THE L_p-APPROXIMATION ORDER OF SURFACE SPLINE INTERPOLATION FOR $1 \le p \le 2$

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Abstract. We show that the L_p-approximation order of surface spline interpolation equals $m+1/p$ for p in the range $1 \le p \le 2$, where m is an integer parameter which specifies the surface spline. Previously it was known that this order was bounded below by $m+1/2$ and above by $m+1/p$. With h denoting the fill-distance between the interpolation points and the domain Ω, we show specifically that the $L_p(\Omega)$-norm of the error between f and its surface spline interpolant is $O(h^{m+1/p})$ provided that f belongs to an appropriate Sobolev or Besov space and $\Omega \subset \mathbb{R}^d$ is open, bounded and has the C^{2m}-regularity property. We also show that the boundary effects (which cause the rate of convergence to be significantly worse than $O(h^{2m})$) are confined to a boundary layer whose width is no larger than a constant multiple of $h\,|\log h|$. Finally, we state numerical evidence which supports the conjecture that the L_p-approximation order of surface spline interpolation is $m+1/p$ for $2 < p \le \infty$.

1. Introduction

Given a scattered set of points $\Xi \subset \mathbb{R}^d$ and data $f_{|\Xi}$, the scattered data interpolation problem is that of finding a 'nice' function s, defined on $\mathbb{R}^d$, which interpolates the data; that is, which satisfies

$$s(\xi) = f(\xi) \text{ for all } \xi \in \Xi. \tag{1.1}$$

Surface spline interpolation is one approach to this problem. It was defined by Duchon [8] as the solution of the following variational problem. For an integer $m > d/2$, let H^m denote the space of distributions f for which $D^\alpha f \in L_2 := L_2(\mathbb{R}^d)$ for all $|\alpha| = m$ and define the seminorm

$$|f|_{H^m} := (2\pi)^{d/2} \sqrt{\sum_{|\alpha|=m} \tau_\alpha \, \|D^\alpha f\|_{L_2}^2},$$

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where the positive integers τ_α are determined by the equation $|x|^{2m} = \sum_{|\alpha|=m} \tau_\alpha x^{2\alpha}$, $x \in \mathbb{R}^d$. Let Π_k denote the space of polynomials over $\mathbb{R}^d$ whose total degree does not exceed k. Duchon proved that if Ξ is any subset of $\mathbb{R}^d$ satisfying

$$q(\Xi) \neq \{0\} \text{ for all } q \in \Pi_{m-1} \backslash 0 \tag{1.2}$$

(ie Ξ is not contained in the zero set of any nontrivial polynomial in Π_{m-1}), then for any $f \in H^m$, there exists a unique $s \in H^m$ which minimizes $|s|_{H^m}$ subject to the interpolation conditions (1.1). The function s is called the *surface spline interpolant* to f at Ξ and will be denoted by $T_\Xi f$. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be the radially symmetric function given by

$$\phi(x) := \begin{cases} |x|^{2m-d} & \text{if } d \text{ is odd} \\ |x|^{2m-d} \log |x| & \text{if } d \text{ is even} \end{cases}.$$

In case Ξ is a finite set satisfying (1.2), Duchon showed that $T_\Xi f$ can be expressed as the unique function which satisfies the interpolation conditions (1.1) and has the form

$$q + \sum_{\xi \in \Xi} \lambda_\xi \phi(\cdot - \xi)$$

where q is a polynomial in Π_{m-1} and the coefficients $\{\lambda_\xi\}$ are subject to the auxiliary conditions

$$\sum_{\xi \in \Xi} \lambda_\xi p(\xi) = 0 \text{ for all } p \in \Pi_{m-1}.$$

We mention that surface spline interpolation can be generalized by replacing ϕ with any radially symmetric function which is conditionally strictly positive definite of order m. In the literature, this generalization goes by the name of *Radial basis function interpolation* for which the reader is referred to the surveys [10], [7] and [11].

In order to discuss the approximation power of surface spline interpolation, let us assume that Ω is an open, bounded subset of $\mathbb{R}^d$ and that the interpolation points Ξ are contained in $\overline{\Omega}$:=closure(Ω). The *fill distance* from Ξ to Ω is defined by

$$h := h(\Xi, \Omega) := \sup_{x \in \Omega} \inf_{\xi \in \Xi} |x - \xi| .$$

Definition. A scattered data interpolation method $f_{|\Xi} \mapsto s$ provides *L_p-approximation of order* $\gamma > 0$ if for every open, connected and bounded $\Omega \subset \mathbb{R}^d$, having a C^∞ boundary, and for every $f \in C^\infty(\mathbb{R}^d)$, we have

$$\|f - s\|_{L_p(\Omega)} = O(h^\gamma) \text{ as } h \to 0.$$

The largest (or supremum of all) such γ is *the L_p-approximation order* of the method.

The above definition is designed to reflect the rate at which s will converge to f, in the $L_p(\Omega)$-norm, under very favorable circumstances. Results about the approximation order found in the literature usually assume much less regarding the smoothness of the boundary and the smoothness of the data function f. However, the assumptions are sufficiently weak to disallow 'special cases' where, for example, f is assumed to satisfy certain boundary conditions, or f is restricted to a certain class of entire functions, or the interpolation points Ξ are assumed to satisfy some quasi-uniform condition.

