Introduction to Shift-Invariant Spaces
I: Linear Independence

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Abstract. Shift-invariant spaces play an increasingly important role in various areas of mathematical analysis and its applications. They appear either implicitly or explicitly in studies of wavelets, splines, radial basis function approximation, regular sampling, Gabor systems, uniform subdivision schemes, and perhaps in some other areas. One must keep in mind, however, that the shift-invariant system explored in one of the above-mentioned areas might be very different from those investigated in other areas. For example, in *splines* the shift-invariant system is generated by elements of compact support, while in the area of *sampling* the shift-invariant system is generated by band-limited elements, i.e., elements whose Fourier transform is compactly supported.

The *theory of shift-invariant spaces* attempts to provide a uniform platform for all these different investigations of shift-invariant spaces. The two main pillars of that theory are the study of the *approximation properties* of such spaces, and the study of *generating sets* for these spaces. Another survey article in this volume (*A Survey on L_2-Approximation Order From Shift-invariant Spaces*, by Kurt Jetter and Gerlind Plonka) provides excellent up-to-date information about the first topic. The present article is devoted to the latter topic.

My goal in this article is to provide the reader with an easy and friendly introduction to the basic principles of that topic. The core of the presentation is devoted to the study of *local principal shift-invariant* spaces, while the more general cases are treated as extensions of that basic setup.

§1. Introduction

A shift-invariant (SI) space is a linear space S consisting of functions (or distributions) defined on $\mathbb{R}^d$ $(d \geq 1)$, that is invariant under lattice translations:

$$f \in S \Longrightarrow E^j f \in S, \quad j \in \mathcal{L}, \tag{1}$$

where E^j is the **shift operator**

$$(E^j f)(x) = f(x - j). \tag{2}$$

The most common choice for $\mathcal{L}$ is the integer lattice $\mathcal{L} = \mathbb{Z}^d$. Here and hereafter we use the notion of a **shift** as a synonym to *integer translation* and/or *integer translate*. Given a set Φ of functions defined on $\mathbb{R}^d$, we say that Φ **generates** the SI space S, if the collection

$$E(\Phi) := (E^j\phi : \ \phi \in \Phi, \ j \in \mathbb{Z}^d) \tag{3}$$

of shifts of Φ is **fundamental** in S, i.e., the span of $E(\Phi)$ is dense in S. Of course, the definition just given assumes that S is endowed with some topology, so we will give a more precise definition of this notion in the sequel.

Shift-invariant spaces are usually defined in terms of their generating set Φ, and they are classified according to the properties of the generating set. For example, a **principal shift-invariant (PSI)** space is generated by a single function, i.e., $\Phi = \{\phi\}$, and a **finitely generated shift-invariant (FSI)** space is generated by a finite Φ. In some sense, the PSI space is the simplest type of SI space. Another possible classification is according to smoothness or decay properties of the generating set Φ. For example, an SI space is **local** if it is generated by a *compactly supported* Φ. Local PSI spaces are, probably, the bread and butter of shift-invariant spaces. At the other end of this classification are the **band-limited** SI spaces; their generators have their Fourier transforms supported in some given compact domain.

Studies in several areas of analysis employ, explicitly or implicitly, SI spaces, and the *Theory of Shift-Invariant Spaces* attempts to provide a uniform platform for all these studies. In certain areas, the SI space appears as an *approximation space*. Precisely, in *Spline Approximation*, local PSI and local FSI spaces are employed, the most notable examples of such spaces are the box splines and the exponential box spline spaces (cf. [7], [9] and the references therein). In contrast, in *radial basis function approximation*, PSI spaces generated by functions of global support is typical; e.g., fundamental solutions of elliptic operators are known to be useful generators there (cf. [10], [18] and the references therein). In *Uniform Sampling*, band-limited SI spaces are the rule (cf. e.g., [29]). *Uniform Subdivision* (cf. [16], [11]) is an example where SI spaces appear in an implicit way: the SI spaces appear there in the analysis, not in the setup. The SI spaces in this area are usually local PSI/FSI, and possess the additional important property of *refinability* (that we define and discuss in the body of this article).

In other areas, the shift-invariant space is the 'building block' of a larger system, or, to put it differently, a multitude of SI spaces is employed simultaneously. In the area of *Weyl-Heisenberg* (WH, also known as *Gabor*) *systems* (cf. [19]), the SI space S is PSI/FSI and is either local or 'near-local' (e.g., generated by functions that decay exponentially fast at ∞; the generators are sometime referred to as 'windows'). The complete system is then of the form $(S_i)_{i\in I}$, with each S_i a *modulation* of S, i.e., the multiplication product of S by a suitable exponential function. Finally, in the area of *Wavelets* (cf. [31], [15], [40]), SI spaces appear in two different ways. First, the *wavelet system* is of the form $(S_i)_{i\in I}$ where all the S_i spaces obtained from a single SI space (which, again, is a PSI/FSI space and is usually local), but this time *dilation* replaces the modulation from the WH case. Second, refinable PSI/FSI spaces are crucial in the construction of wavelet systems via the vehicle of *Multiresolution Analysis*.

There are two foci in the study of shift-invariant spaces. The first is the study of their approximation properties (cf. [9], [4], [5], [6]). The second is the study of the shift-invariant system $E(\Phi)$ as a basis for the SI space it spans. The present article discusses the basics of that latter topic. In view of the prevalence of local PSI spaces in the relevant application areas, we develop first the theory for that case, §2, and then discuss various extensions of the basic theory. Most of the theory presented in the present article was developed in the early 90's, but, to the best of my knowledge, has not been summarized before in a self-contained manner.

The rest of this article is laid out as follows:

§2. Bases for PSI spaces

Let ϕ be a compactly supported function in $L_1(\mathbb{R}^d)$, or, more generally, a compactly supported distribution in $\mathcal{D}'(\mathbb{R}^d)$. We analyse in detail the "basis" properties of the set of shifts $E(\phi)$ (cf. (3)). The compact support assumption simplifies some technical details in the analysis, and, more importantly, allows the introduction and analysis of a fine scale of possible "basis" properties.

2.1. The Analysis and Synthesis Operators

The basic operators associated with a shift-invariant system are the *analysis operator* and the *synthesis operator*. There are several different variants of these operators, due to different choices of the domain of the corresponding map. It is then important to stress right from the beginning that these differences are *very significant*. We illustrate this point in the sequel.

Let

$$\mathcal{Q} \tag{4}$$

be the space of all complex valued functions defined on $\mathbb{Z}^d$. More generally, let Φ be some set of functions/distributions; letting the elements in Φ index themselves, we set

$$\mathcal{Q}(\Phi) := \mathcal{Q} \times \Phi. \tag{5}$$

The space $\mathcal{Q}(\Phi)$ is equipped with the topology of *pointwise convergence* (which makes it into a Fréchet space). For the lion's share of the study below, however, it suffices to treat $\mathcal{Q}$ merely as a linear space.

Given a finite set Φ of compactly supported distributions, the **synthesis operator $\mathcal{T}_\Phi$** is defined by

$$\mathcal{T}_\Phi : \mathcal{Q}(\Phi) \to S_\star(\Phi) : c \mapsto \sum_{\phi\in\Phi} \sum_{j\in\mathbb{Z}^d} c(j,\phi)\ E^j\phi.$$

The notation

$$S_\star(\Phi)$$

that we have just used stands, by definition, for the range of $\mathcal{T}_\Phi$. In this section, we focus on the PSI case i.e., the case when Φ is a singleton $\{\phi\}$. Thus:

$$\mathcal{T}_\phi : \mathcal{Q} \to S_\star(\phi) : c \mapsto \sum_{j\in\mathbb{Z}^d} c(j)\ E^j\phi.$$

Example. If ϕ is the support function of the interval $[0,1]$ (in one dimension), then $S_\star(\phi)$ is the space of all piecewise-constants with (possible) discontinuities at $\mathbb{Z}$. $\square$

Note that, thanks to the compact support assumption on ϕ, the operator $\mathcal{T}_\phi$ is well-defined on the entire space $\mathcal{Q}$. In the sequel, we either consider the operator $\mathcal{T}_\phi$ as above, or inspect its restriction to some subspace $C \subset \mathcal{Q}$ (and usually equip that subspace with a stronger topology). The compact support assumption on ϕ "buys" the largest possible domain, viz., $\mathcal{Q}$, hence the full range of subdomains to inspect. The properties of $\mathcal{T}_\phi$ (or a restriction of it) that are of immediate interest are the *injectivity* of the operator, its *continuity*, and its *invertibility*. Of course, the two latter properties make sense only if we rigorously define the target space, and equip the domain and the target space with appropriate topologies.

Discussion. As mentioned before, the choice of the domain of $\mathcal{T}_\phi$ is crucial. Consider for example the injectivity property of $\mathcal{T}_\phi$: this property, known as the linear independence of $E(\phi)$, is one of the most fundamental properties of the shift-invariant system. On the other hand, the restriction of $\mathcal{T}_\phi$ to, say, $\ell_2(\mathbb{Z}^d)$ is always injective (recall that ϕ is assumed to be compactly supported), hence that restricted type of injectivity is void of any value. Consequently, keeping the domain of $\mathcal{T}_\phi$ 'large' allows us to get meaningful definitions, hence is important for the development of the theory. $\square$

An important alternative to the above is to study the formal adjoint $\mathcal{T}_\phi^*$ of $\mathcal{T}_\phi$, known as the **analysis operator** and defined by

$$\mathcal{T}_\phi^* : f \mapsto (\langle f, E^j \phi\rangle)_{j\in\mathbb{Z}^d}.$$

(Here and elsewhere, we write

$$\langle f, \lambda\rangle$$

for the action of the linear functional λ on the function f.) We intentionally avoided the task of defining the domain of this adjoint: it is defined on the largest domain that can make sense! For example, if $\phi \in L_2(\mathbb{R}^d)$ (and is of compact support), $\mathcal{T}_\phi^*$ is naturally defined on $L_{2,\mathrm{loc}}(\mathbb{R}^d)$. On the other hand, if ϕ is merely a compactly supported distribution, f should be assumed to be a $C^\infty(\mathbb{R}^d)$-function. In any event, and unless the surjectivity of $\mathcal{T}_\phi^*$ is the goal, the target space here can be taken to be $\mathcal{Q}$.

The study of $E(\phi)$ via the analysis operator is done by considering pre-images of certain sequences in the target space. For example, a non-empty pre-image $T_\phi^{*-1}\delta$ of the δ-sequence indicates that the shifts of ϕ can, at least to some extent, be separated. A much finer analysis is obtained by studying properties of the functions in $T_\phi^{*-1}\delta$; first and foremost *decaying* properties of such functions. The desire is to find in $T_\phi^{*-1}\delta$ a function that decays as fast as possible, ideally a compactly supported function. Note that $f \in T_\phi^{*-1}\delta$ if and only if $E(f)$ forms a dual basis to the shifts of ϕ in sense that

$$\langle E^k f, E^j \phi\rangle = \delta_{j,k}.$$

One of the advantages in this complementary approach is that certain functions in $T_\phi^{*-1}\delta$ can be represented explicitly in terms of ϕ, hence their decay properties can be examined directly.

2.2. Basic Theory: Linear Independence in Local PSI Spaces

We say that the shifts of the compactly supported distribution ϕ are **(globally) linearly independent (=:gli)** if $\mathcal{T}_\phi$ is injective, i.e., if the condition

$$(\mathcal{T}_\phi\, c = 0, \quad \text{for some } c \in \mathcal{Q}) \Longrightarrow c = 0$$

holds. The discussion concerning this basic, important, property is two-fold: first, we discuss characterizations of the linear independence property that are useful for checking its validity. Then, we discuss other properties, that are either equivalent to linear independence or are implied by it, and that are useful for the construction of approximation maps into $S_\star(\phi)$.

