A sharp upper bound on the approximation order of smooth bivariate pp functions

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Introduction

It is the purpose of this note to show that the approximation order from the space

$$\Pi^{\rho}_{k,\Delta}$$

of all piecewise polynomial functions in C^{ρ} of polynomial degree $\le k$ on a triangulation Δ of $\mathbb{R}^2$ is, in general, no better than k in case $k < 3\rho + 2$. This complements the result of [BH88] that the approximation order from $\Pi^{\rho}_{k,\Delta}$ for an arbitrary mesh Δ is $k+1$ if $k \ge 3\rho + 2$.

Here, we define the **approximation order** of a space S of functions on $\mathbb{R}^2$ to be the largest real number r for which

$$\operatorname{dist}(f, \sigma_h S) \le \mathrm{const}_f h^r$$

for any sufficiently smooth function f, with the distance measured in the L_p-norm ($1 \le p \le \infty$) on $\mathbb{R}^2$ (or some suitable subset G of $\mathbb{R}^2$), and with the **scaling map** σ_h defined by

$$\sigma_h f := f(\cdot/h).$$

In particular, the approximation order from $\Pi^{\rho}_{k,\Delta}$ cannot be better than $k+1$ regardless of ρ and is trivially $k+1$ in case $\rho = -1$ or 0. Thus, an upper bound of k is an indication of the price being paid for having ρ much larger than 0.

It turns out that the upper bound to be proven here already holds when Δ is a very simple triangulation, viz. the **three-direction mesh**, i.e., the mesh

$$\Delta := \bigcup_{i=1}^{3} \mathbb{R}e_i + \mathbb{Z}^2$$

with

$$e_1 := (1,0), e_2 := (0,1), e_3 := (1,1) = e_1 + e_2.$$

A first result along these lines was given in [BH83$_1$] where it was shown that the approximation order of $\Pi^1_{3,\Delta}$ (with Δ the three-direction mesh) is only 3, which was surprising in view of the fact that all cubic polynomials are contained locally in this space. [J83] showed the corresponding result for C^1-quartics on the three-direction mesh and [BH83$_2$] provided upper and lower bounds for the approximation order of

$$S := \Pi^{\rho}_{k,\Delta}$$

for arbitrary k and ρ.

For $2k - 3\rho \leq 7$, the approximation order of S was completely determined in [J86]. Since it is easy to determine the approximation order of any space spanned by the translates of one box spline ([BH82/3]) with the aid of quasi-interpolants, it is tempting to consider, more generally, **local** approximations from S, i.e., approximations to the given f which are linear combinations of box splines in S, with the restriction that the coefficient of any particular box spline should depend only on the behavior of f near the support of that box spline. The resulting approximation order has been termed the **local approximation order** of S in [BJ]. The local approximation order of S was entirely determined in [J88]. In particular, it is shown there that the local approximation order of S can never be full, i.e., equal $k+1$. It is also conjectured there that the local approximation order equals the approximation order when $k < 3\rho + 2$. In addition, it is shown in [J88] that the approximation order of S is at least k when $k \geq 2\rho + 2$. This, together with the result to be proved here and the result from [J86], gives the precise approximation order for S for $\rho \leq 5$ and all k. Finally, the fact that the approximation order from S is only k when $k = 3\rho + 1$ was demonstrated in [BH88] for $\rho = 1, 2, 3$.

In all of these references cited, only the approximation order with respect to the max-norm was considered.