Duchon [9] has shown that the approximation order of surface spline interpolation is at least $\gamma_p := \min\{m, m - d/2 + d/p\}$ for $1 \le p \le \infty$. He actually showed that $\|f - T_\Xi f\|_{L_p(\Omega)} = o(h^{\gamma_p})$ for all $f \in H^m$ whenever the domain Ω has the cone property (see the following section for the details). Duchon's error analysis was eventually generalized by Wu and Schaback [21] and Wendland [20] to apply to a large family of radial basis function interpolation methods. At the same time, there were efforts to understand the special case when $\Omega = \mathbb{R}^d$ and $\Xi = h\mathbb{Z}^d$. Although this special case is quite different from the desired setup, it was a tempting case because it falls in line with the very successful theory of approximation from shift-invariant spaces (see [5]). It was shown by Buhmann [6] and Jia and Lei [12] that the L_p-approximation order of surface spline interpolation in this special case equals $2m$. Recently, Matveev [16] obtained the same result without assuming Ξ to be the grid $h\mathbb{Z}^d$ (yet still maintaining $\Omega = \mathbb{R}^d$). The reader will notice the rather large discrepancy between the values of γ_p and $2m$, the latter being at least twice the former. For some time, it wasn't clear whether the value γ_p was excessively low due to limitations of Duchon's approach or whether there was some essential difference between the case of a bounded domain Ω and the case where $\Omega = \mathbb{R}^d$. That the latter is in fact the case was shown by the author [13] by proving that the L_p-approximation order of surface spline interpolation does not exceed $m + 1/p$ for $1 \le p \le \infty$. Subsequently, the author [15] has improved Duchon's original error estimate to the extent of showing that the L_p-approximation order of surface spline interpolation is at least $\gamma_p + 1/2$. The estimate $\|f - T_\Xi f\|_{L_p(\Omega)} = O(h^{\gamma_p + 1/2})$ was obtained assuming that Ω has the uniform C^{2m}-regularity property (see [1, p. 67]) and that f belongs to the Besov space $B_{2,1}^{m+1/2}$ (see section 2 for this definition). Noting that this improved lower bound of $\gamma_p + 1/2$ agrees with the upper bound of $m + 1/p$ at $p = 2$, we see that the L_2-approximation order of surface spline interpolation equals $m + 1/2$.

The main purpose of the present contribution is to further improve this lower bound to the extent of showing that the L_p-approximation order of surface spline interpolation equals $m + 1/p$ for $1 \le p \le 2$. Since this is known for $p = 2$, once we establish it for $p = 1$, it will be possible to obtain the result for the range $1 < p < 2$ by interpolation. We will show that if Ω is open, bounded and has the uniform C^{2m}-regularity property, then

$$\|f - T_\Xi f\|_{L_1(\Omega)} \le const(m, \Omega) h^{m+1} \|f\|_{W_2^{m+1}(\mathbb{R}^d)}$$

whenever $f \in W_2^{m+1}(\mathbb{R}^d)$ and h is sufficiently small. Here $W_p^k(A)$ denotes the Sobolev space of all distributions f for which

$$\|f\|_{W_p^k(A)} := \sum_{|\alpha| \le k} \|D^\alpha f\|_{L_p(A)} < \infty.$$

Our approach to this result can be described in fairly simple language. Let us first state clearly our assumptions and a few definitions for later reference.

Assumptions 1.3. We assume that the data function f belongs to $W_2^{m+1}(\mathbb{R}^d)$. We assume Ω is an open, bounded subset of $\mathbb{R}^d$ having the uniform C^{2m}-regularity property (see [1, p. 67]) and Ξ is a finite subset of $\overline{\Omega}$ which satisfies (1.2). We let $h = h(\Xi, \Omega)$ denote the fill distance from Ξ to Ω, and we define Ξ' and Ξ'' by

$$\Xi' := h\mathbb{Z}^d \backslash (\Omega + hB) \text{ and } \Xi'' := \Xi \cup \Xi'.$$

For each $\xi \in \Xi''$, let L_ξ denote the Lagrange function defined by

$$L_\xi = T_{\Xi''} g_\xi,$$

where g_ξ is any function in H^m satisfying $g_\xi(\xi) = 1$ and $g_\xi(\Xi'' \backslash \xi) = \{0\}$.

The error $f - T_\Xi f$ can be written, first of all, as $(f - T_{\Xi''} f) + (T_{\Xi''} f - T_\Xi f)$. Since $\Xi \subset \Xi''$, it follows that $T_\Xi f = T_{\Xi''} T_\Xi f$ and hence we obtain

$$f - T_\Xi f = (f - T_{\Xi''} f) + T_{\Xi''}(f - T_\Xi f) =: I + II.$$

To express the error in the above form was first suggested by Mike Powell and can be found in Bejancu [3]. That I decays like $O(h^{m+1})$ can be obtained by interpolating between results of Duchon [9] and Matveev [16]. For II, we imitate Bejancu's approach, and express II in terms of the Lagrange basis as

$$II = T_{\Xi''}(f - T_\Xi f) = \sum_{\xi \in \Xi''} (f - T_\Xi f)(\xi) L_\xi. \tag{1.4}$$

That the right side converges meaningfully and that equality indeed holds in (1.4) will be shown in section 3. Since $f - T_\Xi$ vanishes on Ξ, the sum in (1.4) is actually over $\xi \in \Xi'$. We will employ a crucial lemma of Matveev which shows that the Lagrange functions L_ξ admit a certain exponential decay. The basic idea in estimating the $L_1(\Omega)$-norm of II is then to say that if $\xi \in \Xi'$ is close to Ω, then $(f - T_\Xi f)(\xi)$ is small (using the results of [15]) and if $\xi \in \Xi'$ is far from Ω, then the $L_1(\Omega)$-norm of L_ξ is small due to its exponential decay. This description is somewhat oversimplified because the sharp error estimate in [15] is in the $L_2(\Omega)$-norm rather than the $L_\infty(\Omega)$-norm, but there is, nevertheless, a somewhat complicated way of organizing the sum in (1.4) so as to obtain the result that the $L_1(\Omega)$-norm of II decays like $O(h^{m+1})$.

