mathbb{G}}(\lambda)$ whose rows and columns are indexed by the set $Q := \cup_{m \in \mathbb{Z}} 2\pi\mathbb{Z}/\alpha^m$ of all α-adic 2π-integers. The entry of $(q, q') \in Q^2$ of $\tilde{\mathbb{G}}(\lambda)$ is defined as before:

$$\tilde{\mathbb{G}}(\lambda)(q, q') := \Psi[\lambda - q, \lambda - q'].$$

Each matrix is considered as an endomorphism of $\ell_2(Q)$ with norm denoted by $\boldsymbol{\mathcal{G}}^*(\lambda)$ and inverse norm $\boldsymbol{\mathcal{G}}^{*-}(\lambda)$. We note that, given any positive integer m, the matrix $\widetilde{G}(\alpha^m \lambda)$ is obtained from $\tilde{\mathbb{G}}(\lambda)$ by deleting all columns and rows indexed by $Q \backslash (2\pi\mathbb{Z}/\alpha^m)$.

Now, for any given finitely supported sequence $a \in \ell_2(Q)$ with support σ, there exists a non-negative integer M such that $\alpha^M \sigma \subset 2\pi\mathbb{Z}$. Since $\Psi[\lambda - q, \lambda - q'] = \Psi[\alpha^M(\lambda - q), \alpha^M(\lambda - q')]$ (for any $(q, q') \in 2\pi\mathbb{Z}/\alpha^M$ and a.e. λ), we have that

$$a^* \tilde{\mathbb{G}}(\lambda) a = \sum_{(q,q') \in Q^2} a(q)\overline{a}(q')\Psi[\lambda - q, \lambda - q'] = \sum_{(j,l) \in (2\pi\mathbb{Z})^2} b(j)\overline{b}(l)\Psi[\alpha^M\lambda - j, \alpha^M\lambda - l] = b^* \tilde{G}(\alpha^M \lambda) b,$$

where $b : 2\pi\mathbb{Z} \to \mathbb{C},\ j \mapsto a(\alpha^{-M} j)$, and, therefore, $\|a\|_{\ell_2(Q)} = \|b\|_{\ell_2(2\pi\mathbb{Z})}$.

Using the above, it is quite straightforward to infer from Result 3.1 the following fiberization result:

$^{\aleph}$ The proof of this result in [RS97b] imposes a mild smoothness condition on the mother wavelet set Ψ. This condition was removed in [CSW].

Corollary 3.2. *Let $X = X(\Psi, \alpha)$ be an affine system, and let $\mathcal{G}^*$ and $\mathcal{G}^{*-}$ be the norm functions defined as above. Then:*

(a) The following conditions are equivalent:
(a1) X is a Bessel system;
(a2) $\mathcal{G}^* \in L_\infty(\mathbb{R})$;
Furthermore, $\|\mathcal{G}^*\|_\infty$ is then the Bessel bound of X.

(b) Assume X to be a Bessel system. Then the following conditions are equivalent:
(b1) X is an L_2-frame;
(b2) $\mathcal{G}^{*-} \in L_\infty(\mathbb{R})$;
Furthermore, $\|\mathcal{G}^{*-}\|_\infty^{-1}$ is the lower frame bound of X.

We also define, for each $m \in \mathbb{Z}$ and each $\lambda \in \mathbb{R}$, the following augmented non-negative self-adjoint matrix:

$$\tilde{\mathbb{G}}_m(\lambda) := (\Psi_m[\lambda - q, \lambda - q'])_{(q,q')\in Q^2}\,, \quad \Psi_m[\lambda, \lambda'] := \sum_{\psi\in\Psi} \overline{\widehat{\psi}}(\alpha^m\lambda)\widehat{\psi}(\alpha^m\lambda')\chi(\lambda, \lambda', m),$$

where

$$\chi(\lambda, \lambda', m) := \begin{cases} 1, & \text{if } \alpha^m(\lambda - \lambda') \in 2\pi\mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Then for any finitely supported sequence $a \in \ell_2(Q)$ and a.e. $\lambda \in \mathbb{R}$,

$$\sum_{m\in\mathbb{Z}} a^*\tilde{\mathbb{G}}_m(\lambda)a = a^*\tilde{\mathbb{G}}(\lambda)a \le \|a\|_{\ell_2}^2\|\mathcal{G}^*\|_\infty.$$

