Affine systems in $L_2(\mathbb{R}^d)$: the analysis of the analysis operator

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ABSTRACT

Discrete affine systems are obtained by applying dilations to a given shift-invariant system. The complicated structure of the affine system is due, first and foremost, to the fact that it is not invariant under shifts. Affine frames carry the additional difficulty that they are "global" in nature: it is the entire interaction between the various dilation levels that determines whether the system is a frame, and not the behaviour of the system within one dilation level.

We completely unravel the structure of the affine system with the aid of two new notions: *the affine product*, and *a quasi-affine system*. This leads to a characterization of affine frames; the induced characterization of tight affine frames is in terms of exact orthogonality relations that the wavelets should satisfy on the Fourier domain. Several results, such as a general oversampling theorem follow from these characterizations.

Most importantly, the affine product can be factored during a multiresolution analysis construction, and this leads to a complete characterization of all tight frames that can be constructed by such methods. Moreover, this characterization suggests very simple sufficient conditions for constructing tight frames from multiresolution. Of particular importance are the facts that the underlying scaling function does not need to satisfy any a priori conditions, and that the freedom offered by redundancy can be fully exploited in these constructions.

AMS (MOS) Subject Classifications: Primary 42C15, Secondary 42C30

Key Words: affine systems, affine product, quasi-affine systems, frames, tight frames, multiresolution analysis, wavelets.

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1. Introduction

1.1. General

The present paper is the last in a series of three, all devoted to the study of *shift-invariant frames* and *shift-invariant stable (=Riesz) bases* for $L_2(\mathbb{R}^d)$, $d \geq 1$, or a subspace of it. In the first paper, [RS1], we studied such bases under the mere assumption that the basis set can be written as a collection of **shifts** (namely, integer translates) of a set of *generators* Φ. The second paper [RS2] analyses the Weyl-Heisenberg frames and Riesz bases. In the present paper, we study applications of the results of [RS1] to wavelet (or affine) frames. Wavelet systems are not shift-invariant, hence the basic analysis of [RS1] cannot be directly applied to this case.

Our original intent was to write a paper on affine Riesz bases and affine frames. The present paper, however, is devoted solely to fundamental affine frames. The primary reason is that the fiberization techniques of [RS1] allowed us to unravel completely the complicated structure of the analysis operator (or more precisely, of the so-called "frame operator") of an affine system, with less success with respect to the relevant synthesis operator. In fact, the current wavelet theory is (implicitly) centered around the synthesis operator, since, initially, the synthesis operator seems to be very attractive: its transformation to the frequency domain can be done by standard Fourier analysis methods, and this leads to a very simple structure when the system is orthonormal or semi-orthonormal. That, in our opinion, is deceptive: as soon as one attempts to study non-orthogonal systems, the painfully complicated structure of this operator emerges, a structure which is easy to reveal and hard to unravel. In addition, the operator does not interact well with multiresolution constructions, in the sense that its basic component, the bracket product, cannot be factored during the construction.

We believe that the study of the analysis operator in this paper results in the first complete systematic intrinsic analysis of affine systems, and, to explain this point of view, we briefly compare the typical results here to the present state-of-the-art in this field. Wavelet theory is currently dominated by the innovative idea of *multiresolution analysis* (=:MRA; cf. [Ma], [Me]). By all accounts, MRA constitutes a major breakthrough in the understanding of affine systems, and even more importantly, for the construction of such systems, some of which seem to be inaccessible without this machinery (with the primary example being the univariate construction of compactly supported orthonormal affine systems with arbitrary smoothness by Daubechies in [D2]). However, the current MRA theory suffers in several important aspects. Firstly, its main body consists of sufficient conditions for obtaining "good" systems, and not of characterizations of such systems. Furthermore, the typical assumptions begin with the imposition of stringent conditions on the refinable space. Added to that, the sufficient conditions are not given intrinsically in terms of the system, but rather, in terms of the algorithm used for its construction: Put it differently, "good" systems, constructed by "bad" methods, are unapproachable. Secondly, almost all existing

MRA results are about irredundant systems: not only that the additional freedom offered by redundancy have not been successfully exploited to date, but, due to their global nature and lack of biorthogonality relations, redundant systems remain, by and large, an unanswered challenge to multiresolution analysis.

In contrast with the above, the results of this paper center around a new *non-constructive intrinsic* analysis of affine systems. It is carried out in any spatial dimension d, for any *integer* dilation matrix s, and any number of wavelets. It results in complete characterizations of fundamental frames and fundamental tight frames together with formulae for the associated frame bounds. The characterizations, as well as the bound formulae, are given in terms of the norms and inverse-norms of a certain family of constant-coefficient non-negative definite self-adjoint infinite-order matrices, referred to hereafter as "fibers". These characterizations, in their essence, cannot distinguish between redundant and irredundant systems; however, other methods may then be employed to characterize irredundancy: in the case of tight frames / orthonormal systems the additional step is straightforward, and a complete characterization of orthonormal affine systems is therefore obtained.

While our theory does not assume and does not suggest any constructive way for obtaining the affine system, it reduces the analysis of systems constructed by multiresolution to simple arithmetic calculations: the main reason for that is that the basic component of the analysis operator, the newly defined *affine product*, can be factored during the MRA construction. The study of MRA constructions can then be carried out without any a priori restrictions on the spatial dimension, the dilation matrix, and/or the number of scaling functions. Furthermore, the scaling functions may or may not be "good" generators for V_0, the number of wavelets may be arbitrarily large (which means that sometimes redundancy is inevitable), and the mask functions are not a-priori restricted in any way (other than being measurable and appropriately periodic). In that generality, we provide a complete characterization of all fundamental tight frames that can be constructed by multiresolution. These characterizations lead to a very simple sufficient condition, given entirely in terms of mask functions, that guarantees the construction to yield a fundamental tight frame. The results here provide an clear evidence to the "power of redundancy": the simple sufficient condition is based on the ability to find a matrix whose first row is given, and whose columns are orthonormal; redundancy allows one to have more rows than columns in that matrix. As an illustration for that power, compactly supported tight affine frames generated by $2m$ univariate splines of order $2m$ are constructed.

In addition, several (seemingly unrelated and none related to MRA) observations now in the literature may be explained and thereby generalized, with the aid of the results here. To mention few examples, Daubechies-Tchamitchian's upper frame bound estimate, [D1], is closely related to the bounding the ℓ_2-norm of a self-adjoint matrix by its ℓ_1-norm, while their lower frame bound estimate corresponds to inverse-norm estimates of a diagonally dominant matrix. Daubechies' and Chui-Shi's bounds in terms of a "Littlewood-Paley type expression" (see [D3], [CS2] and [CS4]) can now be understood as an attempt to estimate the norm and inverse-norm of a Hermitian matrix in terms of its diagonal entries, while Chui-Shi oversampling results [CS1], [CS3] and [CS4], follow at once by observing that the fibers associated with the oversampling system are, up to a normalization factor, submatrices of those associated with the original system.

Our theory also sheds new light on affine systems generated by *band-limited* wavelets. For example, any system whose wavelets satisfy the band assumption used in [DGM] has diagonal fiber matrices. Similarly, any system whose wavelets have the same band as Meyer's wavelets, have block-diagonal fiber matrices, with 2×2 blocks.

During the revision of this paper, we added several additional references to the reference list. First, we became aware of the paper [H] (that was submitted before ours) where a characterization of tight frames similar to Corollary 5.7 is proved. Second, various applications of the theory developed here can be found in [RS3-5] and [GR]. Specifically, multivariate compactly supported tight spline frames are constructed in [RS4], while [GR] proves the existence, for any given dilation matrix, of compactly supported tight affine systems of arbitrarily high smoothness.

1.2. Univariate dyadic systems

We illustrate some of the main observations made in the paper by discussing them in a particularly simple setup, when the spatial dimension d is 1 (i.e., we decompose $L_2(\mathbb{R})$), and the dilations are dyadic. We assume here basic familiarity with wavelet theory, and defer various definitions to the main body of the article.

An *affine system* $X \subset L_2$ is a collection of functions of the form

$$X = \bigcup_{k \in \mathbb{Z}} D^k E(\Psi),$$

where $\Psi \subset L_2(\mathbb{R})$ is finite, $E(\Psi) = \cup_{\psi \in \Psi} E(\psi)$ is the collection of *shifts* i.e., integer translates, of Ψ, and D is the dyadic dilation operator $D : f \mapsto \sqrt{2}\, f(2\cdot)$. The functions in Ψ are the *generators* of X, usually referred to as *(mother) wavelets*. The *analysis operator* T^* is the map

$$T^* : L_2 \to \ell_2(X) : f \mapsto \{\langle f, x \rangle\}_{x \in X}.$$

The system X is a *fundamental frame* if T^* is well-defined, bounded and bounded below. A fundamental frame is *tight* if, up to a scalar multiple, T^* is unitary. The *frame bounds* are the numbers $\|T^*\|^2$, and $1/\|T^{*-1}\|^2$.

We introduce in this paper, and extensively use, the following *affine product*:

$$\Psi[\omega, \omega'] := \sum_{\psi \in \Psi} \sum_{k=\kappa(\omega-\omega')}^{\infty} \widehat{\psi}(2^k \omega) \overline{\widehat{\psi}}(2^k \omega'), \quad \omega, \omega' \in \mathbb{R},$$

where κ is the dyadic valuation:

$$\kappa : \mathbb{R} \to \mathbb{Z} : \omega \mapsto \inf\{k \in \mathbb{Z} : 2^k \omega \in 2\pi\mathbb{Z}\}.$$

(Thus, $\kappa(0) = -\infty$, and $\kappa(\omega) = \infty$ unless ω is 2π-dyadic.) Our convention is that $\Psi[\omega, \omega'] := \infty$ unless we have absolute convergence in the corresponding sum. Throughout the introduction, we always assume that

$$|\widehat{\psi}(\omega)| = O(|\omega|^{-1/2-\delta}), \quad \text{near } \infty, \quad \text{for some } \ \delta > 0,$$

for every wavelet $\psi \in \Psi$. The assumption is so mild (even the Haar function satisfies it!) that we forgo mentioning it in the formal statements of this section. Finally, we set, for $r \geq 0$,

$$H_r := \{f \in L_2 : \ |\operatorname{supp} \widehat{f} \cap [-r, r]| = 0\}.$$

Since the system X is not shift-invariant, and since our fiberization techniques from [RS1] assume this shift-invariance at their outset, we analyse X by associating it with two different shift-invariant systems. The first, and simpler one, is the *truncated affine system* X_0, obtained by simply removing from X the non-shift-invariant part, i.e., the part generated by negative dilations. The truncated system X_0 is primarily useful for the analysis of Riesz basis systems (the case when T^* is surjective): this property cannot be lost while passing to a subsystem, and, in fact, the converse is also true.

It is harder to study redundant fundamental frames (i.e., fundamental frames that are not Riesz bases) with the aid of truncation, and the reason is essential: frames cannot be "locally analysed", meaning that X can be a frame while a subset $Y \subset X$ may not be a frame (for the closed subspace of L_2 that it spans); thus, one is not likely to be able to analyse "frame properties" of X, by analysing analogous properties of subsets of X. This also may explain the fact that, to date, the literature on multiresolution constructions of affine systems (which are very "local" methods in the above sense) contains a wealth of results about orthonormal affine systems, as well as many results on Riesz basis systems, and only a handful, specific, results on frame constructions.

Partial success in connecting between the analysis operators of X and X_0 is obtained upon restricting the latter one to spaces of the form H_r, $r \to \infty$. Our study of that limit process, which is detailed in the paper, reveals a fundamental connection between the affine system X and another shift-invariant system which we call *the quasi-affine system* associated with X, and denote by X^q. It is obtained from X by replacing, for each $\psi \in \Psi$, $k < 0$, and $j \in \mathbb{Z}$, the function $2^{k/2}\psi(2^k \cdot +j)$ that appears in X, by the 2^{-k} functions

$$2^k\psi(2^k(\cdot + \alpha) + j), \quad \alpha = 0, 1, \ldots, 2^{-k} - 1.$$

Note that, while the affine system is dilation-invariant but not shift-invariant, the situation with the quasi-affine system is complementary.