An outline of this paper is as follows. In section 2, we state the definitions and related results from the literature which will be subsequently needed. Section 3 is devoted to justifying (1.4) by showing that $T_{\Xi''} f$ equals its Lagrange expansion $\sum_{\xi \in \Xi} f(\xi) L_\xi$. The main results of the paper are in section 4 where it is shown that the L_p-approximation order of surface spline interpolation is $m + 1/p$ for $1 \le p < 2$. The proof of one key proposition has, for the sake of clarity, been postponed until section 5. Lastly, in section 6, we give a result which shows that surface spline interpolation provides L_∞-approximation of order $2m$, except in a shrinking boundary layer. We also provide some numerical evidence which supports the conjecture that the L_p-approximation order of surface spline interpolation is $m + 1/p$ for $2 < p \le \infty$.

Throughout this paper we use standard multi-index notation: $D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}\cdots\frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$. The natural numbers are denoted $\mathbb{N} := \{1, 2, 3, \dots\}$, and the non-negative integers are denoted $\mathbb{N}_0$. For multi-indices $\alpha \in \mathbb{N}_0^d$, we define $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_d$, while for $x \in \mathbb{R}^d$, we define $|x| := \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$. The Fourier transform of an integrable function f is defined by $\widehat{f}(w) := \int_{\mathbb{R}^d} e_{-w}(x) f(x)\, dx$, where $e_w(x) := e^{iw\cdot x}$. The space of compactly supported C^∞ functions whose support is contained in $A \subset \mathbb{R}^d$ is denoted $C_c^\infty(A)$. The open unit ball in $\mathbb{R}^d$ is denoted $B := \{x \in \mathbb{R}^d : |x| < 1\}$. If μ is a distribution and g is a test function, then the application of μ to g is denoted $\langle g, \mu\rangle$. We employ the notation $const$ to denote a generic constant in the range $(0, \infty)$ whose value may change with each occurrence. An important aspect of this notation is that $const$ depends only on its arguments if any, and otherwise depends on nothing.

2. Results from the Literature

In this section we state the definitions and results from the literature upon which the present contribution builds. The Fourier transform plays an essential role in defining certain function spaces which we will need. For example, it follows from the Plancherel theorem that for $f \in H^m$, $|f|_{H^m} = \left\| |\cdot|^m \widehat{f} \right\|_{L_2(\mathbb{R}^d \backslash 0)}$. For $s \geq 0$, let W^s denote the Sobolev space of $f \in L_2$ for which

$$\|f\|_{W^s} := \left\| (1 + |\cdot|^2)^{s/2} \widehat{f} \right\|_{L_2} < \infty.$$

For $k \in \mathbb{N}_0$ it can be easily shown, using the Plancherel theorem, that $W^k = W_2^k(\mathbb{R}^d)$ (with equivalent norms). Another important family of spaces which are defined using the Fourier transform are the Besov spaces.

Definition 2.1. Let $A_0 := \overline{B}$, and for $k \in \mathbb{N}$, let $A_k := 2^k\overline{B}\backslash 2^{k-1}B$, where $B := \{x \in \mathbb{R}^d : |x| < 1\}$. The Besov space $B_{2,q}^\gamma$, $\gamma \in \mathbb{R}$, $1 \leq q \leq \infty$, is defined to be the set of all tempered distributions f for which $\widehat{f}$ is a locally integrable function and

$$\|f\|_{B_{2,q}^\gamma} := \left\| k \mapsto 2^{k\gamma} \left\| \widehat{f} \right\|_{L_2(A_k)} \right\|_{\ell_q(\mathbb{N}_0)} < \infty.$$

The above defined spaces W^γ and $B_{2,q}^\gamma$ are complete. The following continuous embeddings can be found in [18] (they are also easy to prove from the definitions):

$$\begin{aligned}
&B_{2,q_1}^{s_1} \hookrightarrow B_{2,q_2}^{s_2} \quad \text{if } s_1 > s2, \\
&B_{2,q_1}^{s} \hookrightarrow W^s \hookrightarrow B_{2,q_2}^{s} \quad \text{if } q_1 \leq 2 \leq q_2,\ s \geq 0, \text{ and} \\
&W^{s_1} \hookrightarrow B_{2,q}^{s} \hookrightarrow W^{s_2} \quad \text{if } s_1 > s > s_2 \geq 0.
\end{aligned}$$

In particular, if $s \geq 0$, then $W^s = B_{2,2}^s$ (with equivalent norms).

We begin with results of Matveev. The following theorem is proved in [16, page 130].

Theorem 2.2. *There exists $\nu \in (0,1)$ (depending only on d and m) such that if $a \in \mathcal{A} \subset \mathbb{R}^d$ satisfy $r_2 := h(\mathcal{A}, \mathbb{R}^d) < \infty$ and $r_3 := \operatorname{dist}(a, \mathcal{A}\backslash\{a\}) > 0$, then for all $r > 0$,*

$$\|T_{\mathcal{A}} g\|_{W_2^m(\mathbb{R}^d \backslash (a+rB))} \leq const(d,m) \nu^{r/r_2} r_3^{d/2-m} (1+r_2)^m,$$

where g is any function in H^m satisfying $g(a) = 1$ and $g(\mathcal{A}\backslash\{a\}) = \{0\}$.

An important consequence of this theorem is the exponential decay of our Lagrange functions L_ξ defined in Assumptions 1.3.