We start with the first task: characterizing linear independence in terms of more verifiable conditions. To this end, we recall that, since ϕ is compactly supported, its Fourier transform extends to an entire function. We still denote that extension by $\widehat{\phi}$. The following fairly immediate observation is crucial:

Observation 6. $\ker \mathcal{T}_\phi$ is a *closed shift-invariant* subspace of $\mathcal{Q}$.

In [27], closed SI subspaces of $\mathcal{Q}$ are studied. It is proved there that $\mathcal{Q}$ admits *spectral analysis*, and, moreover, admits *spectral synthesis*. The latter property means that every closed SI subspace of $\mathcal{Q}$ contains a dense exponential subspace (here 'an exponential' is a linear combination of the restriction to $\mathbb{Z}^d$ of products of exponential functions by polynomials). More details on Lefranc's synthesis result, as well as complete details of its proof can be found in [8]. We need here only the much weaker *analysis* part of Lefranc's theorem, which says the following:

Theorem 7. *Every (nontrivial) closed SI subspace of $\mathcal{Q}$ contains an exponential sequence*

$$e_\theta : j \mapsto \mathrm{e}^{\theta \cdot j}, \quad \theta \in \mathbb{C}^d.$$

We sketch the proof below, and refer to [34] for the complete proof.

Proof (sketch): The continuous dual space of $\mathcal{Q}$ is the space $\mathcal{Q}_0$ of all finitely supported sequences, with $\langle \lambda, c \rangle := \sum_{j \in \mathbb{Z}^d} \lambda(j) c(j)$. Given a closed non-zero SI subspace $C \subset \mathcal{Q}$, its annihilator $C^\perp$ in $\mathcal{Q}_0$ is a proper SI subspace. The sequences $C_+^\perp$ in $C^\perp$ that are entirely supported on $\mathbb{Z}_+^d$ can be viewed as polynomials via the association

$$Z : c \mapsto \sum_{j \in \mathbb{Z}_+^d} c(j) X^j,$$

with X^j the standard monomial. Since $C^\perp$ is SI, $Z(C_+^\perp)$ is an ideal in the space of all d-variate polynomials. Since $C^\perp$ is proper, $Z(C_+^\perp)$ cannot contain any monomial. Hilbert's *(Weak) Nullstellensatz* then implies that the polynomials in $Z(C_+^\perp)$ all vanish at some point $\mathrm{e}^\theta := (\mathrm{e}^{\theta_1}, \ldots, \mathrm{e}^{\theta_d}) \in (\mathbb{C} \backslash 0)^d$. One then concludes that the sequence

$$e_\theta : j \mapsto \mathrm{e}^{\theta \cdot j}$$

vanishes on $C^\perp$, hence, by Hahn-Banach, lies in C. □

Being unaware of Lefranc's work, Dahmen and Micchelli proved in [13] that, assuming the compactly supported ϕ to be a continuous function, $\ker \mathcal{T}_\phi$, if non-trivial, must contain an exponential e_θ. Their argument is essentially the same as Lefranc's, save some simplifications that are available due to their additional assumptions on ϕ.

The following characterization of linear independence appears first in [34]:

Theorem 8. *The shifts of the compactly supported distribution ϕ are linearly independent if and only if $\widehat{\phi}$ does not have any 2π-periodic zero (in $\mathbb{C}^d$).*

Proof (sketch): Poisson's summation formula implies that, for any $\theta \in \mathbb{C}^d$,

$$\mathcal{T}_\phi \, e_\theta = 0 \quad \Longleftrightarrow \quad (\widehat{\phi} \text{ vanishes on } -i\theta + 2\pi \mathbb{Z}^d). \tag{9}$$

Therefore, $\ker \mathcal{T}_\phi$ contains an exponential e_θ iff $\widehat{\phi}$ has a 2π-periodic zero. Since $\ker \mathcal{T}_\phi$ is SI and closed, Theorem 7 completes the proof. □

Example. Let ϕ be a univariate exponential B-spline, [17]. The Fourier transform of such a spline is of the form

$$\omega \mapsto \prod_{j=1}^{n} \int_0^1 \mathrm{e}^{(\mu_j - \mathrm{i}\omega)t}\, \mathrm{d}t,$$

with $(\mu_j)_j \subset \mathbb{C}$. One observes that $\widehat{\phi}$ vanishes exactly on the set

$$\cup_j(-\mathrm{i}\mu_j + 2\pi(\mathbb{Z}\backslash 0)).$$

From Theorem 8 it then follows that $E(\phi)$ are linearly dependent iff there exist j and k such that $\mu_j - \mu_k \in 2\pi(\mathbb{Z}\backslash 0)$. □

Example. Let ϕ be the k-fold convolution of a compactly supported distribution ϕ_0. It is fairly obvious that $\mathcal{T}_\phi$ cannot be injective in case $\mathcal{T}_{\phi_0}$ is not. Since the zero sets $\widehat{\phi}$ and $\widehat{\phi}_0$ are identical, Theorem 8 proves that the converse is valid, too: linear independence of $E(\phi_0)$ implies that of $E(\phi)$! □

Among the many applications of Theorem 8, we mention [34] where the theorem is applied to exponential box splines, [24], [44] and [37] where the theorem is used for the study of box splines with rational directions, [20] where discrete box splines are studied, and [12] where convolution products of box splines and compactly supported distributions are considered.

We now turn our attention to the second subject: useful properties of $E(\phi)$ that are implied or equivalent to linear independence. We work initially in a slightly more general setup: instead of studying $E(\phi)$, we treat any countable set F of distributions with **locally finite** supports: given any bounded set Ω, the supports of almost all the elements of F are disjoint of Ω. For convenience, we index F by $\mathbb{Z}^d$: $F = (f_j)_{j\in\mathbb{Z}^d}$. The relevant synthesis operator is:

$$\mathcal{T} : \mathcal{Q} \to \mathcal{D}'(\mathbb{R}^d) \ :\ c \mapsto \sum_{j\in\mathbb{Z}^d} c(j) f_j,$$

and is well-defined, thanks to the local finiteness assumption. The notion of *linear independence* remains unchanged: it is the injectivity of the map $\mathcal{T}$.

We start with the following result of [1]. The proof given follows [49].

Theorem 10. *Let $F = (f_j)_{j\in\mathbb{Z}^d}$ be a collection of compactly supported distributions with locally finite supports. Then the following conditions are equivalent:*

(a) F is linearly independent.

(b) There exists a dual basis to F in $\mathcal{D}(\mathbb{R}^d)$, i.e., a sequence $G := (g_j)_{j\in\mathbb{Z}^d} \subset \mathcal{D}(\mathbb{R}^d)$ such that

$$\langle g_k, f_j\rangle = \delta_{j,k}.$$

(c) For every $j \in \mathbb{Z}^d$, there exists a bounded $A_j \subset \mathbb{R}^d$ such that, if $\mathcal{T}c$ vanishes on A_j, we must have $c(j) = 0$.

Proof (sketch): Condition (c) clearly implies (a). Also, (b) implies (c): with $G \subset \mathcal{D}(\mathbb{R}^d)$ the basis dual to F, we have that $\langle g_j, \mathcal{T}c\rangle = c(j)$, hence we may take A_j to be any bounded open set that contains $\operatorname{supp} g_j$.

(a)$\Longrightarrow$(b): Recall that $\mathcal{Q}_0$ is the collection of finitely supported sequences in $\mathcal{Q}$. It is equipped with the inductive-limit topology, a discrete analog of the $\mathcal{D}(\mathbb{R}^d)$-topology. The only facts required on this topological space $\mathcal{Q}_0$ are that (a) $\mathcal{Q}$ and $\mathcal{Q}_0$ are each the continuous dual space of the other, and (b) $\mathcal{Q}_0$ does not contain proper dense subspaces (cf. [48] for details). From that and the definition of $\mathcal{T}$, one concludes that $\mathcal{T}$ is continuous, that its adjoint is the operator

$$\mathcal{T}^* : \mathcal{D}(\mathbb{R}^d) \to \mathcal{Q}_0 \ : \ g \mapsto (\langle g, f_j \rangle_{j \in \mathbb{Z}^d}),$$

and that $\mathcal{T}^{**} = \mathcal{T}$. Thus, if $\mathcal{T}$ is injective, then $\mathcal{T}^*$ has a dense range, hence must be surjective. This surjectivity implies, in particular, that all sequences supported at one point are in the range of $\mathcal{T}^*$, and (b) follows. $\square$

For the choice $F = E(\phi)$, the sets $(A_j)_j$ are obtained by shifting A_0, and a dual basis G can be chosen to have the form $E(g_0)$. Moreover, if $F = E(\Phi)$, Φ finite, the sets $(A_j)_j$ are still obtained as the shifts of some finitely many compact sets (viz., with $E(G)$ the dual basis of $E(\Phi)$ which is guaranteed to exist by Theorem 10, the finitely many compact sets are the supports of the functions in G). Thus, for this case the sets $(A_j)_j$ are locally finite. With this in hand, [22] concluded the following from Theorem 10:

Corollary 11. *Let Φ be a finite set of compactly supported distributions. If $E(\Phi)$, the set of shifts of Φ, is linearly independent, then every compactly supported $f \in S_\star(\Phi)$ is a finite linear combination of $E(\Phi)$.*

Proof: Let $E(G) \subset \mathcal{D}(\mathbb{R}^d)$ be the basis dual to $E(\Phi)$ from Theorem 10. Given a compactly supported f, we have for every $g \in G$, and for almost every $j \in \mathbb{Z}^d$, that the function $E^j g$ has its support disjoint from $\operatorname{supp} f$. This finishes the proof since, if $f = \mathcal{T}c = \sum_{\phi\in\Phi} \sum_{j \in \mathbb{Z}^d} c(j,\phi) E^j \phi$, then $\langle f, E^j g \rangle = c(j, \phi)$. $\square$

Theorem 10 allows us, in the presence of linear independence, to construct projectors into $S_\star(\phi)$ of the form $\mathcal{T}_\phi \mathcal{T}_g^*$ that are based on compactly supported functions. In fact, the theorem also shows that nothing less than linear independence suffices for such construction.

Corollary 12. *Let $\phi \in L_2(\mathbb{R}^d)$ be compactly supported, and assume $E(\phi)$ to be orthonormal. Then $E(\phi)$ is linearly independent.*

The next theorem summarizes some of the observations made above, and adds a few more:

Theorem 13. *Let ϕ be a compactly supported distribution. Consider the following conditions:*

(gli) global linear independence: $\mathcal{T}_\phi$ is injective.

(ldb) local dual basis: $E(\phi)$ has a dual basis $E(g)$, $g \in \mathcal{D}(\mathbb{R}^d)$.

(ls) local spanning: all compactly supported elements of $S_\star(\phi)$ are finitely spanned by $E(\phi)$.

(ms) minimal support: for every compactly supported $f \in S_\star(\phi)$, there exists some $j \in \mathbb{Z}^d$ such that $\operatorname{supp} \phi$ lies in the convex hull of $\operatorname{supp} E^j f$.

Equality can happen only if $f = cE^j\phi$, for some constant c, and some $j \in \mathbb{Z}^d$.

Then (gli) $\iff$ (ldb)$\Longrightarrow$ (ls)$\Longrightarrow$ (ms). Also, if $S_\star(\phi)$ is known to contain a linearly independent SI basis $E(\phi_0)$, then all the above conditions are equivalent. In particular, a linearly independent generator of $S_\star(\phi)$ is unique, up to shifts and multiplication by constants.

Proof (sketch): The fact that (ls) $\Longrightarrow$ (ms) follows from basic geometric observations. In view of previous results, it remains only to show that, if $S_\star(\phi)$ contains a linear independent $E(\phi_0)$, then (ms) implies (gli). That, however, is simple: since $E(\phi_0)$ is linearly independent, ϕ_0 has minimal support among all compactly supported elements of $S_\star(\phi)$. If ϕ also has that minimal support property, then, since it is finitely spanned by $E(\phi_0)$, it must be a constant multiple of a shift of ϕ_0, hence $E(\phi_0)$ is linearly independent, too.