In particular, for any given $m \in \mathbb{Z}$, and any finite $I \subset \mathbb{Z}$, $a^*(\sum_{m\in I}\tilde{\mathbb{G}}_m(\lambda))a \le \|a\|_{\ell_2}^2\|\mathcal{G}^*\|_\infty$. Hence, we have, for that same I,

$$\|\sum_{m\in I}\tilde{\mathbb{G}}_m(\lambda)\| \le \|\mathcal{G}^*\|_\infty. \tag{3.4}$$

Since, for a.e. $\lambda \in \mathbb{R}$ and each $m \in \mathbb{Z}$, $\widetilde{\mathbb{G}}_m(\lambda)$ is a non-negative self-adjoint matrix, the proof of Lemma 2.3, when combined with (3.4), yields the following observation:

Lemma 3.5. *If $X := X(\Psi, \alpha)$ is an affine Bessel system, then for each $a \in \ell_2(Q)$ and a.e. $\lambda \in \mathbb{R}$,*

$$\sum_{m\in\mathbb{Z}} a^*\tilde{\mathbb{G}}_m(\lambda)a = a^*\tilde{\mathbb{G}}(\lambda)a.$$

Also, we have the following analogue of Corollary 2.4:

Corollary 3.6. *If $X := X(\Psi, \alpha)$ is an affine Bessel system, then the functions*

$$\sum_{m=\kappa(t)}^{\infty}\sum_{\psi\in\Psi}\left|\widehat{\psi}(\alpha^m\cdot)\overline{\widehat{\psi}}(\alpha^m(\cdot + t))\right|, \quad t \in Q, \tag{3.7}$$

lie, each, in $L_\infty(\mathbb{R})$, and form a bounded set there.

3.2. Characterizations of AP functions with the aid of affine systems

Proof of Theorem 1.13. First, we prove the "only if" implication in (a). The proof provides all the necessary details for the proof of (b), too. To begin with, we will establish the AP-frame inequalities for a trigonometric polynomial $f = \sum_\sigma a(\lambda)e_\lambda$ with $\widehat{f}(0) = 0$, where σ is a finite subset of $\mathbb{R}$. Since $\widehat{f}(0) = 0$, we know that $0 \notin \sigma$. First, since $\Psi \subset L_1(\mathbb{R})$, for each $m, n \in \mathbb{Z}$ and each $\psi \in \Psi$,

$$\langle f, \psi_{m,n}\rangle = \sum_{\lambda\in\sigma} a(\lambda)\overline{\widehat{\psi}}(\alpha^m\lambda)e^{i\alpha^m n\lambda}$$

is finite, hence

$$|\langle f, \psi_{m,n}\rangle|^2 = \sum_{(\lambda,\lambda')\in\sigma^2} a(\lambda)\overline{a}(\lambda')\overline{\widehat{\psi}}(\alpha^m\lambda)\widehat{\psi}(\alpha^m\lambda')e^{i\alpha^m n(\lambda-\lambda')}.$$

Since σ is finite, we have that

$$\begin{aligned} F(f,\Psi) &= \sum_{m\in\mathbb{Z}} \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} \sum_{\psi\in\Psi} |\langle f, \psi_{m,n}\rangle|^2 \\ &= \sum_{m\in\mathbb{Z}} \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} \sum_{\psi\in\Psi} \sum_{(\lambda,\lambda')\in\sigma^2} a(\lambda)\overline{a}(\lambda')\overline{\widehat{\psi}}(\alpha^m\lambda)\widehat{\psi}(\alpha^m\lambda')e^{i\alpha^m n(\lambda-\lambda')} \\ &= \sum_{m\in\mathbb{Z}} \sum_{(\lambda,\lambda')\in\sigma^2} a(\lambda)\overline{a}(\lambda') \sum_{\psi\in\Psi} \overline{\widehat{\psi}}(\alpha^m\lambda)\widehat{\psi}(\alpha^m\lambda') \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} e^{i\alpha^m n(\lambda-\lambda')}. \end{aligned}$$

On the other hand,

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} e^{i\alpha^m n(\lambda-\lambda')} = \chi(\lambda,\lambda',m) := \begin{cases} 1, & \text{if } \alpha^m(\lambda-\lambda') \in 2\pi\mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