Corollary 2.3. *Assuming Assumptions 1.3, there exists $c > 0$ (depending only on d, m) such that for $\xi \in \Xi'$, the Lagrange function L_ξ satisfies the decay estimate*

$$|L_\xi(x)| \leq const(d,m) \exp(-c\,|x - \xi|\,/h), \quad x \in \mathbb{R}^d.$$

Proof. It suffices to prove the corollary for the special case $h = 1$ since the general case can then be obtained by scaling both Ξ and Ω and employing the identity $T_{h^{-1}\mathcal{A}}(g(h\cdot)) = (T_{\mathcal{A}} g)(h\cdot)$; so assume $h = 1$. Let $\xi \in \Xi'$, and note that the assumptions in force ensure that $\delta := h(\Xi'', \mathbb{R}^d) \leq (2 + \sqrt{d}/2)$ and $\operatorname{dist}(\xi, \Xi''\backslash\{\xi\}) \geq 1$. By Theorem 2.2, there exists $\nu \in (0,1)$ (depending only on d, m) such that $\|L_\xi\|_{W_2^m(\mathbb{R}^d\backslash(\xi+rB))} \leq const(d,m)\nu^r$ for all $r > 0$. Define $c := -\log \nu > 0$. Let $x \in \mathbb{R}^d\backslash\{\xi\}$ and put $r := |\xi - x|$. Since $m > d/2$, it follows from the Sobolev Embedding Theorem [1, p. 97] that

$$\begin{aligned}|L_\xi(x)| \leq \|L_\xi\|_{L_\infty(\mathbb{R}^d\backslash(\xi+rB))} &\leq const(d,m)\,\|L_\xi\|_{W_2^m(\mathbb{R}^d\backslash(\xi+rB))} \\ &\leq const(d,m)\nu^r = const(d,m) e^{-c|\xi - x|}.\end{aligned}$$

□

In the following, we use $D^m g$ to denote the vector $(D^\alpha g)_{|\alpha|=m}$, and adopt the notation

$$\|D^m g\|_X := \sum_{|\alpha|=m} \|D^\alpha g\|_X\,.$$

The following is proved in [16, Theorem 6].

Theorem 2.4. *If $\mathcal{A} \subset \mathbb{R}^d$ satisfies $\delta := h(\mathcal{A}, \mathbb{R}^d) < \infty$, then for all $1 \leq p \leq \infty$ and $f \in W_p^{2m}(\mathbb{R}^d) \cap H^m$,*

$$\|f - T_{\mathcal{A}} f\|_{L_p(\mathbb{R}^d)} \leq const(d,m,p)\delta^{2m} \left\|D^{2m} f\right\|_{L_p(\mathbb{R}^d)}.$$

The results of Duchon employ domains having the cone property; the following is equivalent to the standard definition of the cone property.

Definition 2.5. A set $\Omega \subset \mathbb{R}^d$ is said to have the *cone property* if there exists $\varepsilon_\Omega, r_\Omega \in (0,\infty)$ such that for all $x \in \Omega$ there exists $y \in \Omega$ such that $|x - y| = \varepsilon_\Omega$ and

$$(1-t)x + ty + r_\Omega t B \subset \Omega \quad \forall t \in [0,1].$$

The following is proved in [9].

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^d$ be open, bounded and have the cone property. There exists $h_0 > 0$ (depending only on $m, d, \varepsilon_\Omega, r_\Omega$) such that if $h := h(\Xi, \Omega) \le h_0$, then*

$$\|f - T_\Xi f\|_{L_p(\Omega)} \le const(d, m, \varepsilon_\Omega, r_\Omega) h^{m-d/2+d/p} |f|_{H^m}$$

for all $f \in H^m$, $2 \le p \le \infty$.

We need the following corollary.

Corollary 2.7. *If $\mathcal{A} \subset \mathbb{R}^d$ satisfies $\delta := h(\mathcal{A}, \mathbb{R}^d) < \infty$, then*

$$\|f - T_\mathcal{A} f\|_{L_2} \le const(d, m) \delta^m |f|_{H^m} \text{ for all } f \in H^m.$$

Proof. Let B denote the unit ball in $\mathbb{R}^d$ centered at the origin. By Theorem 2.6 there exists $h_0 > 0$ such that if $\Xi \subset \overline{B}$ satisfies $h = h(\Xi, B) \le h_0$, then

$$\|f - T_\Xi f\|_{L_2(B)} \le const(d, m) h^m |f|_{H^m} \text{ for all } f \in H^m.$$

Let $\mathcal{A} \subset \mathbb{R}^d$ satisfy $\delta := h(\mathcal{A}, \mathbb{R}^d) < \infty$ and for $r > 0$ put $\mathcal{A}_r := B \cap (\mathcal{A}/r)$. Let $f \in H^m$ and put $g := f - T_\mathcal{A} f$. Then for r sufficiently large

$$\begin{aligned}&\|f - T_\mathcal{A} f\|_{L_2(rB)} = r^{d/2} \|g(r\cdot)\|_{L_2(B)} = r^{d/2} \|g(r\cdot) - T_{\mathcal{A}_r}(g(r\cdot))\|_{L_2(B)} \\ &\le const(d, m) r^{d/2} (\delta/r)^m |g(r\cdot)|_{H^m} = const(d, m) \delta^m |g|_{H^m} \le const(d, m) \delta^m |f|_{H^m}.\end{aligned}$$

The proof is then completed by taking the limit as $r \to \infty$. □

Since $W^m \subset H^m$ and the norm $\|\cdot\|_{W^m}$ is stronger than the seminorm $|\cdot|_{H^m}$, Corollary 2.7 remains valid with $|f|_{H^m}$ replaced by $\|f\|_{W^m}$. Similarly, the case $p = 2$ of Theorem 2.4 remains valid if $\left\|D^{2m} f\right\|_{L_2}$ is replaced by $\|f\|_{W^{2m}}$. Interpolating (see [4, p. 301,302] and [19, p. 39,40]) between these results yields the following.