The uniqueness of the linearly independent generator follows now from the analogous uniqueness property of the minimally supported generator. $\square$

Until now, we have retained the compact support assumption on ϕ. This allowed us to strive for the superior property of linear independence. However, PSI spaces that contain no non-trivial compactly supported functions are of interest, too. How to analyse the set $E(\phi)$ if ϕ is not of compact support? One way, the customary one, is to apply a cruder analysis: one should define the synthesis operator on whatever domain that operator may make sense, and study then this restricted operator. A major effort in this direction is presented in the next subsection, where the notions of stability and frames are introduced and studied. There, only mild decay assumptions on ϕ are imposed, e.g., that $\phi \in L_2(\mathbb{R}^d)$. On the other hand, if ϕ decays at ∞ in a more substantial way, more can be said. For example, if ϕ decays rapidly (i.e., faster than any fixed rational polynomial) then the following is true [34]. Recall that a sequence c is said to have **polynomial growth** if there exists a polynomial p such that $|c(j)| \le |p(j)|$ for all $j \in \mathbb{Z}^d$.

Proposition 14. *Assume that ϕ decays rapidly, and let $\mathcal{T}_{\phi,T}$ be the restriction of the synthesis operator to sequences of (at most) polynomial growth. Then the following conditions are equivalent:*

(a) $\mathcal{T}_{\phi,T}$ is injective.

(b) The restriction of $\mathcal{T}_{\phi,T}$ to $\ell_\infty(\mathbb{Z}^d)$ is injective.

(c) $\widehat{\phi}$ (that is defined now on $\mathbb{R}^d$ only) does not have a (real) 2π-periodic zero.

(d) There is a basis $E(g)$ dual to $E(\phi)$, with g a rapidly decaying, $C^\infty(\mathbb{R}^d)$-function.

Proof (sketch): (a) trivially implies (b). If $\widehat{\phi}$ vanishes identically on $\theta + 2\pi\mathbb{Z}^d$, θ real, then, by Poisson's summation formula, $e_{i\theta} \in \ker \mathcal{T}_{\phi,T}$ (with $e_\vartheta : j \mapsto \mathrm{e}^{\vartheta\cdot j}$). This exponential is bounded (since θ is real), hence (b) implies (c). The fact that (d) implies (a) follows from the relation $\langle \mathcal{T}_{\phi,T}c, E^j g\rangle = c(j)$, a relation that holds for any polynomially growing sequence, thanks to the rapid decay assumptions on ϕ and g.

We prove the missing implication "(c) implies (d)" in the next section (after the proof of Theorem 32), under the assumption that ϕ is a function. If one likes to stick with a *distribution* ϕ, then, instead of proving the missing implication directly, (a) can be proved from (c) as in [34], (see also the proof of the implication (c)$\Longrightarrow$(b) of Theorem 29), and the equivalence of (a) and (d) can be proven by an argument similar to that used in Theorem 10. $\square$

Thus, if ϕ is not of compact support, we settle for notions weaker than linear independence. This approach is not entirely satisfactory, as illustrated in the following example.

Example. Assume that ϕ is compactly supported and univariate. Then, by Theorem 15 below , there is, up to shifting and multiplication by scalars, exactly one linearly independent generator for $S_\star(\phi)$, and one may wish to select this, and only this generator. In contrast, there are many other compactly supported generators of $S_\star(\phi)$ that satisfy all the properties of Proposition 14. Unfortunately, the supports of these seemingly 'good' generators may be as large as one wishes. $\square$

The example indicates that properties weaker than linear independence may fail to distinguish between 'good' generators and 'better' generators. Therefore, an alternative approach to the above (i.e., to the idea of restricting the synthesis operator to a smaller domain) is desired: extending the notion of "linear independence" beyond *local* PSI spaces. For that, we note first that both Theorem 10 and Proposition 14 characterize injectivity properties of $\mathcal{T}_\phi$ in terms of surjectivity properties of $\mathcal{T}^*_\phi$, or, more precisely, in terms of the existence of the "nicely decaying" dual basis. Thus, I suggest the following extension of the linear independence notion.

Definition: linear independence. Let ϕ be any distribution. We say that $E(\phi)$ is **linearly independent** if there exists $g \in \mathcal{D}(\mathbb{R}^d)$ whose shift set $E(g)$ is linearly independent and is a dual basis to $E(\phi)$. $\square$

In view of Theorem 10, this definition is, indeed, an extension of the previous linear independence definition.

Open Problem. Find an effective characterization of the above general notion of linear independence, similar to Theorem 8. $\square$

Proposition 14 deals with the synthesis operator of a rapidly decaying $E(\phi)$. There are cases where ϕ decays faster than rapidly: for example at a certain exponential rate $\rho \in \mathbb{R}^d_+$. Roughly speaking, it means that

$$|\phi(x)| \le c e^{\rho(x)}, \quad \rho(x) := \sum_j \rho_j\, |x_j|,$$

In this case, the synthesis operator is well-defined on all sequences that grow (at most) at that exponential rate. The injectivity of the synthesis operator on this extended domain can be shown, once again, to be equivalent to the lack of 2π-periodic zeros of $\widehat{\phi}$, with $\widehat{\phi}$ the analytic extension of the Fourier transform. This analytic extension is well-defined in the multistrip

$$\{z \in \mathbb{C}^d : \ |\Im z_j| \le \rho_j, \ j = 1, \ldots, d\}.$$

The above strip can be shown to be the spectrum of a suitably chosen commutative Banach algebra (with the Gelfand transform being the Fourier transform), and those basic observations yield the above-mentioned injectivity result, [39].

2.3. Univariate Local PSI Spaces

There are three properties of local PSI spaces that, while not valid in general, are always valid in case the spatial dimension is 1. The first, and possibly the most important one, is the existence of a 'canonical' generator (Theorem 15). The second is the equivalence of the linear independence property to another, seemingly stronger, notion of linear independence, termed here "weak local linear independence". A third fact is discussed in §4.1 .

The following result can be found in [35] (it was already stated without proof in [46]). The result is invalid in two variables: the space generated by the shifts of the characteristic function of the square with vertices $(0,0),(1,1),(0,2),(-1,1)$ is a simple counter-example.

Theorem 15. *Let S be a univariate local PSI space. Then S contains a generator ϕ whose shifts $E(\phi)$ are linearly independent.*

Proof: Let ϕ_0 be a generator of S with support in $[a,b]$. If $E(\phi_0)$ is linearly independent, we are done. Otherwise, $\ker \mathcal{T}_{\phi_0}$ is non-zero, hence, by Theorem 7, contains an exponential $e_\theta : j \mapsto \mathrm{e}^{\theta j}$. We define two distributions

$$\phi_1 := \sum_{j=0}^{\infty} e_\theta(j)\, E^j \phi_0,$$

and

$$-\sum_{j=-\infty}^{-1} e_\theta(j)\, E^j \phi_0.$$

Obviously, ϕ_1 is supported on $[a,\infty)$, and the second distribution is supported on $(-\infty, b-1]$. However, since $e_\theta \in \ker \mathcal{T}_{\phi_0}$, the two distributions are equal, and hence ϕ_1 is supported on $[a, b-1]$. Also, ϕ_0 is spanned by $\{\phi_1, E^1\phi_1\}$, and one concludes that $S_\star(\phi_0) = S_\star(\phi_1)$. Since $[a,b]$ is of finite length, we may proceed by induction until arriving at the desired linearly independent ϕ. $\square$

Combining Theorem 15 and Theorem 13, we conclude that the properties (gli), (ldb), (ls), and (ms) (that appear in Theorem 13) are all equivalent for *univariate* local PSI spaces. In fact, there is another, seemingly stronger, property that is equivalent here to linear independence.

Definition: local linear independence . Let F be a countable, locally finite, family of distributions/functions. Let $A \subset \mathbb{R}^d$ be an open set. We say that F is **locally linearly independent on** A if, for every $c \in \mathcal{Q}$ (and with $\mathcal{T}$ the synthesis operator of F) the condition

$$\mathcal{T}c = 0 \quad \text{on } A$$

implies that $c(f)\,f = 0$ on A, for every $f \in F$. The set F is **weakly locally linearly independent =: (wlli)** if F is locally linearly independent on *some* open, bounded set A. These distributions are **strongly locally linearly independent :=(slli)** is they are locally linearly independent on *any* open set A. □

Note that in the case $\operatorname{supp} f \cap A = \emptyset$, we trivially obtain that $c(f)\,f = 0$ on A. So, the non-trivial part of the definition is that $c(f) = 0$ whenever $\operatorname{supp} f$ intersects A.

Trivially, weak local linear independence implies linear independence. The converse fails to hold already in the PSI space setup: in [1], there is an example of a bivariate ϕ whose shifts are globally linearly independent, but are not weakly locally independent. However, as claimed in [1], these two different notions of independence do coincide in the univariate case. The proof provided here is taken from [36]. Another proof is given in [45].

Proposition 16. *Let ϕ be a univariate distribution supported in $[0, N]$ with $E(\phi)$ globally linearly independent. Then $E(\phi)$ is weakly locally independent. More precisely, $E(\phi)$ is locally linearly independent on any interval A whose length is greater than $N - 1$.*

Proof: To avoid technical "end-point" problems, we assume herein that ϕ is a *function* supported in $[0, N]$, and prove that $E(\phi)$ is locally linearly independent over $[0, N-1]$. For that, we assume that some sequence $c \in \mathcal{Q}\backslash 0$ satisfies $\mathcal{T}_\phi\, c = 0$ on $[0, N-1]$. We will show that this implies the existence of a non-zero sequence, say b, such that $\mathcal{T}_\phi\, b = 0$ a.e.

For $j = 1, ..., N$, let f_j be the periodic extension of $\phi_{|[j-1,j]}$. Note that

$$(17) \qquad (\mathcal{T}_\phi\, b)_{|[j,j+1]} = 0 \quad \Longleftrightarrow \quad \sum_{i=1}^{N} b(j-N+i) f_{N-i+1} = 0.$$

Suppose that we are given some $b \in \mathcal{Q}$, and would like to check whether $\mathcal{T}_\phi\, b = 0$. To that end, let B be the matrix indexed by $\{1, \ldots, N\} \times \mathbb{Z}$ whose (i,j)-entry is $b(j-N+i)$. Then, as (17) shows, the condition $\mathcal{T}_\phi\, b = 0$ is equivalent to $FB = 0$, with F the row vector $[f_N, f_{N-1}, ..., f_1]$. Our aim is to construct such B. Initially, we select $b := c$, and then know that $FB(\cdot, j) = 0$, $j = 0, 1, ..., N-2$, since $\mathcal{T}_\phi\, c = 0$ on $[0, N-1]$.

Let $C = B(1{:}N-1, 0{:}N-1)$ be the submatrix of B made up from the first $N-1$ rows and from columns $0, \ldots, N-1$ of B. Since C has more columns than rows, there is a first column that is in the span of the columns to its left. In other words, $C(\cdot, r) = \sum_{j=0}^{r-1} a(j) C(\cdot, j)$ for some $r > 0$ and some $a(0), \ldots, a(r-1)$. In other words, the sequence $b(-N+1), \ldots, b(r-1)$ satisfies the constant-coefficient difference equation

$$b(i) = \sum_{j=0}^{r-1} a(j) b(i-r+j), \qquad i = r+1-N, \ldots, r-1.$$

Now use this very equation to define $b(i)$ inductively for $i = r, r+1, \ldots$. Then the corresponding columns of our matrix B satisfy the equation

$$B(\cdot, i) = \sum_{j=0}^{r-1} a(j) B(\cdot, i-r+j), \qquad i = r, r+1, \ldots,$$

and, since $FB(\cdot, j) = 0$ for $j = 0, \ldots, r-1$, this now also holds for $j = r, r+1, \ldots$. In other words, $\mathcal{T}_\phi b = 0$ on $[0, \infty)$. The corresponding further modification of b to also achieve $\mathcal{T}_\phi b = 0$ on $(\infty, 0]$ is now obvious. $\square$

2.4. The Space $S_2(\phi)$

The notion of the 'linear independence of $E(\phi)$' is the 'right one' for local PSI spaces. While we were able to extend this notion to PSI (and other) spaces that are not necessarily local, there do not exist at present effective methods for checking this more general notion.