In addition to the notation already defined in the course of the above introduction, we also use the following: We denote by

$$\Pi_k \qquad (\Pi_{<k})$$

the collection of all polynomials of total degree $\leq k$ ($< k$). We denote by

$$\langle y, \cdot \rangle$$

the linear polynomial whose value at $x \in \mathbb{R}^2$ is the scalar product $\langle y, x \rangle$ of y with x. We write

$$D_y := y(1) D_1 + y(2) D_2$$

for the (unnormalized) directional derivative in the direction y, with D_i the partial derivative with respect to the ith argument, $i = 1, 2$. Thus,

$$D_i = D_{e_i},$$

but we use this abbreviation also for $i = 3$, and use, correspondingly, the convenient abbreviation

$$D^a := \prod_{i=1}^{3} D_i^{a(i)},$$

with $a \in \mathbb{Z}_+^3$. For such a, we write

$$|a| := \sum_i a(i).$$

Correspondingly, we write

$$\tau^a := \prod_{i=1}^{3} \tau_i^{a(i)} \quad \textit{and} \quad \nabla^a := \prod_{i=1}^{3} \nabla_i^{a(i)},$$

with

$$\tau_i f := f(\cdot + e_i) \quad \textit{and} \quad \nabla_i := 1 - {\tau_i}^{-1}.$$

Finally, we denote by $p(D) := \sum_\alpha c(\alpha) D^\alpha$ the constant coefficient differential operator associated with the polynomial $p = \sum_\alpha c(\alpha)()^\alpha$. For example,

$$D_i = \langle e_i, D\rangle.$$

Main Result

The main result of this note is the following

Theorem. *The approximation order of* $S := \Pi^\rho_{k,\Delta}$ *(in any* L_p, $1 \le p \le \infty$*) is at best* k *when* $k < 3\rho + 2$, $\rho > 0$ *and* Δ *is the three-direction mesh.*

In this section, we outline the proof, leaving the verification of certain technical Lemmata to a subsequent section.

The proof uses the same ideas with which the special cases $\rho = 1$ and 2 were handled in [BH83$_1$], [J83], and [BH88], respectively, i.e., the construction of a local linear functional which vanishes on $\Pi^\rho_{k,\Delta}$ but does not vanish on some homogeneous polynomial of degree $k+1$ and whose integer translates add up to the zero linear functional. But the construction of the specific linear functional follows the rather different lines of [J86].

To begin with, recall from [BH83$_2$] that the approximation order of S equals that of

$$S_{\text{loc}} := \text{span}\{M_{r,s,t}(\cdot - j) : j \in \mathbb{Z}^2, M_{r,s,t} \in S\}.$$

(To be precise, the proof of Proposition 3.1 in [BH83$_2$] can be modified to show that if r is an upper bound on the approximation order of S_{loc}, then it is also an upper bound on the approximation order of S, while the converse is trivial since $S_{\text{loc}} \subseteq S$.) Here, $M_{r,s,t}$ is the box spline $M(\cdot, \Xi)$, i.e., the distribution $f \mapsto \int_{[0..1)^{r+s+t}} f(\Xi t)\,dt$ (cf., e.g., [BH82/3]), with direction matrix

$$\Xi := [\underbrace{e_1, \ldots, e_1}_{r \text{ times}}, \underbrace{e_2, \ldots, e_2}_{s \text{ times}}, \underbrace{e_3, \ldots, e_3}_{t \text{ times}}].$$

Further, the linear functional will be constructed from linear functionals of the form $f \mapsto \int_T p(D) f$, with

$$T := \{x \in \mathbb{R}^2 : 0 < x(2) < x(1) < 1\}$$

a triangle in the three-direction mesh Δ, and with p a homogeneous polynomial of degree k. Such functionals vanish on $\Pi_{<k}$, hence also vanish on any $M_{r,s,t}$ with $r+s+t-2 < k$. It is proved in [BH83$_2$] that, for $k > 2\rho + 1$, S_{loc} is spanned by the integer translates of the box splines of degree $< k$ in S and the box splines M_α with α in

$$A := A_1 \cup A_2 \cup A_3,$$

where

$$A_1 := \{(k-\rho+1-i, 0, \rho+1+i) : i = 1, \ldots, k-2\rho-1\},$$
$$A_2 := \{(\rho+2-i, i, k-\rho) : i = 1, \ldots, \rho+1\},$$
$$A_3 := \{(0, \rho+1+i, k-\rho+1-i) : i = 1, \ldots, k-2\rho-1\}.$$

(These are exactly the box splines whose restriction to the line $e_1 + \mathbb{R}(e_2 - e_1)$ coincide there with a(n appropriately scaled univariate) B-spline of degree k for the knot sequence in which each of $0, 1/2, 1$ occurs exactly $k-\rho$ times.) This implies that it is sufficient to require our linear functional λ to vanish on $M_\alpha(\cdot - j)$ for $\alpha \in A$ and $j \in \mathbb{Z}^2$ in order to ensure that $\lambda \perp S_{\mathrm{loc}}$.