It is obvious that "basis properties" of X (such as orthogonality) are not preserved while passing to X^q. In contrast, the following basic result, which is a special case of Theorem 5.5, holds:

Theorem 1.1. *An affine system is a fundamental frame if and only if its quasi-affine counterpart is a fundamental frame. Furthermore, the two systems have identical frame bounds. In particular, the affine system is tight if and only if the quasi-affine system is tight.*

We then analyse the affine system X via the so-called "dual Gramian" fibers, $\widetilde{G}(\omega)$, $\omega \in \mathbb{R}^d$ (which may be only almost everywhere defined) of the shift-invariant X^q, [RS1]. Each fiber $\widetilde{G}(\omega)$ is a non-negative definite self-adjoint matrix whose rows and columns are indexed by $2\pi\mathbb{Z}$, and whose (α, β)-entry is

$$\widetilde{G}(\omega)(\alpha, \beta) = \Psi[\omega + \alpha, \omega + \beta].$$

Each matrix is considered as an endomorphism of $\ell_2(2\pi\mathbb{Z})$ with norm denoted by $\mathcal{G}^*(\omega)$ and inverse norm $\mathcal{G}^{*-}(\omega)$. It is understood that $\mathcal{G}^*(\omega) := \infty$ whenever $\widetilde{G}(\omega)$ does not represent a bounded operator, and a similar remark applies to $\mathcal{G}^{*-}(\omega)$. We then conclude from Theorem 1.1 and the results of [RS1] the following:

Theorem 1.2. *Let X be an affine system generated by Ψ. Let $\mathcal{G}^*$ and $\mathcal{G}^{*-}$ be the dual Gramian norm functions defined as above. Then X is a fundamental frame if and only if $\mathcal{G}^*, \mathcal{G}^{*-} \in L_\infty$. Furthermore, the frame bounds of X are $\|\mathcal{G}^*\|_{L_\infty}$ and $1/\|\mathcal{G}^{*-}\|_{L_\infty}$.*

It is easy to conclude the following from the above theorem (cf. Corollary 5.7 for the general case):

Corollary 1.3.

(a) *An affine system X generated by Ψ is a fundamental tight frame with frame bound C if and only if*

$$\Psi[\omega, \omega] = C, \tag{1.4}$$

and

$$\Psi[\omega, \omega + 2\pi + 4\pi j] = 0, \tag{1.5}$$

for a.e. $\omega \in \mathbb{R}$ and $j \in \mathbb{Z}$.

(b) *An affine system X is a complete orthonormal system if and only if (1.5) holds, (1.4) holds with $C = 1$, and Ψ lies on the unit sphere of L_2.*

Note that the diagonal entries of the dual Gramian matrices have the form

$$\Psi[\omega, \omega] = \sum_{\psi\in\Psi} \sum_{k=-\infty}^{\infty} |\widehat{\psi}(2^k\omega)|^2. \tag{1.6}$$

Thus, known estimates for the frames bounds in terms of this expression [D1], [CS1], [CS2] and [CS3], can be accurately viewed as an estimation of the norm and the inverse norm of a non-negative definite matrix via the inspection of its diagonal entries. Furthermore, in complete analogy to semi-orthogonal systems, one can define here *diagonal* affine systems as the case when $\Psi[\omega, \omega'] = 0$, for every $\omega \neq \omega'$. In this case, the frame bounds are entirely determined by (1.6), and a dual frame can be conveniently constructed by "diagonal" division, i.e., dividing each $\widehat{\psi}$ by $\Psi[\omega, \omega]$.

Several applications of the above analysis are described in the paper. Among these, we mention here only the one concerning the construction of tight frames using multiresolution with a single scaling function. Here, we assume $\phi \in L_2$ to be *refinable* with *mask* τ_ϕ, and mean that

$$\widehat{\phi}(2\cdot) = \tau_\phi \widehat{\phi},$$

for some 2π-periodic τ_ϕ, $\lim_{\omega\to 0} \widehat{\phi}(\omega) = 1 = \widehat{\phi}(0)$, and that $\widehat{\phi}$ decays at $\pm\infty$ at a polynomial rate no slower than $1/2 + \delta$, $\delta > 0$.

Given any finite set Ψ in the closed linear span V_1 of the half-shifts of $\phi(2\cdot)$, it is then possible to represent each $\psi \in \Psi$ on the Fourier domain as

$$\widehat{\psi}(2\cdot) = \tau_\psi \widehat{\phi},$$

for some 2π-periodic τ_ψ, assumed hereafter to be (essentially) bounded. We then construct a matrix Δ which has two columns and $1 + \#\Psi$ rows, whose ϕ-row is

$$[\tau_\phi, \tau_\phi(\cdot + \pi)],$$

and with the other rows being

$$[\tau_\psi, \tau_\psi(\cdot + \pi)], \quad \psi \in \Psi.$$

Note that, importantly, we are not assuming the matrix Δ to be square, and that no major assumption has been made so far with respect to ϕ and τ_ϕ. The following is a special case of Corollary 6.7:

Theorem 1.7. *Under the assumptions listed above, if the columns of the matrix Δ are orthonormal for almost every $\omega \in [0, \pi]$, then Ψ generates a fundamental tight affine frame with frame bound* 1.

Note that the construction is "local" but the analysis cannot be so: The shifts of Ψ cannot be expected in general to form a frame for V_1 or a subspace of it. Note also that if Ψ is a singleton, the matrix Δ is 2×2, and the above construction can succeed only if τ_ϕ is a *conjugate quadrature filter* (CQF), i.e.,

$$|\tau_\phi|^2 + |\tau_\phi(\cdot + \pi)|^2 = 1, \quad a.e.$$

Thus, given a CQF τ_ϕ, one may, for example, uses Mallat's construction (see [Ma]) to yield a tight frame generated by a single wavelet. This result (for the present particular setup) is essentially due to [L].

We also remark that the shifts $E(\phi)$ of a refinable function ϕ whose refinement mask is CQF do not necessarily form a frame of $V_0 := D^{-1}V_1$. In fact, if, e.g., $\widehat{\phi}$ vanishes on a null-set only (as is the case when ϕ is a compactly supported, or an exponentially decaying function), then $E(\phi)$ cannot be a redundant frame (see [RS1]). It follows then, in case the CQF mask of the refinable ϕ is finite, $E(\phi)$ is a frame only when it is orthonormal. Hence, the above-detailed construction of tight affine frames is of particular interest since it covers cases when ϕ is a "bad" generator of V_0. In fact, affine frames constructed by MRA from a *frame* $E(\phi)$ are already analysed in the present literature; cf. [LC] and [BL].

Theorem 1.7 does not characterize *all* tight frames constructed by multiresolution. However, such characterization is possible, and is given in Theorem 6.5.

Finally, the following result (which is a special case of Corollary 6.9) concerns the construction of *orthonormal systems*:

Corollary 1.8. *Assume that Δ is a square matrix. Then, the tight frame constructed in Theorem 1.7 is orthonormal if and only if $\|\phi\| = 1$.*

The standard current argument for constructing an orthonormal affine system from multiresolution, assumes that the shifts of the scaling function are orthonormal (cf. [D3]), which forces τ_ϕ to be CQF. However, the above result shows that, given a CQF, an orthonormal system is guaranteed by the mere assumption that the scaling function has norm 1 (recall that one cannot adjust ϕ to have norm 1, since we already assume $\widehat{\phi}(0) = 1$). Under the additional assumption that τ_ϕ is a polynomial, this fact has been established in [D3] for the case discussed in this section, and [LLS] for the general case.

1.3. Compactly supported tight spline frames

Our goal in this paper is confined to developing the basic theory of discrete affine systems. Therefore, applications are discussed because they are either instrumental to wavelet theory (such as the discussion in §6), or as an anecdotal illustration (such as the discussion in §4.3). In particular, no part of this paper is devoted to specific constructions of wavelet systems.

However, it should be undoubtedly clear that constructing tight frames based on results like Theorem 1.7 is extremely simple, if one is willing to use sufficiently many wavelets. The simplest construction we are able to observe is detailed in this subsection.

Let m be a positive integer, and define $\tau_0(\omega) := \cos^{2m}(\omega/2)$. The polynomial τ_0 is the refinement mask of the centered B-spline ϕ of order $2m$:

$$\widehat{\phi}(\omega) = \frac{\sin^{2m}(\omega/2)}{(\omega/2)^{2m}}.$$

We define $2m$ (2π-periodic) wavelet masks by

$$\tau_n(\omega) := \sqrt{\binom{2m}{n}} \sin^n(\omega/2)\cos^{2m-n}(\omega/2), \quad 1 \le n \le 2m,$$

and let $\tau := (\tau_n)_{n=0}^{2m}$. We then observe that, firstly,

$$\langle \tau(\omega), \tau(\omega) \rangle = (\cos^2(\omega/2) + \sin^2(\omega/2))^{2m} = 1,$$

and that, secondly,

$$\langle \tau(\omega), \tau(\omega+\pi) \rangle = (\sin(\omega/2)\cos(\omega/2))^{2m}(1-1)^{2m} = 0.$$

Therefore, the $2m$ wavelets defined by

$$\widehat{\psi}_n(\omega) := i^n \sqrt{\binom{2m}{n}} \frac{\cos^{2m-n}(\omega/4)\sin^{2m+n}(\omega/4)}{(\omega/4)^{2m}}, \quad 1 \le n \le 2m,$$

generate a fundamental tight frame. Note that each of the wavelets is a real valued symmetric (or anti-symmetric) function supported in $[-m, m] = \operatorname{supp}\phi$ and is a spline of degree $2m-1$, smoothness C^{2m-2}, and knots at $\mathbb{Z}/2$.

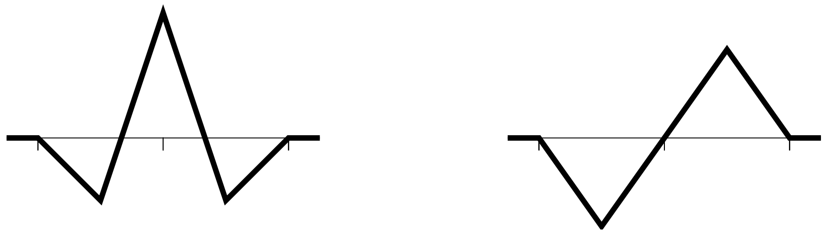

Figure 1. The two wavelets that generate a C^0 piecewise linear tight frame.

The two piecewise-linear wavelets (that correspond to the choice $m = 1$) are drawn is Figure 1.

The extension of the above algorithm to odd order splines is straightforward: one merely needs to replace $2m$ by $2m - 1$ and to insert a factor $\omega \mapsto e^{-i\omega/2}$ into the definition of the various masks.

1.4. Layout of the paper

The rest of the paper is laid out as follows. In §2 we briefly discuss frames and affine systems in L_2, and in §3 present relevant material from [RS1]. In §4 we discuss the relations between an affine system and its truncated affine system. The core of our analysis is in §5, where quasi-affine systems are studied, and where the results of §4 are applied to yield Theorem 1.1 in its general form. Finally, the construction of tight frames via multiresolution is the topic of §6.

2. Frames and affine frames

For a given countable subset $X \subset L_2 := L_2(\mathbb{R}^d)$, the **synthesis operator** $T := T_X$ which is used to reconstruct functions from discrete information is defined by

$$T : \ell_2(X) \to L_2 : c \mapsto \sum_{x \in X} c(x)x. \tag{2.1}$$

For a general X, T_X is well-defined only on the finitely supported elements of $\ell_2(X)$. In case it is bounded on these finitely supported elements, it is then extended by continuity to all of $\ell_2(X)$. In that event, X is said to be a **Bessel system**, and we refer then to the number $\|T_X\|^2$ as the **Bessel bound** of X. The adjoint of T_X^* of T_X is the **analysis operator**

$$T_X^* : L_2 \to \ell_2(X) : f \mapsto (\langle f, x\rangle)_{x \in X}.$$

Of course, the Bessel bound can be equivalently defined as $\|T_X^*\|^2$.

We study in this paper the following possible properties of a given system X.

Definition 2.2. *Let X be a Bessel system. X is a*

(a) **frame** *if $\operatorname{ran} T$ is closed (equivalently, if $\operatorname{ran} T^*$ is closed).*

(b) **Riesz basis** *if it is a frame and T is 1-1; otherwise, the frame X is* **redundant**.

(c) **fundamental frame** *if it is a frame and* T^* *is 1-1.*

Remark. Some of the articles that deal with frames for $L_2(\mathrm{IR})$, reserve the notions of "frame" and "Riesz basis" only to the case that we refer to here as a "fundamental frame" and "fundamental Riesz basis".

If X is a frame, the restriction of T to the orthogonal complement (in $\ell_2(X)$) of $\ker T$ is bounded below, hence invertible. This partial inverse of T is denoted here by T^{-1}, and a similar definition is used to define T^{*-1}. For a frame X, it is customary to refer to the Bessel bound $\|T\|^2$ as the **upper frame bound**. The complementary bound is $\|T^{-1}\|^{-2} = \|T^{*-1}\|^{-2}$ and is sometimes called the **lower frame bound**. Thus, in the instance of a fundamental frame, the frame bounds are the sharpest constants in the inequalities

$$c\|f\|_{L_2}^2 \leq \|T^* f\|_{\ell_2(X)}^2 \leq C\|f\|_{L_2}^2, \quad \forall f \in L_2.$$

A frame whose upper and lower bounds coincide is a **tight frame**. One should note that it is usually easier to handle inverses than pseudo-inverses, and it is thus desired to study the operator that is known to be injective; consequently, the study of a Riesz basis X is best done with the aid of T, and the study of a fundamental frame X is best done with T^*. Indeed, this paper focuses on *fundamental frames*, and exclusively approaches the problem via T^*.