This means that, in case the generator ϕ of the PSI space $S(\phi)$ is *not* compactly supported, we need other, weaker, notions of 'independence'. We have already described some possible notions of this type that apply to generators ϕ that decay exponentially fast or at least decay rapidly.

However, generators ϕ that decay at ∞ at slower rates are also of interest. Two pertinent univariate examples of this type are the *sinc function*

$$\text{sinc} : x \mapsto \frac{\sin(\pi x)}{\pi x},$$

and the *inverse multiquadric*

$$x \mapsto \frac{1}{\sqrt{1+x^2}}.$$

While these functions decay very slowly at ∞, they both still lie in L_2, as well as in any L_p, $p > 1$. It is thus natural to seek a theory that will only assume ϕ to lie in L_2, or more generally, in some L_p space. The two basic notions in that development are the notions of *stability* and *a frame*. For $p = 2$, the notion of stability is also known as the *Riesz basis* property.

In what follows, we assume ϕ to lie in $L_2(\mathbb{R}^d)$. Under this assumption, the PSI space $S_\star(\phi)$ is not well-defined any more, nor is there any hope to define meaningfully the synthesis operator $\mathcal{T}_\phi$. We replace $S_\star(\phi)$ by the PSI space variant

$$S_2(\phi),$$

which is defined as the L_2-closure of the finite span of $E(\phi)$. We also replace the domain $\mathcal{Q}$ of $\mathcal{T}_\phi$ by $\ell_2(\mathbb{Z}^d)$, and denote this restriction by

$$\mathcal{T}_{\phi,2}.$$

Note that, since we are only assuming here that $\phi \in L_2(\mathbb{R}^d)$, we do not know *a priori* that $\mathcal{T}_{\phi,2}$ is well-defined.

Very useful in this context is the **bracket product**: given $f, g \in L_2(\mathbb{R}^d)$, the bracket product $[f, g]$ is defined as follows:

$$[f,g] := \sum_{\alpha \in 2\pi\mathbb{Z}^d} E^\alpha f \, \overline{E^\alpha g}.$$

It is easy to see that $[f,g] \in L_1(\mathbb{T}^d)$. We assign also a special notation for the squareroot of $[\widehat{f}, \widehat{f}]$:

$$\widetilde{f} := \sqrt{[\widehat{f}, \widehat{f}]}. \tag{18}$$

Note that the map $f \mapsto \widetilde{f}$ is a unitary map from $L_2(\mathbb{R}^d)$ into $L_2(\mathbb{T}^d)$.

We collect below a few of the basic facts about the space $S_2(\phi)$, which are taken from [4].

Theorem 19. *Let $\phi \in L_2(\mathbb{R}^d)$. Then:*

(a) The orthogonal projection Pf of $f \in L_2(\mathbb{R}^d)$ onto $S_2(\phi)$ is given by

$$\widehat{Pf} = \frac{[\widehat{f}, \widehat{\phi}]}{[\widehat{\phi}, \widehat{\phi}]} \widehat{\phi}.$$

(b) A function $f \in L_2(\mathbb{R}^d)$ lies in the PSI space $S_2(\phi)$ if and only if $\widehat{f} = \tau\widehat{\phi}$ for some measurable 2π-periodic τ.

(c) The set $\operatorname{supp} \widetilde{\phi} \subset \mathbb{T}^d$ is independent of the choice of ϕ: if $S_2(\phi) = S_2(\psi)$, then, up to a null set, $\operatorname{supp} \widetilde{\phi} = \operatorname{supp} \widetilde{\psi}$.

We call $\operatorname{supp} \widetilde{\phi}$ the **spectrum** of the PSI space $S_2 := S_2(\phi)$, and denote it by

$$\sigma(S_2).$$

Note that the spectrum is defined up to a null set. A PSI space S_2 is **regular** if $\sigma(S_2) = \mathbb{T}^d$ (up to a null set). Note that a *local* PSI space $S_2(\phi)$ is always regular.

The bracket product was introduced in [22] (in a slightly different form), and in [4] in the present form. A key fact concerning the bracket product is the following identity, which is valid for any $\phi, \psi \in L_2(\mathbb{R}^d)$, and every $c \in \ell_2(\mathbb{Z}^d)$, provided, say, that the operators $\mathcal{T}_{\phi,2}$ and $\mathcal{T}_{\psi,2}$ are bounded:

$$(\mathcal{T}_\phi^* \mathcal{T}_\psi \, c)\widehat{\ } = [\widehat{\psi}, \widehat{\phi}] \, \widehat{c}. \tag{20}$$

Lemma 21. *Let $\phi, \psi \in L_2(\mathbb{R}^d)$, and assume that the operators $\mathcal{T}_{\phi,2}$ and $\mathcal{T}_{\psi,2}$ are bounded.*

(a) The kernel $\ker \mathcal{T}^*_{\phi,2}\mathcal{T}_{\psi,2} \subset \ell_2(\mathbb{Z}^d)$ *is the space*

$$K_{\phi,\psi} := \{c \in \ell_2(\mathbb{Z}^d) : \ \operatorname{supp} \widehat{c} \subset \mathbb{T}^d \backslash (\operatorname{supp}[\widehat{\psi}, \widehat{\phi}])\}.$$

(b) The operator $\mathcal{T}^*_{\phi,2}\mathcal{T}_{\psi,2}$ is a projector if and only if $[\widehat{\psi}, \widehat{\phi}] = 1$ on its support.

Proof (sketch): (a) follows from (20), since the latter implies that $\mathcal{T}^*_{\phi,2}\mathcal{T}_{\psi,2}c = 0$ if and only if $\operatorname{supp} \widehat{c}$ is disjoint from

$$\sigma := \operatorname{supp}[\widehat{\psi}, \widehat{\phi}].$$

For (b), note that (20) implies that the range of $\mathcal{T}^*_{\phi,2}\mathcal{T}_{\psi,2}$ is $K^{\perp}_{\phi,\psi} = \{c \in \ell_2(\mathbb{Z}^d) : \operatorname{supp} \widehat{c} \subset \sigma\}$. Thus $\mathcal{T}^*_{\phi,2}\mathcal{T}_{\psi,2}$ is a projector if and only if it is the identity on $(K_{\phi,\psi})^{\perp}$. The result now easily follows from (20). □

2.5. Basic Theory: Stability and Frames in PSI Spaces

We need here to make our setup a bit more general. Thus, we assume F to be any countable subset of $L_2(\mathbb{R}^d)$, and define a corresponding *synthesis map* $T_{F,2}$ as follows:

$$T_{F,2} : \ell_2(F) \to L_2(\mathbb{R}^d) : c \mapsto \sum_{f \in F} c(f) f.$$

The choice $F := E(\phi)$ is of immediate interest here, but other choices will be considered in the sequel.

Definition: Bessel systems, stable bases and frames. Let $F \subset L_2(\mathbb{R}^d)$ be countable.

(a) We say that F forms a Bessel system if $T_{F,2}$ is a well-defined bounded map.

(b) A Bessel system F is a frame if the range of $T_{F,2}$ is closed (in $L_2(\mathbb{R}^d)$).

(c) A frame F is a stable basis if $T_{F,2}$ is injective.

Discussion. The notion of stability effectively says that $T_{F,2}$ is a continuous injective open, hence invertible, map, i.e., that there exist constants $C_1, C_2 > 0$ such that

$$C_1 \|c\|_{\ell_2(F)} \leq \|T_{F,2}c\|_{L_2(\mathbb{R}^d)} \leq C_2 \|c\|_{\ell_2(F)}, \quad \forall c \in \ell_2(F), \tag{22}$$

for every finitely supported c defined on F (hence for every $c \in \ell_2(F)$).

The frame condition is weaker. It does not assume (22) to hold for *all* $c \in \ell_2(F)$, but only for c in the orthogonal complement of $\ker T_{F,2}$. In general, it is hard to compute that orthogonal complement, hence it is non-trivial to implement the definition of a frame via the synthesis operator. However, for the case of interest here, i.e., the PSI system $F = E(\phi)$, computing $\ker T_{F,2}$ is quite simple (cf. Lemma 21).

There is an alternative definition of the frame property, which is more common in the literature. Assume that F is a Bessel system, and let $T^*_{F,2}$ be its analysis operator:

$$T^*_{F,2} : L_2(\mathbb{R}^d) \to \ell_2(F) : g \mapsto (\langle g, f\rangle)_{f\in F}.$$

The equivalent definition of a frame with the aid of this operator is analogous: there exist constants $C_1, C_2 > 0$ such that

$$C_1\|f\|_{L_2(\mathbb{R}^d)} \le \|T^*_{F,2}f\|_{\ell_2(F)} \le C_2\|f\|_{L_2(\mathbb{R}^d)} \tag{23}$$

for every f in the orthogonal complement of $\ker T^*_{F,2}$. While it seems that we have gained nothing by switching operators, it is usually easier to identify the above-mentioned orthogonal complement: it is simply the closure in $L_2(\mathbb{R}^d)$ of the finite span of F.

The constant C_1 (C_2, respectively) is sometimes referred to as the **lower** (**upper**, respectively) **stability/frame bound**.

A third, and possibly the most effective, definition of stable bases/frames goes via a **dual system**: let R be some assignment

$$\mathrm{R} : F \to L_2(\mathbb{R}^d),$$

and assume that F as well as RF are Bessel systems. We then say that RF is a dual system for F if the operator

$$T_{F,2}T^*_{\mathrm{R}F,2} : g \mapsto \sum_{f\in F}\langle g, \mathrm{R}f\rangle f$$

is a *projector*, i.e., it is the identity on the closure of $\operatorname{span} F$. The roles of F and RF in the above definition are interchangeable. We have the following simple lemma:

Lemma 24. *Let $F \subset L_2(\mathbb{R}^d)$ be countable, and assume that F is a Bessel system. Then:*

(a) F is a frame if and only if there exists an assignment $\mathrm{R} : F \to L_2(\mathbb{R}^d)$ such that RF is Bessel, and is a dual system of F.

(b) F is a stable basis if and only if there exists an assignment $\mathrm{R} : F \to L_2(\mathbb{R}^d)$ such that RF is Bessel, and is a dual system for F in the stronger biorthogonal sense: for $f, g \in F$,

$$\langle f, \mathrm{R}g\rangle = \begin{cases} 1, & f = g, \\ 0, & f \ne g \end{cases}$$

(i.e., $T^*_{F,2}T_{\mathrm{R}F,2}$ is the identity operator).

The next result is due to [5] and [41]. It was also established independently by Benedetto and Li (cf. [2]). We use below the convention that

$$0/0 := 0.$$

Theorem 25. *Let $\phi \in L_2(\mathbb{R}^d)$ be given. Then:*

(a) $E(\phi)$ is a Bessel system if and only if $\widetilde{\phi} \in L_\infty(\mathbb{R}^d)$. Moreover, $\|\mathcal{T}_{\phi,2}\| = \|\widetilde{\phi}\|_{L_\infty(\mathbb{R}^d)}$.

(b) Assume $E(\phi)$ is a Bessel system. Then $E(\phi)$ is a frame if and only if $1/\widetilde{\phi} \in L_\infty(\sigma(S_2(\phi)))$. Moreover, $\|{\mathcal{T}_{\phi,2}}^{-1}\| = \|1/\widetilde{\phi}\|_{L_\infty(\mathbb{R}^d)}$ (with ${\mathcal{T}_{\phi,2}}^{-1}$ the pseudo-inverse of $\mathcal{T}_{\phi,2}$).