(1)Lemma. *For $\beta := (1,1,0)$, there exists a set B of $\rho+1$ homogeneous polynomials of degree k so that, on $T + \mathbb{Z}^2$,*

$$p(D)M_\alpha = c_{p,\alpha} \nabla^{\alpha-\beta} M_\beta, \quad p \in B,\ \alpha \in A, \tag{2}$$

with the constants $c_{p,\alpha}$ satisfying

$$c_{p,\alpha} = 0, \qquad \alpha \in A_3.$$

Here and below, we follow the convenient convention that $\nabla^\gamma = 0$ if $\gamma(i) < 0$ for some i.

(3)Lemma. *For $\gamma := (1,0,1)$, there exists a set C of $\rho+1$ homogeneous polynomials of degree k so that, on $T + \mathbb{Z}^2$,*

$$p(D)M_\alpha = c_{p,\alpha} \nabla^{\alpha-\gamma} M_\gamma, \quad p \in C,\ \alpha \in A, \tag{4}$$

with the constants $c_{p,\alpha}$ satisfying

$$c_{p,\alpha} = 0, \qquad \alpha \in A_3.$$

Now note that M_β and M_γ agree on all of $T + \mathbb{Z}^2$ with the characteristic function

$$\chi_T$$

of the triangle T. Thus,

$$p(D)M_\alpha = c_{p,\alpha} \begin{Bmatrix} \nabla^{\alpha-\beta} \\ \nabla^{\alpha-\gamma} \end{Bmatrix} \chi_T \quad \text{on } T + \mathbb{Z}^2, \text{ for } p \in \begin{cases} B \\ C \end{cases}.$$

Further,

$$\nabla_2 \nabla^{\alpha-\beta} = \nabla_2 \nabla_3 \nabla^{\alpha-(1,1,1)} = \nabla_3 \nabla^{\alpha-\gamma}.$$

Thus, if

$$\sum_{p \in B \cup C} w(p) c_{p,\alpha} = 0 \quad \text{for all } \alpha \in A_1 \cup A_2, \tag{5}$$

then

$$JM_\alpha = 0 \quad \text{on } T + \mathbb{Z}^2 \text{ for all } \alpha \in A,$$

with

$$(6) \qquad J := \sum_{p\in B} w(p)\nabla_2 p(D) + \sum_{p\in C} w(p)\nabla_3 p(D)$$

(since $c_{p,\alpha} = 0$ for $p \in B \cup C$ and $\alpha \in A_3$). Here, we may (and do) choose $w \neq 0$, since $\#(B \cup C) = 2\rho + 2 > k - \rho = \#(A_1 \cup A_2)$.

Next, we construct some $g \in \Pi_{k+1}$ for which $Jg = 2$. For this, note that $p(D)\Pi_{k+1} \subset \Pi_1$ for any $p \in B \cup C$, while $\nabla_i = D_i$ on Π_1. This implies that

$$J = \sum_{p\in B\cup C} w(p)\tilde{p}(D) \qquad \textit{on } \Pi_{k+1},$$

with

$$\tilde{p} := p \begin{cases} \langle e_2, \cdot\rangle, & p \in B; \\ \langle e_3, \cdot\rangle, & p \in C. \end{cases}$$

(7)Lemma. *If $k > 2\rho + 1$, then the sets B and C in (1) and (3) can be so chosen that $\{\tilde{p} : p \in B \cup C\}$ is a linearly independent subset of Π_{k+1}.*