The following elementary fact will be used in this paper as the link between tight frames and orthonormal ones.

Proposition 2.3. *Let X be a tight frame in L_2 (not necessarily fundamental) with frame bound* 1. *Then:*
(a) X lies in the closed unit ball of L_2.
(b) X is orthonormal if and only if it lies on the unit sphere of L_2.

Proof: Since X is assumed to be tight with bound 1, TT^* and T^*T are orthogonal projectors. Thus, for every atom $x \in X$, the sequence T^*x cannot exceed in norm the δ-sequence in $\ell_2(X)$ with one-point support in x (since $T^*x = T^*T\delta$, and T^*T is orthogonal). Consequently, $\|T^*x\| \leq 1$, and since the value T^*x assumes at x is $\|x\|^2$, we conclude that $\|x\|^2 \leq 1$ with equality only if $T^*_{X\backslash x}x = 0$. This proves (a), and (b) easily follows. □

In order for X to be fundamental in L_2, it should, necessarily, be infinite. In practice, however, one generates X by applying certain unitary operators to one or few functions, called the **generators** of the system. In the context of *affine (wavelet) systems*, two such operators, *dilation* and *translation*, are employed in the construction of X. Here, the **dilation operator** is meant as

$$D : f \mapsto |\det s|^{1/2} f(s\cdot),$$

with s a $d \times d$ invertible matrix. The matrix s is held fixed throughout the paper, and its specific nature is usually ignored. It is only assumed to satisfy two basic properties: (i) s^{-1} is contractive, and (ii) the entries of s are integer numbers. The first assumption is essential in the affine context. The second is essential for the application of our shift-invariance methods.

The second operator is the shift operator. Here, for a fixed invertible $d \times d$ matrix L, we define the **shift operator** E^j by

$$E^j : f \mapsto f(\cdot + j), \quad j \in L\mathbb{Z}^d,$$

and set, for any function set Φ,

$$E_L(\Phi) := \{E^{Lj}\phi : \ \phi \in \Phi, \ j \in \mathbb{Z}^d\}.$$

Since the extension of our results from the lattice $\mathbb{Z}^d$ to a lattice $L\mathbb{Z}^d$ is purely notational, we always describe our results with respect to the integer lattice. Other lattices enter the discussion only when two different lattices are analysed simultaneously (such as in the context of oversampling).

In these terms, an **affine system** X consists of the orbits obtained by an application of a discrete analog of the affine group to a finite function set Ψ:

$$X := \{D^k E^j \psi = E^{s^{-k}j} D^k \psi : \ \psi \in \Psi, k \in \mathbb{Z}, j \in \mathbb{Z}^d\}. \tag{2.4}$$

We index the function $D^k E^j \psi$ by (ψ, k, j), and identify the index with the function, i.e., we set

$$(\psi, k, j) := D^k E^j \psi. \tag{2.5}$$

Given any discrete lattice $\mathcal{L} \subset \mathbb{R}^d$, the function set X is **$\mathcal{L}$-shift-invariant** if each E^j, $j \in \mathcal{L}$, maps X 1-1 onto itself. The default lattice is always $\mathbb{Z}^d$. In [RS1], it was showed that the synthesis and analysis operators of any shift-invariant X can be decomposed, on the frequency domain, into a collection of constant coefficient (usually infinite-order) matrices, "fibers", termed there the *pre-Gramian*, *Gramian*, and *dual Gramian*. It was proved that the properties of being a Bessel system, a frame, a Riesz basis, and others, can be studied by studying an analogous property for each of the (much simpler) fibers. More details about these fiberization techniques are given in §3. However, at the outset of our study here, one should observe that an affine X is not invariant under any lattice shifts, since only the $s^{-k}\mathbb{Z}^d$-shifts of $D^k\psi$ are included in X, and these shifts become sparser as $k \to -\infty$.

Notations: bracket products. The following bracket product plays a key role in the theory of shift-invariant systems (cf. e.g., [JM], [BDR1,2], [RS1]):

$$[f, g] := \sum_{j \in 2\pi\mathbb{Z}^d} f(\cdot + j)\overline{g}(\cdot + j), \quad f, g \in L_2. \tag{2.6}$$

Among other things, we will require the following elementary fact that follows from Parseval's identity:

$$\|T^*_{E(\phi)} f\| = \|[\widehat{f}, \widehat{\phi}]\|_{L_2(\mathbb{T}^d)}, \quad f, \phi \in L_2. \tag{2.7}$$

In this paper, we introduce another important bracket product: **the affine (or dual) bracket product**. Given $\Psi \in L_2$, and a dilation matrix s, the product is defined as

$$\Psi[\,,\,] : (\omega, \omega') \mapsto \sum_{\psi \in \Psi} \sum_{k=\kappa(\omega-\omega')}^{\infty} \widehat{\psi}(s^{*k}\omega)\overline{\widehat{\psi}}(s^{*k}\omega'), \quad \omega, \omega' \in \mathbb{R}^d. \tag{2.8}$$

Here, the κ-function is defined by

$$\kappa : \mathbb{R}^d \to \mathbb{Z} : \omega \mapsto \inf\{k \in \mathbb{Z} : s^{*k}\omega \in 2\pi\mathbb{Z}^d\}. \tag{2.9}$$

Note that $\kappa(0) = -\infty$, and hence the diagonal of the affine product, denoted hereafter by $\Psi[\,]$, is

$$\Psi[\,] : \omega \mapsto \sum_{\psi\in\Psi} \sum_{k\in\mathbb{Z}} |\widehat{\psi}(s^{*k}\omega)|^2.$$

Also, $\kappa(\omega) = \infty$, unless $\omega \in 2\pi s^{*k}\mathbb{Z}^d$ for some integer k, and hence $\Psi[\omega, \omega'] = 0$, unless $\omega - \omega'$ is s^*-adic. Furthermore, one easily observes that $\Psi[\,,\,]$ is s^*-invariant, i.e.,

$$\Psi[s^*\omega, s^*\omega'] = \Psi[\omega, \omega'], \quad \forall \omega, \omega'. \tag{2.10}$$

3. Preliminaries: dual Gramian fiberization of shift-invariant systems

Given a shift-invariant system $E(\Phi)$, $\Phi \subset L_2$, three matrices, the pre-Gramian, the Gramian and the dual Gramian appear in our fiberization approach in [RS1]. The most relevant to the present context is the **dual Gramian**, which is a decomposition, on the Fourier domain, of the operator TT^*, and is a collection $\widetilde{G}(\omega)$, $\omega \in \mathbb{R}^d$, of non-negative definite self-adjoint matrices. The rows/columns of each matrix are indexed by $2\pi\mathbb{Z}^d$ (or, more generally, by the lattice dual to the lattice of shifts that we use, viz., the lattice $2\pi L^{*-1}\mathbb{Z}^d$, if the shifts are taken from $L\mathbb{Z}^d$), and the entry (α, β) of $\widetilde{G}(\omega)$ is

$$\widetilde{G}(\omega)(\alpha, \beta) = \sum_{\phi\in\Phi} \widehat{\phi}(\omega + \alpha)\overline{\widehat{\phi}}(\omega + \beta).$$

The matrix $\widetilde{G}(\omega)$ is considered as an endomorphism acting on $\ell_2(2\pi\mathbb{Z}^d)$. (Initially, however, we cannot even assert that the entries of $\widetilde{G}(\omega)$ are well-defined in the sense that their sum converges absolutely, let alone that $\widetilde{G}(\omega)$ represents a bounded endomorphism of $\ell_2(2\pi\mathbb{Z}^d)$.)

The following theorem summarizes some of dual Gramian results (cf. Corollary 3.2.2, Theorem 3.3.5, and Theorem 3.4.1 of [RS1]).

Theorem 3.1. *Let X be a system that consists of the shifts of some $\Phi \subset L_2$, with a dual Gramian $\widetilde{G}$. Consider the following functions (if the underlying operator is not well-defined or is unbounded, its norm equals ∞, by definition):*

$$\begin{array}{lcccccccl} \mathcal{G}^* & : & \mathbb{R}^d & \to & \mathbb{R}_+ & : & w & \mapsto & \|\widetilde{G}(w)\| \quad , \\ \mathcal{G}^{*-} & : & \mathbb{R}^d & \to & \mathbb{R}_+ & : & w & \mapsto & \|\widetilde{G}(w)^{-1}\| \quad . \end{array}$$

Then the following is true.

(a) The following conditions are equivalent:

(i) X is a Bessel system.

(ii) $\mathcal{G}^* \in L_\infty$.

Furthermore, the Bessel bound of X is $\|\mathcal{G}^*\|_{L_\infty}$.

(b) Assume X is a Bessel system. Then the following conditions are equivalent.

(i) X is a fundamental frame.

(ii) $\mathcal{G}^{*-} \in L_\infty$.

Furthermore, the lower frame bound is then $1/\|\mathcal{G}^{*-}\|_{L_\infty}$.

(e) Assume X is a fundamental frame . Then the following conditions are equivalent:

(i) X is a tight frame.

(ii) $\widetilde{G} = CI$ a.e. for some constant C (with I the identity matrix).

Furthermore, C is then the frame bound of X.

4. Truncated affine systems

4.1. The connection between an affine system and its truncated counterpart

Let X be a an affine system (cf. (2.4)). Given an integer k, the **truncated affine system** X_k is defined by

$$X_k := \{(\psi, k', j) = D^{k'} E^j \psi \in X : \ k' \geq -k\}, \tag{4.1}$$

(cf. (2.5)). It is clear that X_k is s^k-shift-invariant. We set $X_{k-} := X \backslash X_k$, and abbreviate $T := T_X$, $T_k := T_{X_k}$, and $T_{k-} := T_{X_{k-}}$. For any k, a natural isometry between the spaces $\ell_2(X_0)$ and $\ell_2(X_k)$ is given by

$$(V^k c)(\psi, n, j) := c(\psi, k+n, j).$$

It is evident that

$$T_0 = D^k T_k V^k. \tag{4.2}$$

Since the maps V^k, D^k are norm-preserving, the above relation reveals a rigid connection between the Bessel property and/or Riesz basis property of X and X_0 (see below). The analysis of redundant frames via the above approach is harder: X can be a frame (fundamental or not) while X_0 is not. To overcome this difficulty, we investigate the restriction of the analysis operator to subspaces of L_2. We note that the following theorem and its subsequent corollary hold for general dilation-invariant systems.

Theorem 4.3. *Let X be a an affine system.*

(a) X is a Bessel system if and only if X_k is so, for some/any k. Furthermore, $\|T\| = \|T_k\|$.

(b) X is a Riesz basis if and only if X_k is so, for some/any k. Furthermore, $\|T^{-1}\| = \|{T_k}^{-1}\|$.

(c) Assume that X is a Bessel system, let H be some subspace of $L_2(\mathbb{R}^d)$, and let H' be the closure of $\cup_{k\in\mathbb{Z}} D^k H$. If, for some k, T_k^* is bounded below on $D^{-k}H$, then T^* is bounded below on H', and

$$\|{T^*}_{|_{H'}}{}^{-1}\| \leq \|{T^*_{k|_H}}^{-1}\|.$$

Proof: The relation (4.2) proves that T_{k_0} is bounded (invertible) for some k_0, if and only if T_k is so for every k, and the norms are identical in such a case. The claims in (a,b) now easily follow from the facts that (i) the boundedness and invertibility of T are determined by its action on the finitely supported sequences in $\ell_2(X)$, and (ii) each such sequence lies in some $\ell_2(X_k)$, for sufficiently large k.

In the proof of (c), we assume, without loss, that $k = 0$, and first note that, in view of (a), it may be assumed without loss that X and X_0 are Bessel systems. Now, (4.2) implies that

$$T_0^* = V^{-k} T_k^* D^{-k}.$$

Therefore, T_0^* is bounded below on H if and only if T_k^* is bounded below on $D^{-k}H$, and furthermore,

$$\|(T_0^*{}_{|H})^{-1}\| = \|(T_k^*{}_{|_{D^{-k}H}})^{-1}\|.$$

The boundedness below of $T_k^*{}_{|_{D^{-k}H}}$ implies the boundedness below of the restriction $T^*{}_{|_{D^{-k}H}}$ of T^* to $D^{-k}H$, and thus

$$\|(T^*{}_{|_{D^{-k}H}})^{-1}\| \le \|(T_k^*{}_{|_{D^{-k}H}})^{-1}\| = \|(T_0^*{}_{|H})^{-1}\|.$$

Since k here is arbitrary, (c) follows. □

In general, it is hard for us to apply (c) of Theorem 4.3 for the derivation of explicit conditions for X to be a frame. However, for one specific choice of H, our tools apply. This special, yet very important, case is described in the next result.