That is,

$$F(f,\Psi) = \sum_{m\in\mathbb{Z}} \sum_{(\lambda,\lambda')\in\sigma^2} a(\lambda)\overline{a}(\lambda') \sum_{\psi\in\Psi} \overline{\widehat{\psi}}(\alpha^m\lambda)\widehat{\psi}(\alpha^m\lambda')\chi(\lambda,\lambda',m).$$

By assumption (1.14) of the present theorem,

$$\sum_{m=\kappa(\gamma)}^{\infty} \sum_{\psi\in\Psi} \widehat{\psi}(\alpha^m\cdot)\overline{\widehat{\psi}}(\alpha^m(\cdot+\gamma)) \tag{3.8}$$

converges everywhere for each $\gamma \in Q$. This justifies the change in the summation order in the first equality of the following derivation:

$$\begin{aligned} &\sum_{m\in\mathbb{Z}} \sum_{(\lambda,\lambda')\in\sigma^2} a(\lambda)\overline{a}(\lambda') \sum_{\psi\in\Psi} \overline{\widehat{\psi}}(\alpha^m\lambda)\widehat{\psi}(\alpha^m\lambda')\chi(\lambda,\lambda',m) \\ &= \sum_{(\lambda,\lambda')\in\sigma^2} a(\lambda)\overline{a}(\lambda') \sum_{m\geq\kappa(\lambda-\lambda')} \sum_{\psi\in\Psi} \overline{\widehat{\psi}}(\alpha^m\lambda)\widehat{\psi}(\alpha^m\lambda') \\ &= \sum_{(\lambda,\lambda')\in\sigma^2} a(\lambda)\overline{a}(\lambda')\Psi[\lambda,\lambda']. \end{aligned}$$

Consequently,

$$F(f,\Psi) = \sum_{(\lambda,\lambda')\in\sigma^2} a(\lambda)\overline{a}(\lambda')\Psi[\lambda,\lambda'].$$

Now, we define an equivalence relation $\sim$ on σ by $\lambda \sim \lambda' \iff \lambda - \lambda' \in Q$. Let $\Lambda \subset \sigma$ be a set of representers of the equivalence classes. For $\lambda \in \Lambda$, denote by $\sigma_\lambda \subset \sigma$ the corresponding equivalence class. Since $\Psi[\lambda_1, \lambda_2] = 0$ unless $\lambda_1 \sim \lambda_2$,

$$F(f, \Psi) = \sum_{(\lambda,\lambda')\in\sigma^2} a(\lambda)\overline{a}(\lambda')\Psi[\lambda, \lambda'] = \sum_{\lambda\in\Lambda} \sum_{(\lambda_1,\lambda_2)\in\sigma_\lambda^2} a(\lambda_1)\overline{a}(\lambda_2)\Psi[\lambda_1, \lambda_2] = \sum_{\lambda\in\Lambda} a_\lambda^* \tilde{\mathbb{G}}(\lambda) a_\lambda,$$

where $a_\lambda(\lambda') := \begin{cases} a(\lambda'), & \text{if } \lambda' \in \sigma_\lambda, \\ 0, & \text{otherwise.} \end{cases}$

By assumption (1.14), for each $\lambda \in \Lambda$, each entry of $\tilde{\mathbb{G}}(\lambda)$ is continuous unless $\lambda - p = 0$ for some $p \in Q$. If $\lambda - p = 0$, then, since $a(0) = 0$, we know that $a_\lambda(p) = 0$. In this case, we define $\tilde{\mathbb{G}}(\lambda)'$ to be the submatrix of $\tilde{\mathbb{G}}(\lambda)$ which is obtained by deleting the $\lambda - p$ row and column from $\tilde{\mathbb{G}}(\lambda)$. That makes each entry of $\tilde{\mathbb{G}}(\lambda)'$ continuous. Also, we define a'_λ by $a'_\lambda : Q\backslash\{p\} \to \mathbb{C},\ q \mapsto a_\lambda(q)$. Then

$$a_\lambda^* \tilde{\mathbb{G}}(\lambda) a_\lambda = (a'_\lambda)^* \tilde{\mathbb{G}}(\lambda)' a'_\lambda.$$

Using the above argument, without loss of generality, we can assume that for each $\lambda \in \Lambda$, each entry of $\tilde{\mathbb{G}}(\lambda)$ is continuous. Then, by the same reasonings as in (2.8) and (2.9), for each $\lambda \in \Lambda$ and for any $a_\lambda \in \ell_2(Q)$, we have

$$\frac{\|a_\lambda\|_{\ell_2}^2}{\|\boldsymbol{\mathcal{G}}^{*-}\|_\infty} \le a_\lambda^* \tilde{\mathbb{G}}(\lambda) a_\lambda \le \|\boldsymbol{\mathcal{G}}^*\|_\infty \|a_\lambda\|_{\ell_2}^2. \tag{3.9}$$