To make use of this lemma, we need to restrict attention to the case $k > 2\rho+1$. We do this by, possibly, *decreasing* ρ (and, hence increasing S) to force the inequality $k > 2\rho + 1$. Of course, we must make sure that we still have $k < 3\rho+2$. Assuming that ρ' is the largest integer for which $k > 2\rho' + 1$, we have $k \leq 2\rho' + 3 < 3\rho' + 2$ except, possibly, when $\rho' \leq 1$, hence $k \leq 5$. But, for $k \leq 5$ and $\rho \geq 1$, the approximation order of S is known ([J86], [BH88]) to satisfy our theorem's claim.

Thus, for $k > 5$, we may assume without loss of generality that $k > 2\rho + 1$, hence use the lemma to conclude, from the fact that $w \neq 0$, that $J = q(D)$ on Π_{k+1} for some *nontrivial* homogeneous polynomial q of degree $k+1$. This implies that J maps Π_{k+1} *onto* Π_0, hence $Jg = 2$ for some $g \in \Pi_{k+1}$.

Since $JM_\alpha = 0$ on $T+\mathbb{Z}^2$, and J commutes with any integer shift, it follows that the linear functional

$$\lambda : f \mapsto \int_T Jf$$

vanishes on S_{loc}, but takes the value 1 on that particular polynomial g. Further, λ has the form

$$\lambda = \lambda_2\nabla_2 + \lambda_3\nabla_3$$

with

$$\lambda_i : f \mapsto \int_T p_i(D)f,$$

for some homogeneous polynomials p_i of degree k. This shows that

$$\sum_{j\in\mathbb{Z}^2} \lambda\tau^j = 0,$$

in the sense that, for any compact set, there is some n_0 so that any sum $\sum_{j\in\mathbb{Z}^2\cap[-n..n]^2}\lambda\tau^j$ with $n>n_0$ has no support in that compact set.

We make use of λ in the following more precise fashion. Define

$$H_{i,n} := \sum_{j=1}^{n} \tau_i^j.$$

Then $H_{i,n}\nabla_i = \tau_i^n - 1$. Therefore,

$$\lambda^{(n)} := \lambda \sum_{j\in\mathbb{Z}^3\cap[1..n]^3} \tau^j = \lambda_2(\tau_2^n-1)H_{1,n}H_{3,n} + \lambda_3(\tau_3^n-1)H_{1,n}H_{2,n}$$

has support only in

$$T_n := T + \sum_{j\in\mathbb{Z}^3\cap[0..n]^3} \sum_i j(i)e_i =: T + I,$$

and is, more explicitly, of the form

$$f \mapsto \sum_{j\in I} \int_{T+j} (b(j)p_2(D) + c(j)p_3(D))f,$$

with $b(j), c(j) \in \{-1,0,1\}$ for all j. (Put differently, the mesh functions b and c are first differences of the discrete box spline associated with the three directions e_1, e_2, e_3, hence are piecewise constant.) Since $\tau^j g \in g + \Pi_k$ and $\lambda^{(n)}$ vanishes on Π_k, this implies that $\lambda^{(n)}g = n^3$. Further, as a functional on, say, $\Pi^0_{k+1,\Delta} \subset L_1([-1..2n+1]^2)$, $\lambda^{(n)}$ has norm

$$\|\lambda^{(n)}\| \le \mathrm{const}_k,$$

since, on each $T+j$, any f of interest (i.e., any $f \in S + \operatorname{span} g$) reduces to a polynomial of degree $\le k+1$, hence

$$|\int_{T+j} p_i(D)f| \le \mathrm{const}_k \int_{T+j} |f|$$

with const_k derived from Markov's inequality.