Corollary 4.4. *Let*

$$H_r := \{f \in L_2(\mathbb{R}^d) : \operatorname{supp} \widehat{f} \subset \mathbb{R}^d \backslash \Omega_r\}, \tag{4.5}$$

and where Ω_r the ball of radius r around the origin. Then X is a fundamental frame if, for some $r \ge 0$, T_0^* is bounded, and is also bounded below on H_r. Also, with $T_{0,r}^*$ the restriction of T_0^* to H_r,

$$\|T^{*-1}\| \le \|T_{0,r}^*{}^{-1}\|.$$

Proof: By (a) of Theorem 4.3, T^* is bounded if and only if T_0^* is bounded, and therefore, we may assume without loss that X is a Bessel system here. Now, we invoke (c) of Theorem 4.3, for the choice $H := H_r$. Since s^{*-1} is contractive, $D^{-1}H_r \supset H_{\delta r}$, for some $\delta < 1$. Therefore, $\cup_{n\in\mathbb{Z}} D^n H_r$ is the space of *all* functions whose Fourier transform vanishes on *some* neighborhood of the origin. Since this space is dense in $L_2(\mathbb{R}^d)$, we obtain that T^* is bounded below on the entire $L_2(\mathbb{R}^d)$, i.e., that X is a fundamental frame. □

The converse of the above result is valid as well, but requires us to impose a decay condition (at ∞) on $\widehat{\Psi}$ (with Ψ the generating set of X), which we consider as very mild. To describe this assumption, set, for every $k \in \mathbb{Z}_+$,

$$A_k := \{\alpha \in 2\pi\mathbb{Z}^d : \ |\alpha| > 2^k\},$$

and

$$c(\psi, k) := \| \sum_{\alpha \in A_k} |\widehat{\psi}(\cdot + \alpha)|^2 \|_{L_\infty([-\pi,\pi]^d)}.$$

Our decay assumption on $\widehat{\Psi}$ is as follows:

$$\sum_{\psi \in \Psi} \sum_{k=0}^{\infty} c(\psi, k) < \infty. \tag{4.6}$$

It is elementary to prove that (4.6) is satisfied once $\widehat{\psi}(\omega) = O(|\omega|^{-\rho})$, as $\omega \to \infty$, for some $\rho > d/2$, and every $\psi \in \Psi$. However, there are examples (e.g., Haar wavelets in several dimensions) that satisfy (4.6) while violating that simpler, yet stronger, decay assumption. Whence our decision to stick to the more complicated (4.6).

With the additional assumption (4.6), the condition stated in Corollary 4.4 is *equivalent* to X being a fundamental frame.

Lemma 4.7. *Let X be a fundamental affine frame, generated by a finite set Ψ (cf. (2.4)) that satisfies (4.6). Then, for every $\varepsilon > 0$, there exists a sufficiently large r such that T_0^* is bounded below on H_r, and*

$$\|{T_{0,r}^*}^{-1}\| \le \|{T^*}^{-1}\| + \varepsilon.$$

Proof: First, since X is assumed to be a frame, X is a Bessel system, hence X_0 is a Bessel system, too, by virtue of Theorem 4.3.

Let $T_{0,r}^*$ ($T_{0-,r}^*$ respectively) be the restriction of T_0^* (T_{0-}^* resp.) to H_r. Clearly, for every $f \in H_r$,

$$\|T^* f\|^2 = \|T_{0,r}^* f\|^2 + \|T_{0-,r}^* f\|^2. \tag{4.8}$$

We will show that

$$\|T_{0-,r}^*\| \overset{r\to\infty}{\longrightarrow} 0. \tag{4.9}$$

However, since X is a fundamental frame, we conclude from (4.8) that for every $f \in H_r$,

$$\|T_{0,r}^* f\|^2 = \|T^* f\|^2 - \|T_{0-,r}^* f\|^2 \ge \|{T^*}^{-1}\|^{-2}\|f\|^2 - \|T_{0-,r}^*\|^2\|f\|^2.$$

Thus, given any $\varepsilon > 0$, we can choose r sufficiently large to obtain that $T_{0,r}^*$ is bounded below and that

$$\|{T_{0,r}^*}^{-1}\| \le \|{T^*}^{-1}\| + \varepsilon.$$

Thus, we only need prove (4.9), and, clearly, we may assume there that Ψ is a singleton $\{\psi\}$, as we do, indeed. Here, we fix $k<0$, set $Y_k := D^k E(\psi)$, and compute that (cf. (2.7))

$$\begin{aligned}\|T^*_{Y_k} f\|^2 &= \sum_{\alpha\in\mathbb{Z}^d} |\langle D^k E^\alpha \psi, f\rangle|^2 = \sum_{\alpha\in\mathbb{Z}^d} |\langle E^\alpha\psi, D^{-k} f\rangle|^2 \\ &= \|[\widehat{\psi}, \widehat{D^{-k} f}]\,\|^2_{L_2(\mathbb{T}^d)} = |\det s|^k \|[\widehat{\psi}, \widehat{f}(s^{*k}\cdot)]\|^2_{L_2(\mathbb{T}^d)}.\end{aligned}$$

Since $f\in H_r$, $\widehat{f}(s^{*k}\cdot)$ vanishes on a ball with center at the origin and radius $\delta^k r$, for some $\delta<1$. Thus,

$$\begin{aligned}|\det s|^k \|[\widehat{\psi}, \widehat{f}(s^{*k}\cdot)]\|^2_{L_2(\mathbb{T}^d)} &\le |\det s|^k \|\sum_{|\alpha|\ge\delta^k r} |\widehat{\psi}(\cdot+\alpha)|^2\|_{L_\infty(\mathbb{T}^d)} \|\,[\widehat{f}(s^{*k}\cdot), \widehat{f}(s^{*k}\cdot)]\,\|_{L_1(\mathbb{T}^d)} \\ &= \|\sum_{|\alpha|\ge\delta^k r} |\widehat{\psi}(\cdot+\alpha)|^2\|_{L_\infty(\mathbb{T}^d)}\, \|f\|^2.\end{aligned}$$

Since $\|T^*_{0-,r} f\|^2 = \sum_{k<0} \|T^*_{Y_k} f\|^2$, we therefore conclude that

$$\|T^*_{0-,r}\|^2 \le \sum_{k<0} \|\sum_{|\alpha|\ge\delta^k r} |\widehat{\psi}(\cdot+\alpha)|^2\|_{L_\infty(\mathbb{T}^d)}.$$

Selecting $r=\delta^{-k'}$, $k'\in\mathbb{Z}_+$, the above sum becomes

$$\sum_{k>k'} \|\sum_{|\alpha|\ge\delta^{-k}} |\widehat{\psi}(\cdot+\alpha)|^2\|_{L_\infty(\mathbb{T}^d)}. \tag{4.10}$$

Since we assume (4.6), this last expression is recognized as the tail of a convergent series, hence can be made arbitrarily small by choosing large k' (i.e., large r). □

We summarize our findings concerning the connections between the frame properties of an affine system and its truncated counterpart in the following theorem.

Theorem 4.11. *Let X be an affine Bessel system generated by the finite Ψ. Assume that Ψ satisfies condition (4.6). Then X is a fundamental frame if and only if for some $r\ge 0$, the restriction $T^*_{0,r}$ of the map $T^*_{X_0}$ to H_r is bounded below. Furthermore,*

$$\|T^{*\,-1}_X\| = \lim_{r\to\infty} \|(T^*_{0,r})^{-1}\|.$$

Finally, two immediate consequences of Theorem 4.3 (that are of *negative* nature) are recorded in the following corollary.

Corollary 4.12. *Let X be an affine system, and X_0 be its truncated counterpart. Then:*
(a) X_0 cannot be a fundamental Riesz basis.
(b) X_0 cannot be a tight frame unless $\operatorname{ran} T_X$ *is the orthogonal sum* $\oplus_{k\in\mathbb{Z}} \operatorname{ran} T_{X_k\setminus X_{k-1}}$.

Proof: (a): If X_0 is fundamental in $L_2(\mathbb{R}^d)$, then X, as a proper superset of X_0, cannot be a Riesz basis for $L_2(\mathbb{R}^d)$. By Theorem 4.3, X_0 is not a Riesz basis, either.

(b): Since X_0 is a frame, then, by (c) of Theorem 4.3, X is a frame too, and we have

$$\|T\| \geq 1/\|T^{-1}\| \geq 1/\|T_0{}^{-1}\| = \|T_0\|,$$

with the equality since X_0 is tight, and the penultimate inequality by (c) of Theorem 4.3. Further, (a) of that theorem guarantees $\|T\| = \|T_0\|$, and hence, we arrive at

$$\|T\| = 1/\|T^{-1}\| = 1/\|T_0{}^{-1}\| = \|T_0\|,$$

which shows that X is also a tight frame, and has the same bounds as those of X_0. Now, let $f \in \operatorname{ran} T_0$. Since X and X_0 are tight frames for their range, and with the same frame bounds, and since $f \in \operatorname{ran} T_0 \subset \operatorname{ran} T$, we obtain that $\|T^* f\| = \|T_0^* f\|$. On the other hand, $\|T^* f\|^2 = \|T_0^* f\|^2 + \|T_{0-}^* f\|^2$. Thus, we conclude that T_{0-}^* vanishes on $\operatorname{ran} T_0$, and the result now easily follows. $\square$

There are examples (some can be constructed based on the biorthogonal wavelets obtained in [CDF]) of an affine system X whose corresponding truncated system X_0 is a frame, for which, nonetheless, $\operatorname{ran} T$ is not the orthogonal sum $\oplus_{n\in\mathbb{Z}} \operatorname{ran} T_{X_n\setminus X_{n-1}}$. This means that 'tightness assumption' in (b) of the above corollary cannot be removed.

4.2. Dual Gramian analysis of truncated affine systems

In order to compute the dual Gramian $\widetilde{G}_0$ of the shift-invariant X_0, we need choose a suitable set Φ for which $X_0 = E(\Phi)$. For that, we let Γ_k be the quotient group

$$\Gamma_k := \mathbb{Z}^d / s^k \mathbb{Z}^d.$$

The same notation also stands for any set of representers for this group. Note that Γ_k is of order $|\det s|^k$, and, of course, the fact that s is an *integer* matrix is essential here. Then, the set Φ is defined as

$$\Phi = \{(\psi, k, \gamma) := D^k E^\gamma \psi : \ \psi \in \Psi, \ k \geq 0, \ \gamma \in \Gamma_k\}.$$

It is straightforward to see that, indeed, the shift-invariant set $E(\Phi)$ generated by Φ is exactly the truncated set X_0.

Next, we observe that the Fourier transform of the function $\phi = (\psi, k, \gamma)$ is the function

$$\widehat{\phi} = D_*^k(e_\gamma \widehat{\psi}) = e_{s^{-k}\gamma} D_*^k \widehat{\psi}, \quad D_* : f \mapsto |\det s|^{-1/2} f(s^{*-1}\cdot), \quad e_\gamma : \omega \mapsto e^{i\gamma\cdot\omega},$$

and thus the $(\alpha, \beta) \in 2\pi(\mathbb{Z}^d \times \mathbb{Z}^d)$-entry of $\widetilde{G}_0(\omega)$ has the form

$$\widetilde{G}_0(\omega)(\alpha, \beta) = \sum_{\psi\in\Psi} \sum_{k\geq 0} D_*^k \widehat{\psi}(w + \alpha) \overline{D_*^k \widehat{\psi}(w + \beta)} \sum_{\gamma\in\Gamma_k} e_{s^{-k}\gamma}(\alpha - \beta).$$

The exponential sum is zero unless $e_{s^{*-k}(\alpha-\beta)}$ is the identity character of Γ_k, i.e., unless $-k \geq \kappa(\alpha - \beta)$ (cf. (2.9) for the definition of κ). Consequently,

$$\widetilde{G}_0(\omega)(\alpha, \beta) = \sum_{\psi\in\Psi} \sum_{k=\kappa(\alpha-\beta)}^{0} \widehat{\psi}(s^{*k}(\omega + \alpha)) \overline{\widehat{\psi}(s^{*k}(\omega + \beta))}.$$

Defining the m-**order truncated affine product** by

$$\Psi_m[\omega,\omega'] := \sum_{\psi\in\Psi}\sum_{k=\kappa(\omega-\omega')}^{m} \widehat{\psi}(s^{*k}\omega)\overline{\widehat{\psi}(s^{*k}\omega')},$$

we can write then

$$\widetilde{G}_0(\omega)(\alpha,\beta) = \Psi_0[\omega+\alpha,\omega+\beta]. \tag{4.13}$$

Recall that some of the main results of this section are in terms of the restriction $T^*_{0,r}$ of $T^*_{X_0}$ to H_r. From the dual Gramian representation as detailed in [RS1], we easily conclude that the assumption $f \in H_r$ renders all α-rows and α-columns of the dual Gramian $\widetilde{G}_0(\omega)$ (viewed, say, as a quadratic form) inactive, in the case $|\omega+\alpha| \le r$. This means that the fibers of the dual Gramian representation of $T^*_{0,r}$ are the matrices

$$\widetilde{G}_{0,r}(\omega), \quad \omega \in \mathbb{R}^d,$$

that are obtained from $\widetilde{G}_0(\omega)$ by retaining the entries (α,β) for which $|\omega+\alpha|, |\omega+\beta| > r$, and removing all other entries.