(c) Assume $E(\phi)$ is a frame. Then it is also a stable basis if and only if $\widetilde{\phi}$ vanishes almost nowhere, i.e., if and only if $S_2(\phi)$ is regular.

Proof (sketch): Choosing $\psi := \phi$ in (20), we obtain that, for $c \in \ell_2(\mathbb{Z}^d)$,

$$\|\mathcal{T}_\phi^* \mathcal{T}_\phi\, c\|_{\ell_2(\mathbb{Z}^d)} = (2\pi)^{-d/2} \|\widetilde{\phi}^2 \widehat{c}\|_{L_2(\mathbb{T}^d)}.$$

This yields that $\|\mathcal{T}_\phi^* \mathcal{T}_\phi\,\| = \|\widetilde{\phi}^2\|_{L_\infty}$, and (a) follows.

For (b), assume that $1/\widetilde{\phi}$ is bounded on its support $\sigma(S_2(\phi))$, and define ψ by

$$\widehat{\psi} := \frac{\widehat{\phi}}{\widetilde{\phi}^2} = \frac{\widehat{\phi}}{[\widehat{\phi}, \widehat{\phi}]}.$$

Then ψ lies in $L_2(\mathbb{R}^d)$. Moreover, if $c \in \ell_2(\mathbb{Z}^d)$ and $\widehat{c}$ is supported in $\operatorname{supp} \widetilde{\phi}$, then

$$[\widehat{\psi}, \widehat{\phi}]\widehat{c} = \frac{[\widehat{\phi}, \widehat{\phi}]}{[\widehat{\phi}, \widehat{\phi}]}\widehat{c} = \widehat{c}.$$

In view of (20), this implies that $\widehat{\mathcal{T}_\phi^* \mathcal{T}_\psi}$ is a projector (whose range consists of all the periodic functions that are supported in $\sigma(S_2(\phi))$), hence that $E(\psi)$ is a system dual to $E(\phi)$. Also, $E(\phi)$ is a Bessel system by assumption, while for ψ we have that

$$[\widehat{\psi}, \widehat{\psi}] = \frac{[\widehat{\phi}, \widehat{\phi}]}{[\widehat{\phi}, \widehat{\phi}]^2} = \frac{1}{[\widehat{\phi}, \widehat{\phi}]} \in L_\infty.$$

Hence, by (a), $E(\psi)$ is a Bessel system, too. We conclude from (a) of Lemma 24 that $E(\phi)$ is a frame.

For the converse implication in (b), let $E(\psi)$ be a Bessel system that is dual to $E(\phi)$. Then, by (b) of Lemma 21, $[\widehat{\psi}, \widehat{\phi}] = 1$ on its support. On the other hand, by Cauchy-Schwarz,

$$[\widehat{\phi}, \widehat{\psi}]^2 \le [\widehat{\phi}, \widehat{\phi}][\widehat{\psi}, \widehat{\psi}].$$

This shows that, on $\operatorname{supp}[\widehat{\psi}, \widehat{\phi}]$,

$$[\widehat{\phi}, \widehat{\phi}]^{-1} \le [\widehat{\psi}, \widehat{\psi}].$$

The conclusion now follows from (a) and the fact that $E(\psi)$ is assumed to be Bessel.

As for (c), once $E(\phi)$ is known to be a frame, it is also a stable basis if and only if $\mathcal{T}_{\phi,2}$ is injective. In view of (a) of Lemma 21 (take $\psi := \phi$ there), and the definition of the spectrum of $S_2(\phi)$, that injectivity is equivalent to the non-vanishing a.e. of $\widetilde{\phi}$. □

Note that the next corollary applies, in particular, to any compactly supported $\phi \in L_2(\mathbb{R}^d)$.

Corollary 26. *Let $\phi \in L_2(\mathbb{R}^d)$. If $\widetilde{\phi}$ is continuous, then $E(\phi)$ is a frame (if and) only if it is a stable basis.*

Proof: Since $\widetilde{\phi}$ is continuous and 2π-periodic, the function $1/\widetilde{\phi}$ can be bounded on its support only if it is bounded everywhere. Now apply Theorem 25. □

§3. Beyond local PSI spaces

'Extending the theory of local PSI spaces' might be interpreted as one of the following two attempts. One direction is to extend the setup that is studied; another direction is to extend the tools that were developed. These two directions are clearly interrelated, but not necessarily identical.

When discussing more general setups, there are, again, several different, and quite complementary, generalizations. Once such extension concerns the application of the stability notion to p-norms, $p \neq 2$. This is the subject of §3.1.

Another extension is the study of the linear independence and the related notions in FSI spaces. This is the subject of §3.2 and §3.3.

A third extension is the extension of the notions of stability and frames to FSI spaces. We will introduce in that context (§3.4) the general L_2-tools and briefly discuss the general approach in that direction: starting with the bracket product, we will be led to the theory of *fiberization*, a theory that goes beyond FSI spaces, and goes even beyond general SI spaces.

3.1. L_p-stability in PSI Spaces

We denote by

$$\mathcal{T}_{\phi,p}$$

the restriction of the synthesis operator $\mathcal{T}_\phi$ to $\ell_p(\mathbb{Z}^d)$.

Definition: p-Bessel systems and p-stable bases. Given $1 \leq p \leq \infty$, and $\phi \in L_p(\mathbb{R}^d)$, we say that $E(\phi)$ forms

(a) a p-Bessel system, if $\mathcal{T}_{\phi,p}$ is a well-defined bounded map into $L_p(\mathbb{R}^d)$.

(b) a p-stable basis, if $\mathcal{T}_{\phi,p}$ is bounded, injective and its range is closed in $L_p(\mathbb{R}^d)$.

We first discuss, in the next result, the p-Bessel property: that property is implied by mild decay conditions on ϕ. We provide *characterizations* for the 1- and ∞-Bessel properties, and a *sufficient condition* for the other cases. As to the proofs, the proof of the 1-case is straightforward, and that of the

∞-case involves routine arguments. The sufficient condition for the general p-Bessel property can be obtained easily from the discrete convolution inequality $\|a*b\|_{\ell_p} \le \|a\|_{\ell_1}\|b\|_{\ell_p}$, can also be obtained by interpolation between the $p=1$ and the $p=\infty$ cases, and is due to [22]. Note that for the case $p=2$ the sufficient condition listed below is *not* equivalent to the characterization in Theorem 25.

The following spaces, which were introduced in [22], are useful here and later:

$$(27)\quad \mathcal{L}_p(\mathbb{R}^d) := \{f \in L_p(\mathbb{R}^d) : \|\phi\|_{\mathcal{L}_p(\mathbb{R}^d)} := \|\sum_{j\in\mathbb{Z}^d} |\phi(\cdot+j)|\|_{L_p([0,1]^d)} < \infty\}.$$

Proposition 28.
(a) $E(\phi)$ *is 1-Bessel iff* $\phi \in L_1(\mathbb{R}^d)$. *Moreover,* $\|\mathcal{T}_{\phi,1}\| = \|\phi\|_{L_1(\mathbb{R}^d)}$.
(b) $E(\phi)$ *is* ∞*-Bessel iff* $\phi \in \mathcal{L}_\infty(\mathbb{R}^d)$. *Moreover,* $\|\mathcal{T}_{\phi,\infty}\| = \|\phi\|_{\mathcal{L}_\infty(\mathbb{R}^d)}$.
(c) If $\phi \in \mathcal{L}_p(\mathbb{R}^d)$, $1<p<\infty$, *then* $E(\phi)$ *is p-Bessel and* $\|\mathcal{T}_{\phi,p}\| \le \|\phi\|_{\mathcal{L}_p(\mathbb{R}^d)}$.
□

We now discuss the p-stability property. A complete characterization of this stability property is known again for the case $p=\infty$ (in addition to the case $p=2$ that was analysed in Theorem 25). We start with this case, which is due to [22].

Theorem 29. *Let* $\phi \in \mathcal{L}_\infty(\mathbb{R}^d)$. *Then the following conditions are equivalent:*
(a) $E(\phi)$ *is* ∞*-stable.*
(b) $\mathcal{T}_{\phi,\infty}$ *is injective.*
(c) $\widehat{\phi}$ *does not have a real* 2π*-periodic zero.*

Proof (sketch): The implication (a)$\Longrightarrow$(b) is trivial, while the proof of (b)$\Longrightarrow$(c) has already been outlined in Proposition 14.

(b)$\Longrightarrow$(a) [36]. Assuming (a) is violated, we find sequences $(a_n)_n \subset \ell_\infty(\mathbb{Z}^d)$ such that $a_n(0) = 1 = \|a_n\|_{\ell_\infty(\mathbb{Z}^d)}$, all n, and such that $\mathcal{T}_\phi a_n$ tends to 0 in $L_\infty(\mathbb{R}^d)$. Without loss, $(a_n)_n$ converges pointwise to a sequence a; necessarily $a \ne 0$. Since $(a_n)_n$ is bounded in $\ell_\infty(\mathbb{Z}^d)$, and since $\phi \in \mathcal{L}_\infty(\mathbb{R}^d)$, it follows that $\mathcal{T}_\phi a_n$ converges pointwise a.e. to $\mathcal{T}_\phi a$. Hence $\mathcal{T}_\phi a = 0$, in contradiction to (b).

(c)$\Longrightarrow$(b): If (b) is violated, say, $\mathcal{T}_\phi a = 0$, then $\widehat{a}\widehat{\phi} = 0$. Since $\widehat{a}$ is a pseudo-measure, and $\phi \in \mathcal{L}_\infty(\mathbb{R}^d) \subset L_1(\mathbb{R}^d)$, we conclude that $\widehat{a}$ is supported in the zero set of $\widehat{\phi}$. However, $\widehat{a}$ is periodic and non-zero, hence $\widehat{\phi}$ must have a periodic zero. □

The reader should be warned that the above reduction of stability to injectivity is very much an L_∞-result. For example, for a univariate compactly supported bounded ϕ, $\mathcal{T}_\phi$ is injective on $\ell_p(\mathbb{Z})$ for all $p<\infty$ [34], while certainly $E(\phi)$ may be unstable (in any chosen norm). On the other hand (as follows from some results in the sequel), assuming that ϕ lies in $\mathcal{L}_\infty(\mathbb{R}^d)$, the injectivity of $\mathcal{T}_{\phi,\infty}$ characterizes the p-stability for *all* $1 \le p \le \infty$!

In order to investigate the stability property for other norms, we follow the approach of [22], assume that $\widetilde{\phi}^2 = [\widehat{\phi}, \widehat{\phi}]$ is bounded away from 0 (cf. Theorem 25), and consider the function g defined by its Fourier transform as

$$\widehat{g} := \frac{\widehat{\phi}}{\widetilde{\phi}^2}.$$

We have the following:

Proposition 30. *Let $\phi, g \in \mathcal{L}_p(\mathbb{R}^d) \cap \mathcal{L}_{p'}(\mathbb{R}^d)$, $1 \le p \le \infty$, and p' is its conjugate. Assume that*

$$[\widehat{\phi}, \widehat{g}] = 1. \tag{31}$$

Then $E(\phi)$ is p-stable.

Proof (sketch): From Proposition 28 and the assumptions here, we conclude that $E(\phi)$ as well as $E(g)$ are p-, as well as p'-Bessel systems. Also Poisson's summation formula can be invoked to infer (from (31)) that

$$\langle \phi, E^j g\rangle = \delta_{j,0}, \quad j \in \mathbb{Z}^d,$$

i.e., that the shifts of g are biorthogonal to the shifts of ϕ.