Corollary 2.8. *If $\mathcal{A} \subset \mathbb{R}^d$ satisfies $\delta := h(\mathcal{A}, \mathbb{R}^d) < \infty$, then for all $\gamma \in [m, 2m]$ and $f \in W^\gamma$,*

$$\|f - T_\mathcal{A} f\|_{L_2(\mathbb{R}^d)} \le const(d, m, \gamma) \delta^\gamma \|f\|_{W^\gamma}.$$

In the following two theorems, it is assumed that $\Omega \subset \mathbb{R}^d$ is open, bounded and has the uniform C^{2m}-regularity property (see [1, p.67]). The first theorem is [15, Theorem 6.1] and the second is [15, Theorem 1.5].

Theorem 2.9. *If $f \in B_{2,1}^{m+1/2}$, then for all $|\alpha| = m$, $D^\alpha T_\Omega f \in B_{2,\infty}^{1/2}$ and*

$$\|D^\alpha T_\Omega f\|_{B_{2,\infty}^{1/2}} \le const(\Omega, m) \|f\|_{B_{2,1}^{m+1/2}}.$$

Theorem 2.10. *There exists $h_1 > 0$ (depending only on Ω, m) such that if $f \in B^{m+1/2}_{2,1}$ and $\Xi \subset \overline{\Omega}$ satisfies $h := h(\Xi, \Omega) \leq h_1$, then*

$$|T_\Omega f - T_\Xi f|_{H^m} \leq const(\Omega, m) h^{1/2} \|f\|_{B^{m+1/2}_{2,1}} \text{ and}$$
$$\|f - T_\Xi f\|_{L_p(\Omega)} \leq const(\Omega, m) h^{\gamma_p + 1/2} \|f\|_{B^{m+1/2}_{2,1}} \text{ for all } 1 \leq p \leq \infty,$$

where $\gamma_p := \min\{m, m - d/2 + d/p\}$.

The following is proved in [15, Lemma 6.4]

Lemma 2.11. *Let A be an open subset of $\mathbb{R}^d$ having a bounded boundary and the uniform C^1-regularity property. There exists $\varepsilon > 0$ (depending only on A) such that if $r \in [1, \infty)$, $\gamma \in (0, r]$ and $h \in (0, \varepsilon\gamma/r]$, then*

$$m_d((\partial A + hB) \cap (x + \gamma B)) \leq const(A) h \gamma^{d-1} \quad \forall x \in \mathbb{R}^d,$$

where m_d denotes Lebesgue measure in $\mathbb{R}^d$.

3. Justifying (1.4)

With Assumptions 1.3 in force, our first task is to justify (1.4); that is, to prove

Proposition 3.1. *For all $f \in H^m$,*

$$T_{\Xi''} f = \sum_{\xi \in \Xi''} f(\xi) L_\xi,$$

where the sum on the right converges uniformly and absolutely on compact sets.

Following Matveev [16], we define

$$\|f\|_{H^m} := \|f\|_{L_2(B)} + |f|_{H^m}, \quad f \in H^m,$$

where B denotes the unit ball in $\mathbb{R}^d$ centered at the origin. It is shown in [16, Lemma 4] that $\|\cdot\|_{H^m}$ is a complete norm on H^m and that for all $\zeta > 1$,

$$\|f\|_{W^m_2(rB)} \leq const(d, m, \zeta) \zeta^r \|f\|_{H^m}, \quad f \in H^m, r > 0.$$

Applying the Sobolev embedding theorem then yields

$$\|f\|_{L_\infty(rB)} \leq const(d, m, \zeta) \zeta^r \|f\|_{H^m}, \quad f \in H^m, r > 0. \tag{3.2}$$

One important consequence of this estimate in conjunction with Corollary 2.3 is that for functions $f \in H^m$ the sum $\sum_{\xi \in \Xi''} f(\xi) L_\xi$ converges uniformly and absolutely on compact sets (choose ζ with $1 < \zeta < e^{c/h}$).

It is obvious that the conclusion of Proposition 3.1 holds for functions $f \in H^m$ having compact support (as $T_{\Xi''}$ is a linear operator). We show in the following result that this conclusion also holds for polynomials in Π_{m-1}.

Lemma 3.3. *For all $q \in \Pi_{m-1}$,*

$$T_{\Xi''} q = q = \sum_{\xi \in \Xi''} q(\xi) L_\xi.$$

Proof. We may assume without loss of generality that q is a homogeneous polynomial of degree $k < m$. Since $|q|_{H^m} = 0$, it is clear that $T_{\Xi''} q = q$. Let $\sigma \in C_c^\infty(\mathbb{R}^d)$ be such that $\sigma = 1$ on B. The exponential decay of L_ξ, as described in Corollary 2.3, ensures that

$$\sum_{\xi \in \Xi''} q(\xi) L_\xi(x) = \lim_{\varepsilon \to 0} \sum_{\xi \in \Xi''} \sigma(\varepsilon \xi) q(\xi) L_\xi(x)$$

for all $x \in \mathbb{R}^d$. Note that

$$\left\| D^{2m}[\sigma(\varepsilon \cdot) q] \right\|_{L_\infty} = \varepsilon^{-k} \left\| D^{2m}[\sigma(\varepsilon \cdot) q(\varepsilon \cdot)] \right\|_{L_\infty} = \varepsilon^{2m-k} \left\| D^{2m}[\sigma q] \right\|_{L_\infty} \to 0 \quad \text{as } \varepsilon \to 0.$$