We thus conclude from Theorem 3.1 and Theorem 4.11 the following result:

Theorem 4.14. *Let X be an affine system generated by Ψ. Let $\widetilde{G}_0(\omega)$, and $\widetilde{G}_{0,r}(\omega)$, $\omega \in \mathbb{R}^d$, be the dual Gramian fibers of $T^*_{X_0}$, and $T^*_{0,r}$ as detailed above. Set:*

$$\mathcal{G}^*_0(\omega) := \|\widetilde{G}_0(\omega)\|, \qquad \mathcal{G}^{*-}_{0,r}(\omega) := \|\widetilde{G}_{0,r}(\omega)^{-1}\|. \tag{4.15}$$

Then:

(a) X is a Bessel system iff $\mathcal{G}^*_0 \in L_\infty$. Furthermore, the Bessel bound of X is then $\|\mathcal{G}^*_0\|_{L_\infty}$.

(b) Assume X is a Bessel system. If, for some $r \ge 0$, $\mathcal{G}^{*-}_{0,r} \in L_\infty$, then X is a fundamental frame and its lower frame bound c satisfies

$$1/c \le \lim_{r\to\infty} \|\mathcal{G}^{*-}_{0,r}\|_{L_\infty}. \tag{4.16}$$

(c) If (4.6) holds, and X is a fundamental frame, then $\mathcal{G}^{*-}_{0,r} \in L_\infty$ for all sufficiently large r, and equality holds in (4.16). □

4.3. Oversampling

We sidetrack in this subsection to consider the problem of *oversampling* an affine system. A reader interested in the core development of this article may skip this section without loss of continuity.

Here, "oversampling" means that we replace, in the definition of X, the integer shifts $\mathbb{Z}^d$ by the denser shifts that are taken from the superlattice $L\mathbb{Z}^d$ of $\mathbb{Z}^d$ (thus L^{-1} is an integer matrix). We denote the oversampling system by $X(L)$.

The variant of the oversampling problem that we consider here was initiated by Chui and Shi [CS1], [CS3] and [CS4]: One starts with a fundamental frame X and aims at connecting between the bounds of X and the bounds of the oversampling $X(L)$.

We compare between the dual Gramian $\widetilde{G}_0$ of the truncated affine X_0 and the dual Gramian $\widetilde{G}_0^L$ of the truncated oversampling $X(L)_0$. The latter is computed in the same way we computed $\widetilde{G}_0$ in §4.2, with an appropriate modification due to change of the lattice: it is now indexed by the dual lattice of $L\mathbb{Z}^d$, viz., the sublattice $\mathcal{L} := 2\pi L^{*-1}\mathbb{Z}^d$ of $2\pi\mathbb{Z}^d$, and its entries are

$$\widetilde{G}_0^L(\omega)(\alpha,\beta) = |\det L|^{-1} \sum_{\psi\in\Psi} \sum_{k=\kappa_L(\alpha-\beta)}^{0} \widehat{\psi}(s^{*k}(\omega+\alpha))\overline{\widehat{\psi}}(s^{*k}(\omega+\beta)), \tag{4.17}$$

where

$$\kappa_L(\alpha) := \min\{k : s^{*k}\alpha \in \mathcal{L}\}.$$

Note that the only two differences between the entries here of $\widetilde{G}_0^L$, and those of the dual Gramian $\widetilde{G}_0$ of X_0 are (i): the factor $|\det L|^{-1}$ that appears here, (ii) the different definition of the κ-function. We thus conclude that the dual Gramian $\widetilde{G}_0^L$ of $X(L)_0$ is a submatrix of $|\det L|^{-1}\widetilde{G}_0$, provided that the following "relative primality" condition holds:

$$\kappa = \kappa_L \quad \text{on } \mathcal{L}.$$

It is straightforward to conclude from the definition of the κ and κ_L that this condition is equivalent to

$$L^{*-1}\mathbb{Z}^d \cap s^{*k}\mathbb{Z}^d = s^{*k}L^{*-1}\mathbb{Z}^d, \quad \forall k \geq 0. \tag{4.18}$$

Note that s^* and L^{*-1} are integer matrices.

Analogous observations are valid if we replace, for $r > 0$, $\widetilde{G}_0$ by $\widetilde{G}_{0,r}$, and $\widetilde{G}_0^L$ by $\widetilde{G}_{0,r}^L$ (only that now the comparison should be done fiber by fiber since each fiber has its own set of rows and columns).

Since all dual Gramian matrices are non-negative definite, passing to submatrices of them is norm-reducing as well as inverse-norm reducing. The following results are therefore immediate from Theorem 4.14, when combined with the above observations.

Theorem 4.19. *Let X be a fundamental affine frame generated by Ψ, with a dilation matrix s, with $\mathbb{Z}^d$ as its lattice of shifts, and with frame bounds c, C. Let $X(L)$ be obtained from X by replacing $\mathbb{Z}^d$ by its superlattice $L\mathbb{Z}^d$. Assume that (4.18) holds. Then:*
(a) $X(L)$ is a fundamental frame with upper frame bound $\leq |\det L|^{-1}C$.
(b) If, in addition, Ψ satisfies (4.6), then the lower frame bound of $X(L)$ is $\geq |\det L|^{-1}c$.
(c) In particular, if (4.6) holds and X is tight, then $X(L)$ is tight, too.

Examples.
(1). If $d = 1$, $s = m$, and $L = 1/n$, condition (4.18) reads as

$$n\mathbb{Z} \cap m^k\mathbb{Z} = m^k n\mathbb{Z},$$

and is clearly equivalent to the relative primality of m, n. Thus, this special case of Theorem 4.19 generalizes the corresponding theorem of [CS3].
(2). More generally, let $\mathcal{M}$ be the left-hand-side of (4.18), and let μ be the determinant of (any basis for) $\mathcal{M}$; also, let $a := \det s$, $l := \det L^{-1}$. Then, on the one hand, μ must be divisible by $l.c.m(a^k, l)$, while, on the other hand, since $\mathcal{M}$ is certainly a superlattice of the right-hand-side of (4.18), the equality (4.18) is equivalent to $|\mu| = |a^k l|$. Thus, (4.18) must hold in case $\det s$ and $\det L^{-1}$ are relatively prime:

Corollary 4.20. *Theorem 4.19 holds if we make there, instead of (4.18), the stronger assumption* $g.c.d(\det s, \det L^{-1}) = 1$.

The case when L is scalar in above corollary is essentially proved in [CS4].

The oversamplings discussed so far are "benign": no fundamental change in the structure of the system occurs while passing from X to $X(L)$. In §6, we will briefly revisit this problem and will consider a rather different variant: we choose there the oversampling matrix L as the inverse of the dilation matrix.

5. Quasi-affine systems

The analysis of affine systems by truncation is very useful for computing the upper frame bound. However, it requires a limit process for the capturing of the more challenging lower frame bound. This is particularly painful when we would like to verify that X is tight, or that another system, say $\widetilde{X}$, is dual to X: we need then to verify that the dual Gramian matrices $\widetilde{G}_{0,r}(\omega)$ converge, as $r \to \infty$ to a scalar form; at the same time, no row or column of $\widetilde{G}_0(\omega)$ belongs to all $(\widetilde{G}_{0,r}(\omega))_r$.

These difficulties are overcome by associating X with another shift-invariant system, X^q, referred to as **the quasi-affine system of** X. To recall, X_0 was obtained from X by truncation, i.e., removing all elements (ψ, k, j) (as defined in (2.5)) whose index k is negative. We construct the quasi-affine system in a more subtle way: given $k < 0$, rather than removing from X the $s^{-k}\mathbb{Z}^d$-shift-invariant set

$$\{(\Psi, k, \mathbb{Z}^d)\} := \{(\psi, k, j) : (\psi, j) \in \Psi \times \mathbb{Z}^d\},$$

we replace it by the larger *shift-invariant* system

$$|\det s|^{k/2}\{\Psi, k, s^k\mathbb{Z}^d\}.$$

Thus,

$$X^q := X_0 \cup \{|\det s|^{k/2}(\psi, k, j) : \ \psi \in \Psi, k < 0, j \in s^k\mathbb{Z}^d\}.$$

Our analysis of truncated systems was independent of their dual Gramian analysis: Only after the main results were established, we converted them into dual Gramian language. In contrast, the dual Gramian of the quasi-affine system is our main tool in the derivation of the connections between the affine X and the quasi-affine X^q, hence need be computed at this stage.

In order to compute the dual Gramian $\widetilde{G}^q$ of X^q, we write the quasi-affine system as the union

$$X^q = X_0 \cup Y_1 \cup Y_2 \cup \ldots,$$

with

$$Y_k = |\det s|^{-k/2} E(D^{-k}\Psi).$$

Since we have already computed in the previous section the dual Gramian $\widetilde{G}_0$ of X_0, it remains to compute the dual Gramian of $\cup_{k\geq 1} Y_k$. The natural generators for Y_k (as a shift-invariant system) are $\Phi_k := |\det s|^{-k/2} D^{-k}\Psi$, whose Fourier transforms are $\widehat{\Psi}(s^{*k}\cdot)$. This means that the (α, β)-entry of the dual Gramian of $X^q \backslash X_0$ is

$$\sum_{\psi\in\Psi}\sum_{k=1}^{\infty} \widehat{\psi}(s^{*k}(\omega+\alpha))\overline{\widehat{\psi}}(s^{*k}(\omega+\beta)) = \Psi[\omega+\alpha, \omega+\beta] - \Psi_0[\omega+\alpha, \omega+\beta].$$

Therefore, we obtain from the representation (4.13) of $\widetilde{G}_0$ the following result:

Proposition 5.1. *Given a quasi-affine system X^q generated by Ψ, the (α, β)-entry of the dual Gramian $\widetilde{G}^q(\omega)$ of X^q is the affine product $\Psi[\omega+\alpha, \omega+\beta]$.*

We denote by $\mathcal{G}^*_q(\omega)$ the norm of the fiber $\widetilde{G}^q(\omega)$, and by $\mathcal{G}^{*-}_q(\omega)$ the norm of its inverse (with the usual convention that these numbers can be infinite). Theorem 3.1 affirms that X^q is a Bessel system if and only if $\mathcal{G}^*_q \in L_\infty$, and that X^q is a fundamental frame if and only if $\mathcal{G}^*_q, \mathcal{G}^{*-}_q \in L_\infty$. The key then to the connection between X^q and X lies in the following lemma:

Lemma 5.2. *Let X^q be a quasi-affine system, and let $r \geq 0$. Let $T^*_{X^q}$ be the analysis operator of X^q, and let $T^*_{q,r}$ be its restriction to H_r. Then:*

(a) X^q is a Bessel system (i.e., $T^*_{X^q}$ is bounded) if (and only if) $T^*_{q,r}$ is bounded. The Bessel bound of X^q is then $\|T^*_{q,r}\|^2$.

(b) Assume X^q is Bessel. Then, X^q is a fundamental frame if (and only if) $T^*_{q,r}$ is bounded below (hence invertible). Furthermore, the lower frame bound of X^q is then $\|T^{*\,-1}_{q,r}\|^{-2}$.

Proof: We prove only (a). The proof of (b) is entirely analogous.

As in the case of the truncated affine system, one can easily verify that the dual Gramian representation of $T^*_{q,r}$ is obtained by removing from $\widetilde{G}^q(\omega)$, for each $\omega \in \mathbb{R}^d$, all rows and columns α for which $|\omega+\alpha| > r$. We denote by $\widetilde{G}^q_r(\omega)$ the so obtained fibers. The norm of $T^*_{q,r}$ is then the essential supremum of the map $\omega \to \|\widetilde{G}^q_r(\omega)\|$.