Consider the operator

$$\mathcal{T}^*_{g,p} : L_p(\mathbb{R}^d) \to \ell_p(\mathbb{Z}^d) : f \mapsto (\langle f, E^j g\rangle)_{j \in \mathbb{Z}^d}.$$

Then, $\mathcal{T}^*_{g,p}$ is the adjoint of the operator $\mathcal{T}_{g,p'}$ (for $p = 1$ it is the restriction of the adjoint to $L_1(\mathbb{R}^d)$). Since $E(g)$ is p'-Bessel, it follows that $\mathcal{T}^*_{g,p}$ is bounded. On the other hand, $\mathcal{T}^*_{g,p}\mathcal{T}_{\phi,p}$ is the identity, hence $\mathcal{T}_{\phi,p}$ is boundedly invertible, i.e., $E(\phi)$ is p-stable. □

The following result is due to [22]. We use in that result, for $1 \le p < \infty$, the notation

$$S_p(\phi)$$

for the $L_p(\mathbb{R}^d)$-closure of the finite span of $E(\phi)$ (for $\phi \in L_p(\mathbb{R}^d)$).

Theorem 32. *Let $1 \le p \le \infty$ be given, let p' be its conjugate exponent, and assume that $\phi \in \mathcal{L} := \mathcal{L}_p(\mathbb{R}^d) \cap \mathcal{L}_{p'}(\mathbb{R}^d)$. Then:*

(a) If $\widetilde{\phi}$ does not have (real) zeros, then $E(\phi)$ is p- and p'- stable. In this case, a generator g of a basis dual to $E(\phi)$ lies in $\mathcal{L}$, and, for $p < \infty$, the dual space $S_p(\phi)^*$ is isomorphic to $S_{p'}(\phi)$.

(b) If $\widetilde{\phi}$ has a 2π-periodic zero, then $E(\phi)$ is not p-stable.

Proof (sketch): (a): Poisson's summation yields that the Fourier coefficients of $\widetilde{\phi}^2$ are the inner products $(\langle \phi, E^j\phi\rangle)_{j\in\mathbb{Z}^d}$. The assumption $\phi \in \mathcal{L}$ implies that $\phi \in \mathcal{L}_2(\mathbb{R}^d)$, and that latter condition implies that the above Fourier coefficients are summable, i.e., $\widetilde{\phi}^2$ lies in the Wiener algebra $\mathcal{A}(\mathbb{T}^d)$.

Now, if the continuous function $\widetilde{\phi}$ does not vanish, then, since $\phi \in L_2(\mathbb{R}^d)$, g defined by $\widehat{g} = \widehat{\phi}/\widetilde{\phi}^2$ is also in $L_2(\mathbb{R}^d)$, and we have $[\widehat{\phi}, \widehat{g}] = 1$. However, by Wiener's Lemma, $1/\widetilde{\phi}^2 \in \mathcal{A}(\mathbb{T}^d)$, too. This means that $g = \mathcal{T}_\phi\, a$ for some $a \in \ell_1(\mathbb{Z}^d)$. From that it follows that $g \in \mathcal{L}$ and is bounded, hence, by Proposition 30, $E(\phi)$ is p-stable. The p'-stability is obtained by symmetry, which directly implies, for $p < \infty$, that $S_p(\phi)^* = S_{p'}(\phi)$. Incidentally, we have proved that a dual basis of $E(\phi)$ lies, indeed, in $\mathcal{L}$.

We refer to [22] for the proof of (b). □

Proof of the implication (c)$\Longrightarrow$(d) in Proposition 14: If ϕ decays rapidly, $\widetilde{\phi}$ is infinitely differentiable. It, further, vanishes nowhere in case $\widehat{\phi}$ does not have a 2π-periodic zero. Thus, the Fourier coefficients a of $\widetilde{\phi}^{-2}$ are rapidly decaying, hence the function g defined by $\widehat{g} = \widehat{\phi}/\widetilde{\phi}^2$ is rapidly decaying, too. □

3.2. Local FSI Spaces: Resolving Linear Dependence, Injectability

An FSI space S is almost always given in terms of a generating set Φ for it. In many cases, the generating set has unfavorable properties. For example, $E(\Phi)$ might be linearly dependent (in the sense that $\mathcal{T}_\Phi\, c = 0$, for some non-zero $c \in \mathcal{Q}(\Phi)$, i.e.,

$$\sum_{j \in \mathbb{Z}^d \times \Phi} c(j, \phi)\, E^j \phi = 0.) \tag{33}$$

Theorem 15 provides a remedy to this situation for *univariate local* PSI spaces: if the compactly supported generator ϕ of S has linearly dependent shifts, we can replace it by another compactly supported generator, whose shifts are linearly *independent*.

The argument extends to univariate local FSI spaces, and that extension is presented in the sequel. The essence of these techniques extend to spaces that are *not* local (cf. e.g., [32]. I should warn the reader that it may not be trivial to see the connection between the factorization techniques of [32] and those that are discussed here; nonetheless, a solid connection does exist), but definitely not to shift-invariant spaces in *several* dimensions. The attempt to find an alternative method that is applicable in several dimensions will lead us, as is discussed near the end of this subsection, to the notion of *injectability*.

We start with the following result from [5]:

Lemma 34. *Let $S_2(\Phi)$ and $S_2(\Psi)$ be two local FSI spaces. Then the orthogonal projection, of $S_2(\Psi)$ into $S_2(\Phi)$, as well as the orthogonal complement of this projection, are each local (FSI) spaces, i.e., each is generated by compactly supported functions.*

Proof (sketch): The key for the proof is the observation (cf. [5]) that, given any compactly supported $f \in L_2(\mathbb{R}^d)$, and any FSI space S that is generated by a compactly supported vector $\Phi \subset L_2(\mathbb{R}^d)$, there exist trigonometric polynomials τ_f, $(\tau_\phi)_{\phi\in\Phi}$ such that

$$\tau_f \widehat{Pf} = \sum_{\phi\in\Phi} \tau_\phi \widehat{\phi}. \tag{35}$$

Here, Pf is the orthogonal projection of f on $S_2(\Phi)$.

Now, let g be the inverse transform of $\tau_f \widehat{Pf}$. From (35) (and the fact that Φ is compactly supported) we get that g is of compact support, too. From (b) of Theorem 19, it follows that $S_2(g) = S_2(Pf)$. Thus, $S_2(Pf)$ is a *local* PSI space. Varying f over Ψ, we get the result concerning projection.

When proving the claim concerning complements, we may assume without loss (in view of the first part of the proof) that $S_2(\Psi) \subset S_2(\Phi)$. Let now P denote the orthogonal projector onto $S_2(\Psi)$. Then, by (35), given ϕ in Φ, there exists a compactly supported g, and a trigonometric polynomial τ such that $\widehat{g} = \tau\widehat{P\phi}$. Thus also $\tau(\widehat{\phi} - \widehat{P\phi})$ is the transform of a compactly supported function g_1 (since ϕ is compactly supported by assumption). By (b) of Theorem 19, $S_2(g_1) = S_2(\phi - P\phi)$, and the desired result follows. □

The first part of the next result is taken from [5]. The second part is due to R.Q. Jia.

Corollary 36.
(a) *Every local FSI space $S_2(\Phi)$ is the orthogonal sum of local PSI spaces.*
(b) *Every local univariate shift-invariant space $S_\star(\Phi)$ is generated by a compactly supported Ψ whose shift set $E(\Psi)$ is linearly independent.*

Proof (sketch): Note that in part (a) we tacitly assume that $\Phi \subset L_2(\mathbb{R}^d)$. Let $f \in \Phi$. By Lemma 34, the orthogonal complement of $S_2(f)$ in $S_2(\Phi)$ is a *local* PSI space. Iterating with this argument, we obtain the result.

Part (b) now follows for $\Phi \subset L_2(\mathbb{R})$: we simply write then $S_2(\Phi)$ as an orthogonal sum of local PSI spaces, apply Theorem 15, and use the fact orthonormality implies linear independence. If Φ are merely distributions (still with compact support), then we can reduce this case to the former one by convolving Φ with a suitable smooth compactly supported mollifier. □

In more than one dimension, it is usually impossible to resolve the dependence relations of the shifts of Φ. This is already true in the case of a single generator (cf. the discussion in §2.3.) One of the possible alternative approaches (which I learned from the work of Jia, cf., e.g., [21]; the basic idea can already be found in the proof of Lemma 3.1 of [23]), is to embed the given SI space in a larger SI space that has generators with 'better' properties.

Definition 37. Let Φ be a finite collection of compactly supported distributions. We say that the local FSI space $S_\star(\Phi)$ is **injectable** if there exists another finite set Φ_0 of compactly supported distributions so that:
(a) $S_\star(\Phi) \subset S_\star(\Phi_0)$, and
(b) Φ_0 has linearly independent shifts.

The injectability assumption is quite mild. For example, if Φ consists of (compactly supported) *functions* then $S_\star(\Phi)$ is injectable: one can take Φ_0 to be any basis for the finite dimensional space $\mathrm{span}\{\chi E^j\phi : \ \phi \in \Phi, \ j \in \mathbb{Z}^d\}$, where χ is the support function of the unit cube.

At the time this article is written, I am not entirely convinced that the injectability notion is the right one for the general studies of local FSI spaces, and for several reasons. First, if some of the entries of the compactly supported Φ are merely distributions, it is not clear how to inject $S_\star(\Phi)$ into a better space. Moreover, the above-mentioned canonical injection of a *function* Φ into $S_\star(\Phi_0)$ is not smoothness-preserving. I.e., while the entries of Φ may be smooth, the entries of Φ_0 are not expected to be so. We do not know of a general technique for smoothness-preserving injection.

My last comment in this context is about the actual notion of 'linear independence'. The analysis in §2.2 and §2.3 provides ample evidence that this notion is the right one in the context of local *principal* shift-invariant spaces. The same cannot be said about local *finitely-generated* SI spaces, as the following example indicates.

Example. Let $\Phi := \{\phi_1, \phi_2\}$ be a set of two compactly supported functions, and assume that $E(\Phi)$ is linear independent. Then, with f any finite linear combination of $E(\phi_1)$, the set $\{\phi_1, \phi_2 + f\}$ also has linearly independent shifts. However, we can select f to have very large support, hence to enforce large support on $\phi_2 + f$. Consequently, the generators of a linearly independent $E(\Phi)$ may have as large supports as one wishes them to have, in stark contrast with the PSI space counterpart (Theorem 13). $\square$

Thus we need, in the context of FSI space theory, a notion that is somewhat stronger than linear independence, and that takes into account the support size of the various elements, as well as an effective characterization of this property, as effective perhaps as that of linear independence that appears in the next section.

3.3. Local FSI Spaces: Linear Independence

Despite of the reservations discussed in the previous subsection, linear independence is still a basic notion is the theory of local FSI spaces, and the injectability assumption provides one at times with a very effective tool. The current subsection is devoted to the study of the linear independence property via the injectability tool. The basic reference on this matter is [23]. The results here are derived under the assumption that the FSI space $S_\star(\Phi)$ is injectable into the FSI space $S_\star(\Phi_0)$ (whose generators have linearly independent shifts). Recall from the last subsection that every *univariate* local FSI space is injectable (into itself), and that, in higher dimensions, local FSI spaces that are generated by compactly supported *functions* are injectable as well. It is very safe to conjecture that the results here are valid for spaces generated by compactly supported distributions, and it would be nice to find a neat way to close this small gap.

Thus, we are given a local FSI space $S_\star(\Phi)$, and assume that the space is injectable, i.e., it is a subspace of the local FSI space $S_\star(\Phi_0)$, and that Φ_0 have linear independent shifts. In view of Corollary 11, we conclude that there exists, for every $\phi \in \Phi$, a *finitely supported* $c_\phi \in \mathcal{Q}(\Phi_0)$, such that

$$\mathcal{T}_{\Phi_0} c_\phi = \phi.$$

We then create a matrix Γ, whose columns are the vectors c_ϕ, $\phi \in \Phi$ (thus the columns of Γ are indexed by Φ, the rows are indexed by Φ_0, and all the entries are finitely supported sequences defined on $\mathbb{Z}^d$).