Hence, by Theorem 2.4,

$$\left\| \sigma(\varepsilon \cdot) q - \sum_{\xi \in \Xi''} \sigma(\varepsilon \xi) q(\xi) L_\xi \right\|_{L_\infty} \to 0 \text{ as } \varepsilon \to 0.$$

□

Let $\mathcal{H}^m$ denote the closure of $C_c^\infty(\mathbb{R}^d) + \Pi_{m-1}$ in H^m. We now show that the conclusion of Proposition 3.1 holds for all $f \in \mathcal{H}^m$. That $T_{\Xi''}$ is a bounded operator on H^m can be seen by noting, for $f \in H^m$, that $|T_{\Xi''} f|_{H^m} \le |f|_{H^m}$ and that

$$\begin{aligned} \|T_{\Xi''} f\|_{L_2(B)} &\le \|f\|_{L_2(B)} + \|f - T_{\Xi''} f\|_{L_2(\mathbb{R}^d)} \\ &\le \|f\|_{L_2(B)} + const(m, \Omega, \Xi)\, |f|_{L_2(\mathbb{R}^d)} \le const(m, \Omega, \Xi)\, \|f\|_{H^m}, \end{aligned}$$

where we have used Corollary 2.7 in the second inequality.

Lemma 3.4. *If $f \in \mathcal{H}^m$, then*

$$T_{\Xi''} f = \sum_{\xi \in \Xi''} f(\xi) L_\xi.$$

Proof. Let $f \in \mathcal{H}^m$, say $f_n \to f$ in H^m with $f_n \in C_c^\infty(\mathbb{R}^d) + \Pi_{m-1}$. With c as in Corollary 2.3, let ζ satisfy $1 < \zeta < e^{c/h}$ and note that estimate (3.2) yields

$$\max_{x \in \mathbb{R}^d} \frac{|f_n(x) - f(x)|}{\zeta^{|x|}} \to 0 \text{ as } n \to \infty.$$

It now follows from Corollary 2.3 that

$$\sum_{\xi \in \Xi''} f_n(\xi) L_\xi \to \sum_{\xi \in \Xi''} f(\xi) L_\xi$$

uniformly on compact sets as $n \to \infty$. Since $T_{\Xi''}$ is a bounded operator on H^m, it follows that $T_{\Xi''} f_n \to T_{\Xi''} f$ in H^m. Moreover, in view of (3.2), we see that $T_{\Xi''} f_n \to T_{\Xi''} f$ uniformly on compact sets as $n \to \infty$. Our desired conclusion now follows from the fact that $T_{\Xi''} f_n = \sum_{\xi \in \Xi''} f_n(\xi) L_\xi$ for all n. □

So, with Lemma 3.4 in view, in order to prove Proposition 3.1, it suffices to prove that $H^m = \mathcal{H}^m$.

Lemma 3.5. $H^m = \mathcal{H}^m$.

Proof. Since H^m is complete and contains both $C_c^\infty(\mathbb{R}^d)$ and Π_{m-1}, it is clear that $\mathcal{H}^m \subset H^m$, so we concentrate on the opposite inclusion. Let $\sigma \in C_c^\infty(2B)$ satisfy $\sigma = 1$ on B and $\|\sigma\|_{L_\infty} = 1$. Let $f \in H^m$, and define the distribution $\widehat{\nu}$ by

$$\langle g, \widehat{\nu} \rangle := \int_{\mathbb{R}^d \setminus 0} (g - \sigma P_{m-1} g) \widehat{f}, \quad g \in C_c^\infty(\mathbb{R}^d),$$

where $P_{m-1}g$ denotes the Taylor polynomial of degree $m-1$ to g at 0. Note that the integrand is absolutely integrable since it is compactly supported and

$$|g(w) - \sigma P_{m-1} g(w)| = O(|w|^m) \text{ as } |w| \to 0.$$

Since $\widehat{\nu}$ is the sum of a compactly supported distribution and an integrable function, it follows that $\widehat{\nu}$ is a tempered distribution and that ν has at most polynomial growth. Note that if $|\alpha| = m$ and $g \in C_c^\infty(\mathbb{R}^d)$, then

$$\begin{aligned} |\langle g, (D^\alpha \nu)\widehat{\ } \rangle| = |\langle ()^\alpha g, \widehat{\nu} \rangle| &\le \int_{\mathbb{R}^d \setminus 0} |w|^m \, |g(w)| \left| \widehat{f}(w) \right| dw \\ &\le \|g\|_{L_2} \left\| |\cdot|^m \widehat{f} \right\|_{L_2(\mathbb{R}^d \setminus 0)} = \|g\|_{L_2} |f|_{H^m} . \end{aligned}$$

By the Riesz representation theorem and the Plancherel theorem, $D^\alpha \nu \in L_2$ for all $|\alpha| = m$; consequently, $\nu \in H^m$. Since $\widehat{\nu} = \widehat{f}$ on $\mathbb{R}^d \setminus 0$ and $\nu, f \in H^m$, it follows that $f = \nu + q$ for some $q \in \Pi_{m-1}$. We will show that $\nu \in \mathcal{H}^m$. For $\varepsilon > 0$, define ν_ε by

$$\langle g, \widehat{\nu_\varepsilon} \rangle := \int_{\mathbb{R}^d \setminus \varepsilon B} (g - \sigma P_{m-1} g) \widehat{f}, \quad g \in C_c^\infty(\mathbb{R}^d).$$