Fix $\omega \in \mathbb{R}^d \backslash (2\pi\mathbb{Z}^d)$. Then there exists a positive integer k such that, with $\omega_k := s^{*k}\omega$,

$$\text{dist}(\omega_k, 2\pi s^{*k}\mathbb{Z}^d) > r. \tag{5.3}$$

Using the s^*-invariance of the affine bracket product (2.10), we see that

$$\widetilde{G}^q(\omega)(\alpha, \beta) = \widetilde{G}^q(\omega_k)(s^{*k}\alpha, s^{*k}\beta);$$

i.e., $\widetilde{G}^q(\omega)$ coincides with the submatrix of $\widetilde{G}^q(\omega_k)$ that corresponds to the indices $2\pi s^{*k}\mathbb{Z}^d$. Moreover, thanks to (5.3), that submatrix in not only a submatrix of $\widetilde{G}^q(\omega_k)$ but also of the smaller matrix $\widetilde{G}^q_r(\omega_k)$. Since passing to a submatrix is norm-reducing (as well as inverse-norm reducing, as needed for the proof of (b)) on non-negative definite matrices, we therefore conclude that

$$\mathcal{G}^*_q(\omega) := \|\widetilde{G}^q(\omega)\| \leq \|\widetilde{G}^q_r(\omega_k)\| \leq \|T^*_{q,r}\|^2.$$

This being true for almost every ω (i.e., every ω with the exclusion of the null-set $2\pi\mathbb{Z}^d$), we conclude that

$$\|T^*_{X^q}\|^2 = \|\mathcal{G}^*_q\|_{L_\infty} \leq \|T^*_{q,r}\|^2.$$

Since increasing the domain of any operator can only increase its norm, the converse implication and inequality are trivial. This proves (a). □

The above lemma shows that, when analysing a quasi-affine system, we may safely restrict attention to any space of the form H_r. The next lemma (which is closely related to Lemma 4.7) states that, in that event, the difference between the quasi-affine X^q and the truncated affine X_0 is "negligible".

Lemma 5.4. *Let X^q be a quasi-affine system generated by Ψ. Assume that Ψ satisfies (4.6). Then, for every $\varepsilon > 0$, there exists sufficiently large r, such that, with $Y := X^q \backslash X_0$, and with $T^*_{Y,r}$ the restriction of T^*_Y to H_r,*

$$\|T^*_{Y,r}\| < \varepsilon.$$

We postpone the proof of the lemma to the end of this section, and move to the main theorem of this paper.

Theorem 5.5. *Let X be an affine system generated by Ψ, and let X^q be its quasi-affine counterpart. Assume that Ψ satisfies (4.6). Then:*

(a) X is a Bessel system if and only if X^q is a Bessel system. Furthermore, the two systems have the same Bessel bound.

(b) X is a fundamental frame if and only if X^q is a fundamental frame. Furthermore, the two systems have the same frame bounds.

In particular, X is a fundamental tight frame if and only if X^q is a fundamental tight frame.

Proof: (a): If X^q is a Bessel system with Bessel bound C_q, then certainly its subset X_0 is a Bessel system with Bessel bound $C \leq C_q$. Invoking Theorem 4.3, we conclude that X is a Bessel system, too, and its Bessel bound is C, as well, hence is $\leq C_q$. Note that we have not used (4.6) in this part of the proof.

Conversely, assume that X is a Bessel system with bound C. Then, Theorem 4.3, X_0 is a Bessel system, too, and with the same bound C; *a fortiori*, $T^*_{0,r}$ (=the restriction of $T^*_{X_0}$ to H_r) is bounded, and its norm is $\leq \sqrt{C}$, for whatever r we choose. We now choose r large enough to ensure that, Lemma 5.4, $\|T^*_{Y,r}\|^2 \leq \varepsilon$, with $T^*_{Y,r}$ as in that lemma. Consequently, for every $f \in H_r$,

$$\|T^*_{X^q} f\|^2 = \|T^*_{X_0} f\|^2 + \|T^*_Y f\|^2 \leq (C + \varepsilon)\|f\|^2.$$

This proves that the restriction of $T^*_{X^q}$ to H_r is bounded (and its norm is $\le \sqrt{C+\varepsilon}$), which implies, Lemma 5.2, that X^q is a Bessel system with bound $\le \sqrt{C+\varepsilon}$. Since ε was arbitrary, we obtain the desired result.

(b): in view of (a), we may assume without loss that X^q and X are Bessel systems with the same Bessel bounds. Now, suppose that X^q is a fundamental frame, with lower frame bound c_q. Invoking Lemma 5.4, we find r sufficiently large such that, in that lemma's notations, $\|T^*_{Y,r}\|^2 \le \varepsilon$. Then, for every $f \in H_r$,

$$\|T^*_{X_0} f\|^2 = \|T^*_{X^q} f\|^2 - \|T^*_Y f\|^2 \ge (c_q - \varepsilon)\|f\|^2.$$

Assuming, without loss, that $c_q - \varepsilon > 0$, Theorem 4.11 can be invoked to yield that that T^*_X is a fundamental frame, and with frame bound $c \ge c_q - \varepsilon$. Thus, $c \ge c_q$.

Finally, we assume that X is a fundamental frame and with lower frame bound c. Theorem 4.11 then implies that, for any given ε, we can find r such that

$$\|T^*_{X_0} f\|^2 \ge (c-\varepsilon)\|f\|^2, \quad \forall f \in H_r.$$

Since X^q is a superset of X_0, then we trivially obtain from the above that

$$\|T^*_{X^q} f\|^2 \ge (c-\varepsilon)\|f\|^2, \quad \forall f \in H_r.$$

Thus, $T^*_{X^q}$ is bounded below on H_r, therefore, Lemma 5.2, X^q is a fundamental frame with lower frame bound $\ge c - \varepsilon$. We conclude that $c_q \ge c$, and this completes the proof of (b). $\square$

Theorem 5.5, when combined with Theorem 3.1, provides the following complete characterization of fundamental affine frames:

Theorem 5.6. *Let X be an affine system generated by Ψ. Assume that Ψ satisfies (4.6). Let $\widetilde{G}^q$ be a dual Gramian of the associated quasi-affine system, as described in Proposition 5.1, with norm-function $\mathcal{G}^*_q$, and inverse-norm function $\mathcal{G}^{*-}_q$. Then:*

(a) X is a Bessel system if and only if $\mathcal{G}^*_q \in L_\infty$. Furthermore, the Bessel bound is $\|\mathcal{G}^*_q\|_{L_\infty}$.

(b) X is a fundamental frame if and only if $\mathcal{G}^*_q, \mathcal{G}^{*-}_q \in L_\infty$. Furthermore, the lower frame bound is $1/\|\mathcal{G}^{*-}_q\|_{L_\infty}$.

Our characterization of (fundamental) tight affine frames is now immediate: by Theorem 3.1 and Theorem 5.5, X is fundamental and tight if and only if the dual Gramian $\widetilde{G}^q$ is a.e., the scalar matrix CI, with C the frame bound, i.e., if and only is $\Psi[\omega, \omega'] = C\delta_{\omega,\omega'}$. However, there are essentially only two cases here: the diagonal case $\omega = \omega'$, and the case when $\kappa(\omega - \omega') = 0$. The other required conditions are easily derived from this latter case using the affine invariance (2.10) of the affine product.

Corollary 5.7. *Let X be an affine system generated by Ψ. Assume that (4.6) holds. Then X is a fundamental tight frame if and only if, for almost every $\omega, \omega' \in \mathbb{R}^d$,*

$$\Psi[\omega, \omega'] = C\delta_{\omega,\omega'};$$

equivalently,

$$\Psi[\omega,\omega]=C, \quad \textit{and} \;\; \Psi[\omega,\omega+\alpha]=0, \qquad \textit{for a.e. } \omega, \textit{ and every } \alpha\in 2\pi(\mathbb{Z}^d\backslash s^*\mathbb{Z}^d).$$

Remark. The "diagonal condition" in the above characterization is

$$\Psi[\omega,\omega]=\sum_{\psi\in\Psi}\sum_{k\in\mathbb{Z}}|\widehat{\psi}(s^{*k}\omega)|^2=C.$$

This, indeed, is well-known as a *necessary condition* for a tight frame (cf. [D3]). □

Remark. The last corollary implies that functions Ψ whose Fourier transforms are positive a.e. cannot generate tight frames.

From the characterization of tight frames, one obtains the following useful characterization of orthonormal wavelets.

Corollary 5.8. *Let X be an affine system generated by Ψ. Assume that (4.6) holds. Then the following statements are equivalent:*
(i) The affine set X is an orthonormal basis of $L_2(\mathbb{R}^d)$.
(ii) Each $\psi\in\Psi$ has norm 1, and

$$\Psi[\omega,\omega]=1, \quad \textit{and} \;\; \Psi[\omega,\omega+\alpha]=0, \qquad \textit{for a.e. } \omega, \textit{ and every } \alpha\in 2\pi(\mathbb{Z}^d\backslash s^*\mathbb{Z}^d).$$

Proof: Obviously, X lies on the unit sphere of L_2 whenever Ψ does so. Therefore, the result follows directly from Corollary 5.7 and Proposition 2.3. □

Remark. It is important to understand that, even if X forms *an orthonormal basis* for L_2, X^q is still only a tight frame: irredundancy is lost while passing from X to X^q! On the other hand, if X, indeed, is orthonormal and fundamental, then the shift-invariance of X_0 implies that not only $X_0 \perp (X\backslash X_0)$, but also, $X_0 \perp (X^q\backslash X_0)$. This means that, not only X^q is tight for L_2, but also $X^q\backslash X_0$ is a tight frame for the orthogonal complement $X_0^\perp$ of X_0. In case X is derived from multiresolution, $X_0^\perp$ is the familiar scaling function space V_0. Hence we obtain the following oversampling result:

Corollary 5.9. *If Ψ is a collection of orthonormal wavelets constructed with respect to a scaling function space V_0, and if they satisfy (4.6), then the shift-invariant system*

$$\{|\det s|^{k/2}E^jD^k\psi:\ \psi\in\Psi, j\in\mathbb{Z}^d, k<0\}$$

is a tight frame for V_0.

Note that the corollary does not assume any particular way for obtaining the wavelets from the multiresolution. In fact, even the *length* of V_0 (i.e., the minimal number of scaling functions whose shifts span V_0) is only assume here to be finite. □

Proof of Lemma 5.4. While it seems plausible that the statement here is weaker than that of Lemma 4.7, we did not find a way to derive it directly from that lemma, hence provide a separate (and very similar) proof. In fact, the proof *does* show that this is a weaker statement.

We may assume without loss that Ψ is a singleton. Then, we let Y_k be the integer shifts of $|\det s|^{k/2} D^k \psi$. Now, we fix $k < 0$, and $f \in H_r$. By (2.7), $\|T^*_{Y_k} f\| = \|[\widehat{\psi}(s^{*-k}\cdot), \widehat{f}]\|_{L_2(\mathbb{T}^d)}$. Taking into account the fact that $f \in H_r$, we obtain from Hölder's inequality that

$$\|[\widehat{\psi}(s^{*-k}\cdot), \widehat{f}]\|^2_{L_2(\mathbb{T}^d)} \le \| \sum_{|\cdot+\alpha|\ge r} |\widehat{\psi}(s^{*-k}(\cdot+\alpha))|^2 \, [\widehat{f}, \widehat{f}]\|_{L_1(\mathbb{T}^d)}.$$

Since $\|[\widehat{f}, \widehat{f}]\|_{L_1(\mathbb{T}^d)} = \|f\|^2$, we then conclude that

$$\|T^*_{Y_k} f\|^2 \le \| \sum_{|\cdot+\alpha|\ge r} |\widehat{\psi}(s^{*-k}(\cdot+\alpha))|^2\|_{L_\infty([-\pi,\pi]^d)} \, \|f\|^2$$

Since $|s^{*-k}(\cdot+\alpha)| > \delta^{-k} r$, for some $\delta > 1$, whenever $|\cdot+\alpha| > r$, and since $s^{*-k} 2\pi\mathbb{Z}^d \subset 2\pi\mathbb{Z}^d$, we conclude that

$$\| \sum_{|\cdot+\alpha|\ge r} |\widehat{\psi}(s^{*-k}(\cdot+\alpha))|^2\|_{L_\infty([-\pi,\pi]^d)} \le \| \sum_{|\cdot+\alpha|>\delta^{-k} r} |\widehat{\psi}(\cdot+\alpha)|^2\|_{L_\infty([-\pi,\pi]^d} =: c_k.$$

Now, (4.6) implies that the series $\sum_{k\ge 0} c_k$ converges. However, for the choice $r := \delta^{-k_0}$, the above argument proves that

$$\|T^*_Y f\|^2 = \sum_{k=-\infty}^{-1} \|T^*_{Y_k} f\|^2 \le \sum_{k\ge -k_0} c_k \|f\|^2,$$

i.e., that $\|T^*_{Y,r}\| \le \sum_{k\ge -k_0} c_k$. □

6. Tight frames and orthonormal bases constructed by multiresolution

Since its introduction by Mallat and Meyer (cf. [Ma], [Me]), multiresolution has always been the prevalent approach for the construction of "good" affine systems (primarily with respect to the dilation matrix $s = 2I$). In the constructions that we are aware of, the cardinality of Ψ has always been $|\det s| - 1$, and the major effort was devoted to selecting Ψ from the refinable space in a way that the resulting affine system inherits the known "good" properties (orthonormality, Riesz basis) of the shifts $E(\phi)$, where ϕ is the scaling function. This, however, *cannot* be carried over, and *need not* be carried over to the frame constructions. Cannot, since there are intrinsic limitations here. For example, [RS1] shows that the only way to obtain redundant frames of the form $E(\Phi)$, Φ finite and compactly supported, is by adding redundant generators to a shift-invariant Riesz basis $E(\Phi_0)$. Need not, since our results suggest (and the construction in §1.3 demonstrates) that successful constructions of affine frames, even tight ones, may be carried out under minimal or no assumptions on the scaling function and its mask.