It is useful to consider each of the above sequences as a (Laurent) polynomial, and to write the possible dependence relations among $E(\Phi)$ as formal power series. Let, thus, A be the space of formal power series in d variables. I.e., $a \in A$ has the form

$$a = \sum_{j \in \mathbb{Z}^d} a(j) X^j,$$

with X^j the formal monomial. Recall that

$$\mathbb{Z}^d \supset \operatorname{supp} a := \{j \in \mathbb{Z}^d : \ a(j) \neq 0\}.$$

Let

$$A_0$$

be the ring of all finitely supported d-variate power series (i.e., Laurent polynomials). Given a finite set Φ, let

$$A(\Phi)$$

be the free A_0-module consisting of $\#\Phi$ copies of A. We recall that the z-transform is the linear bijection

$$Z : \mathcal{Q} \to A : c \mapsto \sum_{j \in \mathbb{Z}^d} c(j) X^j.$$

Applying the z-transform (entry by entry) to our matrix Γ above, we obtain a matrix

$$M,$$

whose entries are in A_0. We consider this matrix M an A_0-homomorphism between the module $A(\Phi)$ and the module $A(\Phi_0)$, i.e.,

$$M \in \operatorname{Hom}_{A_0}(A(\Phi), A(\Phi_0)).$$

It is relatively easy then to conclude the following:

Lemma 38. *In the above notations,*
(a) If $E(\Phi)$ is linearly independent, then M is injective.
(b) The converse is true, too, provided that $E(\Phi_0)$ is linearly independent.

Thus, the characterization of linear independence in (injectable) local FSI spaces is reduced to the characterization of injectivity in $\mathrm{Hom}_{A_0}(A(\Phi), A(\Phi_0))$. We provide below the relevant result, which can be viewed either as a spectral analysis result in $\mathrm{Hom}_{A_0}(A(\Phi), A(\Phi_0))$, or as an extension of the *Nullstellensatz* to modules. The following result is due [23]. The Jia-Micchelli proof reduces the statement in the theorem below to the case studied in Theorem 7 by a tricky Gauss elimination arguement. The proof provided here is somewhat different, and employs the Quillen-Suslin Theorem, [33], [47].

Discussion 39: The Quillen-Suslin Theorem. We briefly explain the relevance of this theorem to our present setup. The Quillen-Suslin Theorem affirms a famous conjecture of J.P Serre (cf. [26]) that every projective module over polynomial ring is free. The extension of that result to Laurent polynomial rings is mentioned in Suslin's paper, and was proved by Swan. A simple consequence of that theorem is that a every row $(w_1, \ldots, w_m)$ of Laurent polynomials that do not have a common zero in $(\mathbb{C}\backslash 0)^d$, can be extended to a square A_0-valued matrix W which is non-singular everywhere, i.e., $W(\xi)$ is non-singular for every $\xi \in (\mathbb{C}\backslash 0)^d$. A very nice discussion of the above, together with a few more references, can be found in [22]. □

The *Nullstellensatz* for Free Modules. *Let A be the space of formal power series in d-variables. Let*

$$A_0$$

be the ring of all finitely supported d-variate power series. Given a positive integer n, let

$$A^n$$

be the free A_0-module consisting of n copies of A. Let

$$M_{m\times n} \in \mathrm{Hom}_{A_0}(A^n, A^m)$$

be an A_0-valued matrix. Then M is injective if and only if there does not exist $\xi \in (\mathbb{C}\backslash 0)^d$ for which $\mathrm{rank}\, M(\xi) < n$.

Here, $M(\xi)$ is the constant-coefficient matrix obtained by evaluating each entry of M at ξ.

Proof (sketch): The 'only if' follows immediately from the fact that, for every ξ as above, there exists a_ξ in A such that $a_0 a_\xi = a_0(\xi)$, for every $a_0 \in A_0$. Indeed, if $M(\xi)$ is rank-deficient, we can find a vector in $c \in \mathbb{C}^n\backslash 0$ such that $M(\xi)c = 0$, and we get that $a_\xi c \in \ker M$.

We prove the converse by induction on n. For $n = 1$, we let I be the ideal in A_0 generated by the entries of the (single) column of M. If $a \in \ker M \subset A^1 = A$, then $a_0 a = 0$ for every $a_0 \in I$. By an argument identical to that used in the proof of Theorem 7, we conclude that, if $\ker M \neq 0$, then all the polynomials in I must vanish at a point $\xi \in (\mathbb{C}\backslash 0)^d$, hence $M(\xi) = 0$.

So assume that $n > 1$, and that $a \in \ker M\backslash 0$. We may assume without loss that the entries of the first column of M do not have a common zero $\xi \in (\mathbb{C}\backslash 0)^d$ (otherwise, we obviously have that $\operatorname{rank} M(\xi) < n$). Thus, by the classical *Nullstellensatz*, we can form a combination of the rows of M, with coefficients w_i in A_0, so that the resulting row u has the constant 1 in its first entry. Then, the entries w_i cannot have a common zero in $(\mathbb{C}\backslash 0)^d$, and therefore (cf. Discussion 39) the row vector $w := (w_i)$ can be extended to an $m \times m$ A_0-valued matrix W that is non-singular at every $\xi \in (\mathbb{C}\backslash 0)^d$. Set $M_1 := WM$.

Since the $(1,1)$-entry of M_1 is the constant 1, we can and do use Gauss elimination to eliminate all the entries in the first column (while preserving, for every $\xi \in (\mathbb{C}\backslash 0)^d$, the rank of $M_1(\xi)$). From the resulting matrix, remove its first row and its first column, and denote the matrix so obtained by M_2. M_2 has $n-1$ columns. Also, since $\ker M \subset \ker M_1$, we conclude that $\ker M_2 \neq \{0\}$ (since otherwise $\ker M_1$ contains an element whose only non-zero entry is the first one, which is absurd, since the $(1,1)$-entry of M_1 is 1). Thus, by the induction hypothesis, there exists $\xi \in (\mathbb{C}\backslash 0)^d$ such that $\operatorname{rank} M_2(\xi) < n-1$. It then easily follows that $\operatorname{rank} M_1(\xi) < n$, and since $W(\xi)$ is non-singular, it must be that $\operatorname{rank} M(\xi) < n$, as claimed. □

By converting back the above result to the language of shift-invariant spaces, we get the following result, [23]. Note that the result is not a complete extension of Theorem 8, due to the injectability assumption here.

Theorem 40. *Let Φ be a finite set of compactly supported distributions, and assume that $S_\star(\Phi)$ is injectable. Then $E(\Phi)$ is linearly dependent if and only if there exists a linear combination $\phi_\star$ of Φ for which $E(\phi_\star)$ is linearly dependent, too.*

3.4. L_2-Stability and Frames in FSI Spaces, Fiberization

One of the main results in the theory of local SI spaces is the characterization of linear independence. A seemingly inefficient way to state the PSI case (Theorem 8) of this result is as follows: 'let ϕ be a compactly supported distribution, $\widehat{\phi}$ its Fourier transform. Given $\omega \in \mathbb{C}^d$, let C_ω be the one-dimensional subspace of $\mathcal{Q}$ spanned by the sequence

$$\phi_\omega : 2\pi\mathbb{Z}^d \to \mathbb{C} : \alpha \mapsto \widehat{\phi}(\omega + \alpha). \tag{41}$$

Let G_ω be the map

$$G_\omega : \mathbb{C} \to C_\omega : c \mapsto c\phi_\omega.$$

Then $\mathcal{T}_\phi$ is injective (i.e., $E(\phi)$ is linearly independent) if and only if each of the maps G_ω is injective'.

Armed with this new perspective of the linear independence characterization, we can now find with ease a similar form for the characterization of the linear independence in the FSI space setup. We just need to change the nature of the 'fiber' spaces C_ω: Given a finite vector of compactly supported functions Φ, and given $\omega \in \mathbb{C}^d$, we define (cf. (41))

$$C_\omega := \operatorname{span}\{\phi_\omega :\ \phi \in \Phi\},$$

and

$$G_\omega : \mathbb{C}^\Phi \to C_\omega : c \mapsto \sum_{\phi\in\Phi} c(\phi)\phi_\omega. \tag{42}$$

Then, the characterization of linear independence for local FSI spaces, Theorem 40 (when combined with Thoerem 8) says that $\mathcal{T}_\Phi$ is injective if and only if each G_ω, $\omega \in \mathbb{C}^d$, is injective.

The discussion above represents a general principle that turned out to be a powerful tool in the context of shift-invariant spaces: *fiberization.* Here, one is given an operator and is interested in a certain property of the operator T (e.g., its injectivity or its boundedness). The goal of fiberization is to associate the operator with a large collection of much simpler operators G_ω (=: fibers), to associate each one of them with an analogous property P_ω, and to prove that T satisfies P iff each G_ω satisfies the property P_ω (sometimes in some uniform way).

The idea of fiberization appears implicitly in many papers on SI spaces from the early 90's (e.g., [22], [23], [5]). It was formalized first in [41], and was applied in [42] to Weyl-Heisenberg systems, and in [43] to wavelet systems. We refer to [40] for more details and references.

In principle, the fiberization techniques of [41] apply to the operator $\mathcal{T}_\Phi\mathcal{T}_\Phi^*$, for some (finite of countable) $\Phi \subset L_2(\mathbb{R}^d)$, as well as to the operator $\mathcal{T}_\Phi^*\mathcal{T}_\Phi$. The first approach is *dual Gramian analysis*, while the second is *Gramian analysis.* We provide in this subsection a brief introduction to the latter, by describing its roots in the context of the FSI space $S_2(\Phi)$.

We start our discussion with the **Gramian matrix** $G := G_\Phi$ which is the analog of the function $\widetilde{\phi}^2$ (cf. (18)). The Gramian is an $L_2(\mathbb{T}^d)$-valued matrix indexed by $\Phi\times\Phi$, and its (φ,ϕ) entry is

$$[\widehat{\phi},\widehat{\varphi}](\omega) = \sum_{\alpha\in 2\pi\mathbb{Z}^d} \widehat{\phi}(\omega+\alpha)\overline{\widehat{\varphi}(\omega+\alpha)}.$$

In analogy to the PSI case (cf. the proof of Theorem 25), a dual basis for Φ may be given by the functions whose Fourier transforms are

$$G^{-1}\widehat{\Phi},$$

provided that the above expression represents well-defined functions.

In case $\Phi \subset \mathcal{L}_2(\mathbb{R}^d)$, the entries of G, hence the determinant $\det G$, all lie in the Wiener algebra $\mathcal{A}(\mathbb{R}^d)$. If $\det G$ vanishes nowhere, one obtains that the functions whose Fourier transforms are given by $G^{-1}\widehat{\Phi}$ can be written each as

$\mathcal{T}_\Phi\, c$, for some $c \in \ell_1(\mathbb{Z}^d) \times \Phi$. The functions obtained in this way, thus, lie in $L_2(\mathbb{R}^d)$, and in this way, [22] extend Theorem 32 to FSI spaces: the crucial PSI condition (that $\widetilde{\phi}$ vanishes nowhere) is replaced by the condition that the Gramian is non-singular everywhere. That non-singularity is equivalent to the injectivity of the map G_ω (cf. (42)) for every *real* $\omega \in \mathbb{R}^d$.