Let now $h := 1/n$ and set $\sigma : f \mapsto f(\cdot/h)$. We are interested in a lower bound for the $L_p(G)$-distance of g from $S_h := \sigma S$. Since $\|f\|_1(G') \le \mathrm{const}_{G'}\|f\|_p(G') \le \mathrm{const}_{G'}\|f\|_p(G)$ for any bounded subset G' of G, it is sufficient to restrict attention to $p=1$ and bounded G. Moreover, after a translation and a scaling, we may assume that the domain G of interest contains $[-h..(2n+1)h]^2$. Then $\|\lambda^{(n)}\sigma^{-1}\| \le \mathrm{const}_k h^{-2}$, and $\lambda^{(n)}\sigma^{-1} \perp S_h$, while $\lambda^{(n)}\sigma^{-1}g = \lambda^{(n)}g(\cdot h) = h^{k+1}\lambda^{(n)}g = h^{k-2}$. Consequently,

$$\operatorname{dist}_1(g, S_h) \ge \lambda^{(n)}\sigma^{-1}g/\|\lambda^{(n)}\sigma^{-1}\| \ge h^{k-2}/(\mathrm{const}_k h^{-2}) = \mathrm{const}\, h^k,$$

for some h-independent positive const. This finishes the proof of the theorem.

Proof of the technical lemmata

We take B and C from the set of polynomials

$$p_a := \prod_{i=1}^{3} \langle e_i, \cdot \rangle^{a(i)}$$

with $a \in \mathbb{Z}_+^3$, $|a| = k$.

For the computation of $p_a(D)M_\alpha$, we rely entirely on the differentiation formula [BH82/3]

$$D_\xi M(\cdot, \Xi) = \nabla_\xi M(\cdot, \Xi \backslash \xi)$$

valid for any particular direction ξ from the direction set Ξ for the box spline $M(\cdot, \Xi)$, and on the fact that the (closed) support of the box spline $M(\cdot, \Xi)$ is the set

$$\sum_{\xi \in \Xi} [0..1]\xi.$$

We choose B to consist of the $\rho+1$ polynomials p_a with $a(3) = k-\rho$. Then $a(3) \geq \alpha(3)$ for any $\alpha \in A$, hence

$$(8) \qquad p_a(D)M_\alpha = \nabla_3^{\alpha(3)} p_{a(1),a(2),a(3)-\alpha(3)}(D) M_{\alpha(1),\alpha(2),0}.$$

Since $\alpha(2) = 0$ for $\alpha \in A_1$ and $\alpha(1) = 0$ for $\alpha \in A_3$, this shows that $p_a(D)M_\alpha$ has no support in $T + \mathbb{Z}^2$ when $\alpha \in A_1 \cup A_3$, hence (2) holds for this case with $c_{p,\alpha} = 0$. For the remaining case, $\alpha \in A_2$, we have $\alpha(3) = k - \rho = a(3)$, and therefore, more explicitly than (8),

$$p_a(D)M_\alpha = \nabla_3^{\alpha(3)} D_1^{a(1)} D_2^{a(2)} M_{\alpha(1),\alpha(2),0},$$

and this has support in $T + \mathbb{Z}^2$ if and only if $a(i) < \alpha(i)$ for $i = 1, 2$. Since $a(1) + a(2) = \alpha(1) + \alpha(2) - 2$, this condition is met if and only if $\alpha = a + \beta$ with $\beta = (1, 1, 0)$, and in that case we get

$$p_a(D)M_\alpha = \nabla^{\alpha-\beta} M_\beta.$$

This finishes the proof of (1)Lemma.

The verification of (3)Lemma proceeds analogously. We choose C to consist of the $\rho + 1$ polynomials p_a with $a(2) = k - \rho$. Then $a(2) \geq \alpha(2)$ for any $\alpha \in A$, hence

$$(9) \qquad p_a(D)M_\alpha = \nabla_2^{\alpha(2)} p_{a(1),a(2)-\alpha(2),a(3)}(D) M_{\alpha(1),0,\alpha(3)}.$$