Since $C_c^\infty(\mathbb{R}^d)$ is dense in W_2^m and the norm $\|\cdot\|_{W_2^m}$ is stronger than $\|\cdot\|_{H^m}$, it follows that $W_2^m \subset \mathcal{H}^m$. Consequently, since $\nu_\varepsilon \in W_2^m + \Pi_{m-1}$, it follows that $\nu_\varepsilon \in \mathcal{H}^m$ for all $\varepsilon > 0$. We will show that

$$\|\nu - \nu_\varepsilon\|_{H^m} \to 0 \text{ as } \varepsilon \to 0.$$

Note that

$$\langle g, \widehat{\nu} - \widehat{\nu_\varepsilon} \rangle = \int_{\varepsilon B \setminus 0} (g - \sigma P_{m-1} g) \widehat{f}, \quad g \in C_c^\infty(\mathbb{R}^d).$$

For the seminorm we have

$$|\nu - \nu_\varepsilon|_{H^m} = \left\| |\cdot|^m \widehat{f} \right\|_{L_2(\varepsilon B \setminus 0)} \to 0 \text{ as } \varepsilon \to 0.$$

If $x \in B$, then $|(e_x - P_{m-1} e_x)(w)| \le const(d,m) \, |w|^m$ and consequently

$$\begin{aligned} |(\nu - \nu_\varepsilon)(x)| = (2\pi)^{-d} \, |\langle e_x, \widehat{\nu} - \widehat{\nu_\varepsilon} \rangle| &\le const(d,m) \int_{\varepsilon B \setminus 0} |w|^m \left| \widehat{f}(w) \right| dw \\ &\le const(d,m) \, \|1\|_{L_2(\varepsilon B)} \left\| |\cdot|^m \widehat{f} \right\|_{L_2(\varepsilon B \setminus 0)} \to 0 \text{ as } \varepsilon \to 0. \end{aligned}$$

Therefore, $\|\nu - \nu_\varepsilon\|_{H^m} \to 0$ and hence $f = \nu + q \in \mathcal{H}^m$. $\square$

4. The Error Analysis

Let Assumptions 1.3 be in force. As mentioned in the introduction, we express the error between f and its surface spline interpolant $T_\Xi f$ as

$$f - T_\Xi f = (f - T_{\Xi''} f) + (T_{\Xi''} f - T_\Xi f). \tag{4.1}$$

The first term on the right side of (4.1) is easily estimated:

Lemma 4.2.

$$\|f - T_{\Xi''} f\|_{L_1(\Omega)} \le const(m, \Omega) h^{m+1} \|f\|_{W^{m+1}}.$$

Proof. Since Ω is bounded, we have $\|f - T_{\Xi''} f\|_{L_1(\Omega)} \le const(\Omega) \|f - T_{\Xi''} f\|_{L_2}$. Appealing to Corollary 2.8, with $\gamma = m+1$, and noting that $h(\Xi'', \mathbb{R}^d) \le (2 + \sqrt{d}/2)h$ completes the proof. $\square$

We desire a similar estimate for the latter term on the right side of (4.1). Since $\Xi \subset \Xi''$ and by Proposition 3.1, this term can be expressed as

$$T_{\Xi''} f - T_\Xi f = T_{\Xi''}(f - T_\Xi f) = \sum_{\xi \in \Xi''} (f - T_\Xi f)(\xi) L_\xi = \sum_{\xi \in \Xi'} (f - T_\Xi f)(\xi) L_\xi, \tag{4.3}$$

where the last equality arises because $f - T_\Xi f = 0$ on Ξ. Our plan for estimating the $L_1(\Omega)$-norm of the latter sum in (4.3) is to partition the sum over Ξ' into countably many sub-sums each involving points of Ξ' which are roughly equidistant to Ω. We then carefully estimate the $L_1(\Omega)$-norm of each sub-sum and finally sum the obtained estimates. Our partitioning of Ξ' employs parallel domains to Ω which we now define.

For $\varepsilon > 0$, let Ω_ε denote the parallel domain

$$\Omega_\varepsilon := \Omega + \varepsilon B,$$

where $B := \{x \in \mathbb{R}^d : |x| < 1\}$. Since Ω has the uniform C^{2m}-regularity property, it follows that Ω has the cone property as well. We show that this property is passed on to Ω_ε.

Proposition 4.4. *Let $\varepsilon > 0$. If Ω has the cone property with parameters ε_Ω and r_Ω, then so does Ω_ε.*

Proof. Let $x \in \Omega_\varepsilon$, say $x = x_0 + x_1$ where $x_0 \in \Omega$ and $|x_1| < \varepsilon$. Since $x_0 \in \Omega$, there exists $y_0 \in \Omega$ satisfying $|x_0 - y_0| = \varepsilon_\Omega$ and

$$(1-t)x_0 + t y_0 + r_\Omega t B \subset \Omega \quad \forall t \in [0,1].$$

Put $y = y_0 + x_1 \in \Omega_\varepsilon$. Then $|x - y| = \varepsilon_\Omega$ and since $|x_1| < \varepsilon$, we have

$$(1-t)x + ty + r_\Omega t B = [(1-t)x_0 + t y_0 + r_\Omega t B] + x_1 \subset \Omega_\varepsilon \quad \forall t \in [0,1].$$

$\square$

We define some parallel bands around Ω.