We wish to discuss in more detail the L_2-stability and frame notions, for general Φ in $L_2(\mathbb{R}^d)$. Then the entries of G, hence its determinant, may not be continuous. The extension of the L_2-results to FSI spaces cannot make use of the mere non-singularity of G (on $\mathbb{R}^d$). Instead, one inspects the norms of the operators

$$G_\omega : \ell_2(\Phi) \to \ell_2(\Phi) : v \mapsto G(\omega)v.$$

We also recall the notion of a **pseudo-inverse**: for a linear operator L on a finite-dimensional (inner product) space, the pseudo-inverse L^{-1} of L is the unique linear map for which $L^{-1}L$ is the orthogonal projector with kernel $\ker L$. If L is non-negative Hermitian (like G_ω is), then $\|L^{-1}\| = 1/\lambda_+$, with λ_+ the smallest non-zero eigenvalue of L.

The result stated below was established in [5] (stability characterization) and in [41] (frame characterization). The reference [5] contains a characterization of the so-called *quasi-stability* which is a slightly stronger notion than the notion of a frame (and which coincides with the frame notion in the PSI case).

In the statement of the result below, we use the norm functions

$$\mathcal{G} : \omega \mapsto \|G_\omega\|,$$

and

$$\mathcal{G}^{-1} : \omega \mapsto \|{G_\omega}^{-1}\|.$$

Recall that ${G_\omega}^{-1}$ is a pseudo-inverse, hence is always well-defined.

Theorem 43. *Let Φ be a finite vector of $L_2(\mathbb{R}^d)$ functions. Then:*

(a) $E(\Phi)$ is a Bessel system (i.e., $\mathcal{T}_{\Phi,2}$ is bounded) iff $\mathcal{G} \in L_\infty(\mathbb{R}^d)$. Moreover, $\|\mathcal{T}_{\Phi,2}\|^2 = \|\mathcal{G}\|_{L_\infty}$.

(b) Assume that $E(\Phi)$ is a Bessel system. Then $E(\Phi)$ is a frame iff $\mathcal{G}^{-1} \in L_\infty(\mathbb{R}^d)$. Moreover, the square of the lower frame bound (cf. (22)) is then $1/\|\mathcal{G}^{-1}\|_{L_\infty}$.

(c) $E(\Phi)$ is stable iff it is a frame and in addition $S_2(\Phi)$ is regular, *i.e., G_ω is non-singular a.e.*

§4. Refinable shift-invariant spaces

Refinable shift-invariant spaces are used in the construction of wavelet systems via the vehicle of *multiresolution analysis*. It is beyond the scope of this article to review, to any extent, the rich connections between shift-invariant space theory on the one hand and refinable spaces (and wavelets) on the other hand. I refer to [40] for some discussion of these connections.

Definition 44. *Let N be a positive integer. A compactly supported distribution $\phi \in \mathcal{D}'(\mathbb{R}^d)$ is called* N**-refinable** *if $\phi(\cdot/N) \in S_\star(\phi)$.*

Another possible definition of refinability is given on the Fourier domain: an $L_2(\mathbb{R}^d)$-function ϕ is **refinable** if there exists a bounded 2π-periodic τ such that, a.e. on $\mathbb{R}^d$,

$$\widehat{\phi}(N\omega) = \tau(\omega)\widehat{\phi}(\omega).$$

The definitions are not equivalent, but are closely related and are both used in the literature. We will be primarily interested in the case of a univariate compactly supported $L_2(\mathbb{R})$-function ϕ with globally linearly independent shifts. For such a case, the above two definitions coincide, and the **mask** function τ is a trigonometric polynomial.

Our discussion here is divided into two parts: in §4.1, we present a remarkable property of 2-refinable univariate local PSI spaces: for such spaces, the basic property of global linear independence (that can always be achieved by a suitable selection of the generator of the space, cf. Theorem 15) implies the much stronger property of *local* linear independence. Unfortunately, this result does not extend to *any* more general setup.

A major problem in the context of refinable functions is the identification of their properties by a mere inspection of the mask function. While we do not attempt to address that topic (this article is devoted exclusively to the intrinsic properties of SI spaces), we give, in §4.2, a single example that shows how the basic tools and results about SI spaces help in the study of that problem.

4.1. Local Linear Independence in Univariate Refinable PSI Spaces

A strong independence relation is that of *local linear independence* (cf. §2.3). It is well-known that univariate polynomial B-splines satisfy this property. On the other hand, the support function of the interval $[0, 1.5]$ is an example of a function whose shifts are (gli) (hence (wlli)), but are not (slli), and thus, local linear independence is properly stronger than its global counterpart, even in the univariate context. It is then remarkable to note that, for a 2-*refinable* univariate compactly supported ϕ, global independence and local independence are equivalent.

The theorem below is due to [28]. For a function ϕ with *orthonormal* shifts, it was proved before by Meyer, [30].

Theorem 45. *Let ϕ be a univariate refinable function whose shifts satisfy the local spanning property (ls) (cf. Theorem 13). Then, $E(\phi)$ is locally linearly independent.*

Proof: By Theorem 15, the local spanning property is equivalent to the global linear independence property, and we will use that latter property in the proof.

It will be convenient during the proof to use an alternative notation for the synthesis operator $\mathcal{T}_\phi$. Thus, we set

$$\phi *' : \mathcal{Q} \to S_\star(\phi) : c \mapsto \mathcal{T}_\phi\, c,$$

i.e., $\phi *' := \mathcal{T}_\phi$.

We assume that ϕ is supported in $[0, N]$ and that $\psi = \phi *' a$, with the sequence a supported in $\{0, 1, 2, ..., N\}$ (and, thus, ψ is supported in $[0, 2N]$); we also assume that the shifts of ϕ are *locally* linearly independent over some interval $[0, k]$, and that the even shifts of ψ are (globally) linearly independent. Under all these assumptions, we prove that the even shifts of ψ are locally linearly independent over $[0, k]$, as well.

The theorem will follow from the above: for $\psi := \phi(\cdot/2)$, the global linear independence of $E(\phi)$ is equivalent to the global linear independence of the even shifts of ψ. Thus, assuming the shifts of ϕ are locally independent over $[0, k]$, the above claim (once proved) would imply that the even shifts of ψ are locally independent over that set, too, and this amounts to the local independence of the shifts of ϕ over $[0, k/2]$. Starting with $k := N - 1$ (i.e., invoking Proposition 16), we can then proceed until the interval is as small as we wish.

Let $f = \sum_{j\in\mathbb{Z}} b'(2j)\psi(\cdot - 2j)$. Assuming f to vanish on $[0, k]$, we want to show that $b'(2j) = 0$, $-N < j < k/2$. Since $\psi \in S_\star(\phi)$, $f \in S_\star(\phi)$. Thus, $f = \phi *' c$. The local linear independence of $E(\phi)$ over $[0, k]$ implies that $c(-N + 1) = c(-N + 2) = \ldots = c(k - 1) = 0$. We define

$$f_1 := \sum_{j\le -N} c(j)\phi(\cdot - j) =: \phi *' c_1, \quad f_2 := \sum_{j\ge k} c(j)\phi(\cdot - j) =: \phi *' c_2.$$

Since c_2 vanishes on $j < k$, (and assuming without loss that $a(0) \neq 0$), we can find a sequence b_2 supported also on $j \ge k$ such that $c_2 = a * b_2$. Then,

$$f_2 = \phi *' c_2 = \phi *' (a * b_2) = (\phi *' a) *' b_2 = \psi *' b_2.$$

By the same argument (and assuming $a(N) \neq 0$), we can find a sequence b_1 supported on $\{-2N, -2N - 1, ...\}$ such that $c_1 = a * b_1$, hence, as before,

$$f_1 = \psi *' b_1.$$

Thus, we found that $f = \psi *' (b_1 + b_2)$, with $b := b_1 + b_2$ vanishing on $-2N < j < k$. Since also $f = \psi *' b'$, we conclude that $\lambda := b - b'$ lies in $\ker \psi *'$. This leads to

$$0 = \psi *' \lambda = (\phi *' a) *' \lambda = \phi *' (a * \lambda).$$

Since $E(\phi)$ is linearly independent, $a * \lambda = 0$. Further, λ vanishes at all odd integers in the interval $(-2N, k)$. If λ vanishes also at all even integers in that interval, so does b' and we are done, since this is exactly what we ought to prove. Otherwise, since $\dim \ker a* = N$, and the interval $(-2N, k)$ contains at least N consecutive odd integers we must have (cf. Lemma 46 below) $\theta \in \mathbb{C}\backslash 0$ such that $\pm\theta \in \mathrm{spec}(a*)$, i.e., such that the two exponential sequences

$$\mu_1 : j \mapsto \theta^j, \quad \mu_2 : j \mapsto (-\theta)^j$$

lie in $\ker a*$. But, then, $\mu := \mu_1 + \mu_2 \in \ker a*$, and is supported only on the even integers. Since $\psi *' \mu = \phi *' (a * \mu) = 0$, this contradicts the global linear independence of the 2-shifts of ψ. $\square$

Lemma 46. *Let $a : \mathbb{Z} \to \mathbb{C}$ be a sequence supported on $[0, N]$. If $0 \neq \lambda \in \ker a*$, and λ vanishes at N consecutive even (odd) integers, then there exists $\theta \in \mathbb{C}\backslash 0$ such that the sequences $j \mapsto \theta^j$, and $j \mapsto (-\theta)^j$ lie both in $\ker(a*)$, i.e., $\{\pm\theta\} \subset \operatorname{spec}(a*)$.*

Proof (sketch): $\dim \ker a* \leq N$. Let $\theta \in \operatorname{spec}(a*)$, and assume that $-\theta$ is not there. Then, there exists a difference operator T supported on $N-1$ consecutive even points that maps $\ker a*$ onto the one-dimensional span of $j \mapsto \theta^j$. Since $T\lambda$ vanishes at least at one point, it must be that λ lies in the span of the other exponentials in $\ker(a*)$. □

Thus, for a univariate 2-refinable compactly supported function, we have the following remarkable result (compare with Theorem 13):

Corollary 47. *Let ϕ be a univariate 2-refinable compactly supported function. Then the properties (slli), (gli), (ldb), (ls) and (ms) are all equivalent for this ϕ.*

Theorem 45 does not extend to generators that are refinable by dilation factor $N \neq 2$. To see that, consider, for any integer $N \geq 2$ the refinable function ϕ_N defined as follows:

$$\widehat{\phi}_N(N\omega) = \tau_N(\omega)\widehat{\phi}_N(\omega),$$

with the Fourier coefficients $t_N(k)$ of the 2π-periodic trigonometric polynomial τ_N defined by

$$t_N(k) := \begin{cases} 1/2, & k \in \{0, \ldots, 2(N-1)\}\backslash\{N-1\}, \\ 1, & k = N-1, \\ 0, & \text{otherwise.} \end{cases}$$

The resulting refinable ϕ is supported in $[0, 2]$, has globally linearly independent shifts, and has linearly *dependent* shifts on the interval $(\frac{1}{N}, \frac{N-1}{N})$, which is non-empty for every $N \geq 3$. The case $N = 3$ appears in [14].

4.2. The Simplest Application of SI Theory to Refinable Functions

We close this article with an example that shows how general SI theory may be applied in the study of refinable spaces. The example is taken from [38].

Suppose that ϕ is a univariate, compactly supported, N-refinable distribution with trigonometric polynomial mask τ. Suppose that we would like, by inspecting τ only, to determine whether the shifts of ϕ are linearly independent. We can invoke to this end Theorem 15. By this theorem, there exists $\phi_0 \in S_\star(\phi)$, such that (i): $\phi = \mathcal{T}_{\phi_0} c$, for some finitely supported c (defined on $\mathbb{Z}$), and (ii): $E(\phi_0)$ is linearly independent. One then easily conclude that (i): ϕ_0 is also refinable, with a trigonometric polynomial mask t, and (ii):

$$\tau = t\frac{\widehat{c}(N\cdot)}{\widehat{c}}. \tag{48}$$

This leads to a characterization of the linear independence property of the univariate $E(\phi)$ in terms of the no-factorability of τ in the form (48), [38]. That characterization leads then easily to the characterization of the linear independence property in terms of the distribution of the zeros of τ [25], [38], [50].

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