Thus, our method for constructing tight frames from multiresolution does not make any pre-assumptions on the scaling function, and, at least theoretically, should work for almost any scaling function ϕ.

The results in this section equally apply to the case when the refinable space is PSI (i.e., singly generated), or, more generally, FSI (that is, finitely generated). To simplify the presentation, we first discuss fully the PSI case, and only then sketch the possible generalizations to FSI setups.

6.1. Multiresolution with a single scaling function

The setup is as follows: ϕ in $L_2(\mathbb{R}^d)$ is given, and V_0 be the closed linear span of the shifts $E(\phi)$ of ϕ. Further, ϕ is assumed **refinable** (= is **a scaling function** = is **a father wavelet**) which means that $V_1 := D(V_0)$ is a superspace of V_0. The underlying idea of multiresolution is to select, in some clever way, the generators Ψ of the affine system X from the space V_1, in a way that their shifts $E(\Psi)$ will "complement" the shifts of ϕ. For notational convenience, we set

$$\Psi' := \Psi \cup (\phi).$$

The assumption $\Psi' \subset V_1$, is equivalent, [BDR1], to the equality

$$\widehat{\psi}(s^* \cdot) = \tau_\psi \widehat{\phi}, \quad \psi \in \Psi', \tag{6.1}$$

for some measurable $\tau := (\tau_\psi)_{\psi \in \Psi'}$ whose components are each $2\pi\mathbb{Z}^d$-periodic. The function τ_ϕ is the **refinement mask**, and the other τ_ψ's are the **wavelet masks**.

A key role in the analysis below is played by the following 2π-periodic function, which we term **the fundamental function of multiresolution**, and which is defined on $\mathbb{R}^d \backslash (2\pi\mathbb{Z}^d)$ by

$$\Theta(\omega) := \sum_{k=0}^{\infty} \Theta_k(\omega),$$

with

$$\Theta_k(\omega) := |\tau_\Psi(s^{*k}\omega)|^2 \prod_{j=0}^{k-1} |\tau_\phi(s^{*j}\omega)|^2,$$

and where

$$|\tau_\Psi|^2 := \sum_{\psi \in \Psi} |\tau_\psi|^2.$$

Note that the fundamental function depends on τ_ϕ, and on the aggregate τ_Ψ, but not on the individual wavelet masks.

In order to analyse the construction of tight frames by multiresolution, we naturally invoke the characterization of tight frames given in Corollary 5.7. The fundamental function of multiresolution enters the discussion when we substitute the various masks into the relevant affine products. Precisely, we have:

Lemma 6.2. *Assume that $\omega \in \mathbb{R}^d$, and $\omega' \in \omega + 2\pi\mathbb{Z}^d$. If $\Psi[\omega,\omega]$ and $\Psi[\omega',\omega']$ are finite, then*

$$\Psi[\omega,\omega'] - \Psi_0[\omega,\omega'] = \Theta(\omega)\widehat{\phi}(\omega)\overline{\widehat{\phi}}(\omega'). \tag{6.3}$$

where $\Psi_0[\,,\,]$ is the truncated affine product (cf. (4.13) and its preceding display). In particular, $\Theta(\omega)$ is finite.

Proof: Since we assume $\Psi[\,]$ to be finite at ω, ω', Hölder inequality guarantees the sum that defines $\Psi[\omega,\omega']$ to be absolutely convergent.

Our assumption on ω' clearly implies that $\tau(s^{*j}\omega) = \tau(s^{*j}\omega')$, for every $j \geq 0$. This, combined with the definition of Ψ and the refinability of ϕ readily implies that, for $k \geq 0$,

$$\sum_{\psi\in\Psi} \widehat{\psi}(s^{*k+1}\omega)\overline{\widehat{\psi}}(s^{*k+1}\omega') = \Theta_k(\omega)\widehat{\phi}(\omega)\overline{\widehat{\phi}}(\omega').$$

Summing the above over $k = 0, 1, 2, \ldots$ we obtain the result. □

¿From that, we get the following characterization of fundamental tight frames that can be constructed by multiresolution. In that characterization, it is useful to consider, for a given $t > 0$, the following bilinear form (defined on $\mathbb{C}^{\Psi'}$):

$$\langle v, v'\rangle_t := t v_\phi \overline{v'_\phi} + \sum_{\psi\in\Psi} v_\psi \overline{v'_\psi}\,,$$

and to abbreviate

$$\mathcal{Z} := 2\pi(s^{*-1}\mathbb{Z}^d/\mathbb{Z}^d). \tag{6.4}$$

Theorem 6.5. *Let ϕ be a refinable function, Ψ a finite set of wavelets, and τ the corresponding refinement-wavelet mask as above. Assume that (i) ϕ satisfies (4.6), (ii) $\widehat{\phi}(0) := \lim_{\omega\to 0}\widehat{\phi}(\omega) = 1$, and (iii) the mask τ is essentially bounded. Then Ψ generates a fundamental tight affine frame with bound C if and only if the following two conditions hold:*

(a) For a.e. ω, $\lim_{n\to-\infty}\Theta(s^{*n}\omega) = C$.

(b) For a.e. $\omega, \omega' \in \mathbb{R}^d$, if $\kappa(\omega - \omega') = 1$, then

$$\langle \tau(\omega), \tau(\omega')\rangle_{\Theta(s^*\omega)} = 0,$$

unless $\widehat{\phi}$ vanishes identically on either $\omega + 2\pi\mathbb{Z}^d$ or $\omega' + 2\pi\mathbb{Z}^d$.

In particular, in case $\Theta = 1$ a.e., X is a fundamental tight affine frame if the vectors τ and $E^\nu\tau$ are perpendicular a.e., for every $\nu \in \mathcal{Z}\backslash 0$.

Proof: We invoke Corollary 5.7 (the fact that Ψ satisfies (4.6) follows from assumptions (i) and (iii) of the present theorem). We start with the studying of the diagonal affine product $\Psi[\omega,\omega]$. Since, for all $n \in \mathbb{Z}$,

$$\Psi[s^{*n}\omega, s^{*n}\omega] - \Psi_0[s^{*n}\omega, s^{*n}\omega] = \sum_{\psi\in\Psi}\sum_{k>0} |\widehat{\psi}(s^{*n+k}\omega)|^2 = \Psi[\omega,\omega] - \Psi_n[\omega,\omega],$$

we obtain from Lemma 6.2 (after noting that (6.3) holds as long as the left hand side of that identity is well defined) that

$$\Psi[\omega,\omega]-\Psi_n[\omega,\omega]=\Theta(s^{*n}\omega)|\widehat{\phi}(s^{*n}\omega)|^2.$$

Letting $n\to-\infty$, we conclude (from the fact that $\widehat{\phi}(0)=1$) that $\Psi[\omega,\omega]=\lim_{n\to-\infty}\Theta(s^{*n}\omega)$. Thus, the diagonal condition in Corollary 5.7 is equivalent to assumption (a) here.

Next, adopting (a) (without loss), we know that $\Psi[\,]$ is finite a.e., and hence we may invoke now Lemma 6.2 to conclude that, if $\omega,\omega'\in\mathbb{R}^d$, and $\kappa(\omega-\omega')=0$, then

$$\Psi[\omega,\omega']=\Theta(\omega)\widehat{\phi}(\omega)\overline{\widehat{\phi}}(\omega')+\sum_{\psi\in\Psi}\widehat{\psi}(\omega)\overline{\widehat{\psi}}(\omega').$$

We iterate now once again with the refinement equation and the wavelet definition to obtain:

$$\Psi[\omega,\omega']=\langle\tau(s^{*-1}\omega),\tau(s^{*-1}\omega')\rangle_{\Theta(\omega)}\ \widehat{\phi}(s^{*-1}\omega)\overline{\widehat{\phi}}(s^{*-1}\omega'). \tag{6.6}$$

Now, fix $\omega_0\in\mathbb{R}^d$ and $\nu\in 2\pi(\mathbb{Z}^d\backslash s^*\mathbb{Z}^d)$. We vary ω over $\omega_0+2\pi s^*\mathbb{Z}^d$, and we vary ω' over $\omega_0+\nu+2\pi s^*\mathbb{Z}^d$. Regardless of the specific choice of ω,ω', $\kappa(\omega-\omega')=0$, and the above computation of $\Psi[\omega,\omega']$ is valid. Furthermore,

$$\langle\tau(s^{*-1}\omega),\tau(s^{*-1}\omega')\rangle_{\Theta(\omega)}=\langle\tau(s^{*-1}\omega_0),\tau(s^{*-1}(\omega_0+\nu))\rangle_{\Theta(\omega)}.$$

Thus, for $\Psi[\omega,\omega']$ to be 0 for each of the above ω,ω' it is necessary and sufficient that one of the following holds: either $\widehat{\phi}$ vanishes on $s^{*-1}\omega_0+2\pi\mathbb{Z}^d=s^{*-1}\omega+2\pi\mathbb{Z}^d$, or $\widehat{\phi}$ vanishes on $s^{*-1}(\omega_0+\nu)+2\pi\mathbb{Z}^d=s^{*-1}\omega'+2\pi\mathbb{Z}^d$, or $\langle\tau(s^{*-1}\omega),\tau(s^{*-1}\omega')\rangle_{\Theta(\omega)}=0$. Since $\kappa(s^{*-1}\omega-s^{*-1}\omega')=1$, this triple condition is equivalent to (b).

If $\Theta=1$ almost everywhere, then (a) certainly holds. Furthermore, in this case $\langle,\rangle_{\Theta(\omega)}$ is the usual inner product, and hence the perpendicularity assumption assumed on τ implies the satisfaction of (b) as well, hence X is a tight frame. $\square$

Note that the theorem allows the construction of tight frames in two steps: in the first, one determines the aggregate $|\tau_\Psi|$, to guarantee, say, that $\Theta=1$ (or, at least, that (a) holds). Only then, one may proceed to construct the individual masks (τ_ψ) with the given aggregate $|\tau_\Psi|$, so that they satisfy the orthogonality condition (b).

In practice, it may be hard to select $(\tau_\psi)_\psi$ so that the fundamental function Θ is 1. For this reason, it is worth emphasizing the following important special case of Theorem 6.5, of which Theorem 1.7 is still a special case:

Corollary 6.7. *Let ϕ be a refinable function, Ψ a finite set of wavelets, and τ the corresponding refinement-wavelet mask as above. Assume that ϕ satisfies (4.6), and* $\lim_{\omega\to 0}\widehat{\phi}(\omega)=1$. *If, for a.e. ω, and every $\nu\in\mathcal{Z}$ (cf. (6.4))*

$$\langle\tau(\omega),\tau(\omega+\nu)\rangle=\delta_\nu,$$

then Ψ generates a fundamental tight frame with frame bound 1.

Proof: We prove the corollary by verifying that conditions (a) and (b) of Theorem 6.5 hold here.

First, denoting $a_k := |\tau_\phi(s^{*k}\omega)|^2$, we see that

$$\sum_{k=0}^{m}(1-a_k)\prod_{j=0}^{k-1} a_j = 1 - \prod_{j=0}^{m} a_j.$$

Since $\Theta(\omega)$ is the limit, as $m \to \infty$, of the above expression, we obtain that

$$\Theta(\omega) = 1 - \prod_{j=0}^{\infty} a_j. \tag{6.8}$$

The infinite product here can be shown to vanish a.e. on the set

$$K := \{\omega \in \mathbb{R}^d : \widehat{\phi}_{|\omega+2\pi\mathbb{Z}^d} \neq 0\},$$

and hence $\Theta = 1$ on K. Since we assume that $\widehat{\phi}$ is continuous at the origin and does not vanish there, K contains some neighborhood of the origin. Consequently, (a) of Theorem 6.5 holds here (with $C = 1$).