Since $\alpha(1) = 0$ for $\alpha \in A_3$, this shows that $p_a(D)M_\alpha$ has no support in $T + \mathbb{Z}^2$ when $\alpha \in A_3$, hence (4) holds for this case with $c_{p,\alpha} = 0$. For the remaining case, i.e., for $\alpha \in A_1 \cup A_2$, we make use of the fact that $D_2 = D_3 - D_1$ to write (9) in the form

$$p_a(D)M_\alpha = \nabla_2^{\alpha(2)} \sum_j c_j D_1^{j(1)} D_3^{j(3)} M_{\alpha(1),0,\alpha(3)},$$

with the sum over all j of the form $(a(1)+r, 0, a(3)+t)$ with $r+t = a(2)-\alpha(2)$. Thus, $j(1)+j(3) = \alpha(1)+\alpha(3)-2$, hence the only terms with some support in $T+\mathbb{Z}^2$ are of the form $j(i) = \alpha(i)-1$ for $i = 1, 3$, and in that case,

$$D_1^{j(1)} D_3^{j(3)} M_{\alpha(1),0,\alpha(3)} = \nabla^{\alpha(1)-1,0,\alpha(3)-1} M_\gamma.$$

As to (7)Lemma, we note first that $\tilde{B} := \{\tilde{p} : p \in B\}$ is linearly independent since it consists of the sequence

$$\langle e_2, \cdot\rangle \langle e_3, \cdot\rangle^{k-\rho} \{\langle e_1, \cdot\rangle^j \langle e_2, \cdot\rangle^{\rho-j} : j = 0, \ldots, \rho\},$$

and e_1, e_2 form a basis for $\mathbb{R}^2$. Analogously, $\tilde{C} := \{\tilde{p} : p \in C\}$ is linearly independent since it consists of the sequence

$$\langle e_2, \cdot\rangle^{k-\rho} \langle e_3, \cdot\rangle \{\langle e_1, \cdot\rangle^j \langle e_3, \cdot\rangle^{\rho-j} : j = 0, \ldots, \rho\},$$

and e_1, e_3 form a basis for $\mathbb{R}^2$. Thus it is sufficient to prove that $\operatorname{span} \tilde{B}$ has only trivial intersection with $\operatorname{span} \tilde{C}$. But this follows from the facts (obtainable by substituting $e_3 - e_2$ for e_1 and collecting terms) that

$$\tilde{B} \subset \operatorname{span}\{\langle e_2, \cdot\rangle^{1+j} \langle e_3, \cdot\rangle^{k-j} : j = 0, \ldots, \rho\}$$

and

$$\tilde{C} \subset \operatorname{span}\{\langle e_2, \cdot\rangle^{k-j} \langle e_3, \cdot\rangle^{1+j} : j = 0, \ldots, \rho\},$$

since $k - \rho > \rho + 1$, by assumption.

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References

[BH82/3] C. de Boor and K. Höllig, B-splines from parallelepipeds, *J. d'Anal. Math.* **42** (1982/3), 99–115.

[BH83$_1$] C. de Boor and K. Höllig, Approximation order from bivariate C^1-cubics: A counterexample, *Proc. Amer. Math. Soc.* **87** (1983), 649–655.

[BH83$_2$] C. de Boor and K. Höllig, Bivariate box splines and smooth pp functions on a three-direction mesh, *J. Comput. Applied Math.* **9** (1983), 13–28.

[BH88] C. de Boor and K. Höllig, Approximation power of smooth bivariate pp functions, *Math. Z.* **197** (1988), 343–363.

[BJ] C. de Boor and R.Q. Jia, Controlled approximation and a characterization of local approximation order, *Proc.Amer.Math.Soc.* **95** (1985), 547-553.

[J83] R.Q. Jia, Approximation by smooth bivariate splines on a three-direction mesh, in *Approximation Theory IV*, C.K. Chui, L.L. Schumaker and J. Ward eds., Academic Press, New York, 1983, pp. 539–545.

[J86] R.Q. Jia, Approximation order from certain spaces of smooth bivariate splines on a three-direction mesh, *Trans. Amer.Math.Soc.* **295** (1986), 199–212.

[J88] R.Q. Jia, Local approximation order of box splines, *Scientia Sinica* **31** (1988), 274–285.