Therefore, for a trigonometric polynomial $f \in AP$ with $\widehat{f}(0) = 0$, we have the requisite inequalities:

$$\frac{\|f\|_{AP}^2}{\|\boldsymbol{\mathcal{G}}^{*-}\|_\infty} \le F(f, \Psi) \le \|\boldsymbol{\mathcal{G}}^*\|_\infty \|f\|_{AP}^2. \tag{3.10}$$

Next, let f be a general almost periodic function with $\widehat{f}(0) = 0$, and let $\sigma = \{\lambda_1, \lambda_2, \cdots\}$ be the norm spectrum of f. Then as in the WH case, there is a trigonometric polynomial sequence $\{f_s\}_{s=1}^\infty$ which converges uniformly to f. Similarly to (2.11), we have

$$F(f, \Psi) = \sum_{m\in\mathbb{Z}} \lim_{s\to\infty} \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} \sum_{\psi\in\Psi} |\langle f_s, \psi_{m,n}\rangle|^2. \tag{3.11}$$

Again, by the Fatou's lemma, $F(f, \Psi)$ is bounded above by $\|\mathcal{G}^*\|_\infty^2 \|f\|_{AP}^2$.

The proof of the lower bound of $F(f, \Psi)$ is also similar to the WH case, hence is only sketched. With the equivalence relation $\sim$, and the set of representers Λ as before, we define, for each $s \in \mathbb{N}$,

$$\Lambda_s := \{\lambda \in \Lambda : \lambda - Q \cap \sigma^s \ne \emptyset\}.$$

Then,

$$\begin{aligned} F(f, \Psi) &= \sum_{m\in\mathbb{Z}} \lim_{s\to\infty} \sum_{\lambda\in\Lambda_s} \sum_{(\lambda_1,\lambda_2)\in\sigma_\lambda^2} a^s(\lambda_1)\overline{a^s}(\lambda_2)\Psi_m[\lambda_1, \lambda_2] \\ &= \sum_{m\in\mathbb{Z}} \lim_{s\to\infty} \sum_{\lambda\in\Lambda_s} (a_\lambda^s)^* \tilde{\mathbb{G}}_m(\lambda) a_\lambda^s, \end{aligned}$$

where $a_\lambda^s : Q \to \mathbb{C}, q \mapsto \begin{cases} a^s(\lambda - q), & \text{if } \lambda - q \in \sigma_\lambda^s, \\ 0, & \text{otherwise.} \end{cases}$ Also, $\lim_{s\to\infty} a_\lambda^s = a_\lambda := \begin{cases} a(\lambda - q), & \text{if } \lambda - q \in \sigma_\lambda, \\ 0, & \text{otherwise.} \end{cases}$

Thus, using (3.9) and Lemma 3.5, and invoking a dominated convergence argument, we obtain

$$F(f, \Psi) = \sum_{\lambda\in\Lambda} \sum_{m\in\mathbb{Z}} a_\lambda^* \tilde{\mathbb{G}}_m(\lambda) a_\lambda = \sum_{\lambda\in\Lambda} a_\lambda^* \tilde{\mathbb{G}}(\lambda) a_\lambda \ge \frac{\sum_{\lambda\in\Lambda} \|a_\lambda\|_{\ell_2}^2}{\|\boldsymbol{\mathcal{G}}^{*-}\|_\infty} = \frac{\|f\|_{AP}^2}{\|\boldsymbol{\mathcal{G}}^{*-}\|_\infty}. \tag{3.12}$$

Therefore, for any $f \in AP$ with $\widehat{f}(0) = 0$,

$$\frac{1}{\|\boldsymbol{\mathcal{G}}^{*-}\|_\infty} \|f\|_{AP}^2 \le F(f, \Psi) \le \|\boldsymbol{\mathcal{G}}^*\|_\infty \|f\|_{AP}^2.$$

The remaining parts of the proof here follow almost verbatim the reasonings of the WH case, hence are omitted. (Note that for the "if" implication in (a) and (b), it is enough to show that (3.9) is valid a.e. $\lambda \in \mathbb{R}$.) $\square$

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