As to condition (b) of that theorem, this condition is vaciously valid for $w \notin K$, hence we assume without loss that $\omega \in K$. Then, for a.e. $\omega \in K$, $\Theta(\omega) = 1$; at the same time, (6.8) implies that $a_0(\Theta(s^*\omega) - 1) = \Theta(\omega) - 1 = 0$, hence one of the two must hold: (i) $\Theta(s^*\omega) = 1$, or (ii) $\tau_\phi(\omega) = 0$. Under either of these two assumptions, the requirement $\langle \tau(\omega), \tau(\omega')\rangle_{\Theta(s^*\omega)} = 0$ in Theorem 6.5 is equivalent to $\langle \tau(\omega), \tau(\omega')\rangle = 0$, that we assume here. □

Remark. Note that the above corollary requires Ψ to have a minimal cardinality of $|\det s| - 1$. Moreover, when $\#\Psi = |\det s| - 1$, the matrix

$$\Delta := (E^\nu \tau_\psi)_{\psi\in\Psi',\nu\in\mathcal{Z}}$$

is square, and the column orthogonality assumption then implies that the matrix is unitary, and in particular that

$$\sum_{\nu\in\mathcal{Z}} |E^\nu \tau_\phi|^2 = 1, \quad a.e.$$

Refinement masks that the satisfy the above are known as **conjugate quadrature filters** (CQF). Thus, in essence, every unitary extension of the row $(E^\nu \tau_\phi)_{\nu\in\mathcal{Z}}$ of a CQF mask results in a column τ that whose masks defines wavelets that generate tight frames. Several constructive methods of such unitary extensions are described [RiS1], [RiS2], and [JS], as a part of an effort to construct multivariate orthonormal wavelets. Conversely, a generating set Ψ that consists of $|\det s| - 1$ functions which is constructed as above, can form a tight frame only if τ_ϕ is CQF. However, if we use more than $|\det s| - 1$ generators, there does not seem to be any a-priori restriction on the mask τ_ϕ (other than the most basic conditions, such as $\tau_\phi(0) = 1$).

We now turn our attention to orthonormal systems. First, it is easy to conclude (say, from the analysis of [BDR2]) that for X constructed from a PSI multiresolution to be orthonormal, it is necessary that we do not have more than $|\det s| - 1$ wavelets. Second, Corollary 5.8 characterizes all fundamental tight frames that are orthonormal. However, since the additional assumption in that corollary is in terms of the constructed wavelets, and not in terms of the masks and/or the scaling function, it is worth making the following remark:

Corollary 6.9. *Let X be a fundamental tight frame generated by Ψ whose cardinality is $|\det s|-1$, and which is constructed by multiresolution in the way detailed in Corollary 6.7. Then the following statements are equivalent:*
(a) X *is orthonormal.*
(b) $\|\phi\| = 1$.

Proof: Since τ (the refinement-wavelet mask) is assumed to be unit a.e., we easily conclude (by integrating the equality $|\widehat{\phi}|^2 = (\sum_{\psi\in\Psi'} |\tau_\psi|^2)|\widehat{\phi}|^2 = \sum_{\psi\in\Psi'} |\widehat{\psi}(s^*\cdot)|^2$) that

$$|\det s|^{-1}(\|\widehat{\phi}\|^2 + \sum_{\psi\in\Psi} \|\widehat{\psi}\|^2) = \|\widehat{\phi}\|^2.$$

Thus,

$$\sum_{\psi\in\Psi} \|\psi\|^2 = (|\det s| - 1)\|\phi\|^2. \tag{6.10}$$

Now, if X is orthonormal, each ψ has norm 1, and (since with assume to have exactly $|\det s| - 1$ wavelets) we obtain that $\|\phi\| = 1$.

Conversely, since X is a tight frame with frame bound 1, then, Proposition 2.3, X, hence Ψ, lies in the closed unit ball of L_2, and therefore $\sum_{\psi\in\Psi} \|\psi\|^2 \le \#\Psi = |\det s| - 1$. However, upon assuming $\|\phi\| = 1$, we obtain from (6.10) that equality holds in the last inequality, and hence that Ψ lies on the unit sphere of L_2. By Corollary 5.8, X is orthonormal. □

Discussion. It is easy to generate examples of fundamental tight frames that cannot be constructed by the unitary extension principle; moreover, these frames may be orthonormal, while $\|\phi\| \neq 1$.

For example, let ϕ_0 be a refinable function and let Ψ be a wavelet set that is derived from ϕ_0 by MRA.

We now switch to another generator, ϕ, of V_0 defined by

$$\widehat{\phi} := t\widehat{\phi_0},$$

for some 2π-periodic t that vanishes on a null-set only, and that satisfies, $\lim_{\omega\to 0} t(\omega) = t(0) = 1$. Denoting by $\widetilde{\tau}$ the original refinement-wavelet mask, the new refinement-wavelet mask, τ (with respect to the same wavelet set Ψ) satisfies

$$\tau_\phi = \frac{t(s^*\cdot)}{t}\widetilde{\tau}_{\phi_0},$$

and

$$\tau_\psi = \frac{1}{t}\widetilde{\tau}_\psi, \quad \psi \in \Psi.$$

Denoting by Θ_0 the fundamental function of the original MRA construction, and by Θ the fundamental function of the modified MRA construction, it is easy to see that

$$\Theta = \Theta_0/|t|^2.$$

With that in hand, one observes that $(\Theta_0, \widetilde{\tau})$ satisfy conditions (a,b) of Theorem 6.5 if and only if (Θ, τ) satisfy these conditions. This must be so, since the theorem *characterizes* the tightness of the system generated by Ψ and both MRA constructions result at the same wavelet set Ψ.

However, unless t is unitary, at most one the fundamental functions Θ, Θ_0 can be constant, which means that at least one of the two constructions cannot be performed by the unitary extension recipe of Corollary 6.7. Furthermore, since t is more or less arbitrary, it is clear that we can choose it to guarantee $\|\phi\| \neq 1$, regradless of the fact whether Ψ generates an orthonormal system. $\square$

Example. We illustrate the above discussion with a simple example. Let $\phi_1 := \chi_{[0,1]}$, $\phi_2 := \chi_{[0,2]}/2$, and $\phi_3 := \chi_{[0,3]}/3$, where χ_Ω is the support function of Ω. All three functions are refinable, have mean-value 1, and their shifts span the same refinable space V_0; the orthonormal Haar wavelet system can thus be derived from the MRA based on either of these three functions.

The mask of ϕ_1 is CQF and the unitary extension leads here, indeed, to that Haar wavelet. The mask of ϕ_3 is also CQF, but the unitary extension cannot yield the Haar wavelets (e.g., since $\|\phi_3\| \neq 1$), though, of course, one obtains a tight frame. Finally, the mask of ϕ_2 is *not* CQF. The previous discussion shows that the MRAs constructions that lead to the Haar wavelet from either ϕ_2 or ϕ_3 cannot invoke the unitary extension principle: the two underlying fundamental functions are not constant. $\square$

Oversampling, continued. We continue the analysis of the oversampling procedure that was outlined in §4.3. We now assume that the oversampling L is the inverse s^{-1} of the dilation matrix s. This, of course, violates condition (4.18). Indeed, as is pointed out in [CS3], the oversampling of the univariate dyadic orthonormal Haar system by 2 does not yield a tight frame. As a matter of fact, the following result shows, in particular, that oversampling by a factor of 2 of any dyadic affine system, which is generated from MRA by a compactly supported scaling function, can never yield a tight frame; this is regardless whether the original system is a frame or not.

Proposition 6.11. *Let ϕ be a refinable function, and Ψ a finite subset of V_1. Assume that Ψ satisfies (4.6), that $\Psi[\,]$ is finite a.e., and that $\widehat{\phi}$ vanishes almost nowhere. Let X be the affine system generated by Ψ, and let Y be the oversampling of X with respect to the lattice $s^{-1}\mathbb{Z}^d$. Then Y is not a fundamental tight frame.*

Remarks. We first stress that X is not assumed to be frame, *a fortiori* it is not assumed to be a tight frame. Also, the proof below shows that the condition $\widehat{\phi} \neq 0$ a.e. can be relaxed; however, without any restriction on $\operatorname{supp} \widehat{\phi}$ the statement is not valid: the univariate wavelet that is derived from the sinc-function (and whose Fourier transform is the support function of $[-2\pi, -\pi] \cup [\pi, 2\pi]$) generates an orthonormal dyadic affine system. Oversampling by an integer amount results in a fundamental semi-orthonormal tight frame.

Proof: The new lattice of shifts is $s^{-1}\mathbb{Z}^d$, hence its dual is $2\pi s^*\mathbb{Z}^d$. Thus, in a way entirely analogous to (4.17), we find that the dual Gramian fibers of the quasi-affine Y^q are indexed by $2\pi s^*\mathbb{Z}^d$, and the $(0,\alpha)$-entry being

$$|\det s| \sum_{\psi \in \Psi} \sum_{k=\kappa_{s^{-1}}(\alpha)}^{\infty} \widehat{\psi}(s^{*k}(\omega))\overline{\widehat{\psi}}(s^{*k}(\omega + \alpha)).$$

We choose a non-diagonal entry $(0,\alpha)$ with $\kappa_{s^{-1}}(\alpha)=0$ i.e., $\alpha = s^*\alpha_0$, for some $\alpha_0 \in 2\pi(\mathbb{Z}^d\backslash s^*\mathbb{Z}^d)$. From that it follows that, with $\omega_0 := s^{*-1}\omega$, our $(0,\alpha)$-entry of the dual Gramian of Y^q is

$$|\det s| \sum_{\psi\in\Psi} \sum_{1}^{\infty} \widehat{\psi}(s^{*k}(\omega_0))\overline{\widehat{\psi}}(s^{*k}(\omega_0+\alpha_0)).$$

This expression was computed in Lemma 6.2, and was shown to be

$$|\det s|\Theta(\omega_0)\widehat{\phi}(\omega_0)\overline{\widehat{\phi}}(\omega_0+\alpha_0).$$

Since we assume that, up to a null set, $\operatorname{supp}\widehat{\phi} = \mathbb{R}^d$, it follows, Corollary 5.7, that, if Y is tight, $\Theta = 0$ a.e. However, this is absurd since each summand of Θ is non-negative and the first summand is $|\tau_\Psi|^2$: if this summand is 0 a.e., all our wavelets are 0. □

6.2. Multiresolution with several scaling functions

Here, we assume that the space V_0 is FSI and refinable. This means by definition that the shifts $E(\Phi)$, $\Phi \subset L_2$ finite, are fundamental in V_0, and that $V_1 := D(V_0)$ is a superspace of V_0.

Regardless of any further assumptions, this implies that

$$\widehat{\Phi}(s^*\cdot) = \tau_\Phi\widehat{\Phi},$$

for some $\Phi\times\Phi$ matrix τ_Φ, whose entries are measurable and 2π-periodic. The wavelets Ψ are constructed with the aid of another matrix, τ_Ψ whose entries are 2π-periodic and measurable, and whose order is $\Psi\times\Phi$, that is

$$\widehat{\Psi}(s^*\cdot) = \tau_\Psi\widehat{\Phi}.$$

The augmented matrix τ has now the order of $(\Phi\cup\Psi)\times\Phi$.

The arithmetic manipulations presented in the previous section can be carried *verbatim* to the FSI setup, with an appropriate conversion of the various expressions. For example, the orthogonality conditions expressed in Corollary 6.7 should now read as $\tau^*(\omega)\tau(\omega+\nu) = \delta_\nu I$, with I the $\Phi\times\Phi$ identity matrix. The function Θ_k is replaced by the $\Phi\times\Phi$ matrix

$$\tau_\Phi(\omega)^*\tau_\Phi(s^*\omega)^*\ldots\tau_\Phi(s^{*k-1}\omega)^*\tau_\Psi(s^{*k}\omega)^*\tau_\Psi(s^{*k}\omega)\tau_\Phi(s^{*k-1}\omega)\ldots\tau_\Phi(\omega).$$

The fundamental function Θ is, thus, a non-negative definite $\Phi\times\Phi$ matrix, and should be interpreted in Lemma 6.2 as a bilinear form.

We checked, for example, the details of Corollary 6.7: while the product $\prod_{j=0}^{m} a_j$ that appears in the proof of that corollary is now a matrix product, and may not converge to 0, it suffices to show that this product converges to 0 as a bilinear form acting on a fixed vector pair $(\widehat{\Phi}(\omega),\widehat{\Phi}(\omega'))$, something that follows easily. Further, the continuity assumption on $\widehat{\phi}$ at the origin should be replaced by the assumption that $\lim_{\omega\to 0}(\widehat{\Phi}^*\Theta\widehat{\Phi})(\omega) = 1$.

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