Definition 4.5. For $k \in \mathbb{N}$, let F_k be defined by

$$F_k := \{x \in \mathbb{R}^d : hk \le \operatorname{dist}(x, \Omega) < h(k+1)\} = \Omega_{h(k+1)} \backslash \Omega_{hk}.$$

Lemma 4.6. *Let c be as in Corollary 2.3. If $\xi \in \Xi' \cap F_k$, then*

$$\|L_\xi\|_{L_1(\Omega)} \le const(d,m) h^d k^{d-1} e^{-ck}.$$

Proof. Let $\xi \in \Xi' \cap F_k$. Since $\Omega - \xi$ is contained in $\mathbb{R}^d \backslash hkB$, it follows from Corollary 2.3 that

$$\begin{aligned}\|L_\xi\|_{L_1(\Omega)} &\le const(d,m) \left\| e^{-c|\cdot|/h} \right\|_{L_1(\mathbb{R}^d \backslash hkB)} = const(d,m) \int_{hk}^\infty t^{d-1} e^{-ct/h}\, dt \\ &= const(d,m) h^d \int_k^\infty t^{d-1} e^{-ct}\, dt \le const(d,m) h^d k^{d-1} e^{-ck},\end{aligned}$$

where we have used integration by parts repeatedly to obtain the last inequality. □

Assumptions 1.3 ensure that Ω is bounded and has the uniform C^{2m}-regularity property. The following lemma, however, requires only that Ω be bounded and have the uniform C^1-regularity property.

Lemma 4.7. *For all $k \in \mathbb{N}$, $\#(\Xi' \cap F_k) \le const(\Omega) h^{1-d} k^d$.*

Proof. First note that for distinct $j, l \in \mathbb{Z}^d$, the balls $j + B/2$ and $l + B/2$ are disjoint, and hence

$$\begin{aligned}(4.8)\quad \#(\Xi' \cap F_k) = \frac{m_d((\Xi' \cap F_k) + hB/2)}{m_d(hB/2)} &= const(d) h^{-d} m_d((\Xi' \cap F_k) + hB/2) \\ &\le const(d) h^{-d} m_d(\partial\Omega + h(k+2)B),\end{aligned}$$

as $(\Xi' \cap F_k) + hB/2 \subset \partial\Omega + h(k+2)B$. Let ρ be the smallest natural number for which $\Omega \subset \rho B$. Let $\varepsilon > 0$ be as in Lemma 2.11 with $A = \Omega$, and assume, without loss of generality, that $\varepsilon \le \rho$. We consider first the case when $h(k+2) \le \varepsilon$. Put $\gamma := r := 2\rho$ and $x := 0$, and note that $(\partial\Omega + h(k+2)B) \cap (x + \gamma B) = \partial\Omega + h(k+2)B$. It thus follows from Lemma 2.11 that $m_d(\partial\Omega + h(k+2)B) \le const(\Omega) h(k+2)$. Hence, by (4.8), $\#(\Xi' \cap F_k) \le const(\Omega) h^{1-d} k$. Turning now to the remaining case, assume that $h(k+2) > \varepsilon$. Then $\partial\Omega + h(k+2)B \subset (\rho + h(k+2))B \subset const(\Omega) hkB$, and therefore $m_d(\partial\Omega + h(k+2)B) \le const(\Omega)(hk)^d \le const(\Omega) hk^d$. Employing (4.8), we see that $\#(\Xi' \cap F_k) \le const(\Omega) h^{1-d} k^d$. □

Proposition 4.9. *If g is a function defined on $\Xi' \cap F_k$, then*

$$\left\| \sum_{\xi \in \Xi' \cap F_k} g(\xi) L_\xi \right\|_{L_1(\Omega)} \le const(m, \Omega) h^{(d+1)/2} k^{3d/2-1} e^{-ck} \|g\|_{\ell_2(\Xi' \cap F_k)}, \quad k \in \mathbb{N}.$$

Proof. It follows from Lemma 4.6 that the left side of our inequality is bounded by $const(d,m)h^d k^{d-1} e^{-ck} \|g\|_{\ell_1(\Xi' \cap F_k)}$. Using the Cauchy-Schwarz inequality, we obtain $\|g\|_{\ell_1(\Xi' \cap F_k)} \leq \sqrt{\#(\Xi' \cap F_k)}\, \|g\|_{\ell_2(\Xi' \cap F_k)}$, and finally complete the estimate by applying Lemma 4.7. □

The proof of the following proposition is rather long and technical; for the sake of continuity we postpone it to section 5.

Proposition 4.10. *There exists $h_0 > 0$ (depending only on m and Ω) such that if $h < h_0$, then*

$$\|f - T_\Xi f\|_{\ell_2(\Xi' \cap F_k)} \leq const(m,\Omega) h^{m+(1-d)/2} k^{m+d/2} \|f\|_{W^{m+1}}, \quad k \in \mathbb{N}.$$

Lemma 4.11. *Let $h_0 > 0$ be as in Proposition 4.10. If $h < h_0$, then*

$$\|T_{\Xi''} f - T_\Xi f\|_{L_1(\Omega)} \leq const(m,\Omega) h^{m+1} \|f\|_{W^{m+1}}.$$

Proof. Assume $h < h_0$. Then

$$\begin{aligned}
\|T_{\Xi''} f - T_\Xi f\|_{L_1(\Omega)} &= \left\| \sum_{\xi \in \Xi'} (f - T_\Xi f)(\xi) L_\xi \right\|_{L_1(\Omega)}, \text{ by (4.3)}, \\
&\leq \sum_{k=1}^{\infty} \left\| \sum_{\xi \in \Xi' \cap F_k} (f - T_\Xi f)(\xi) L_\xi \right\|_{L_1(\Omega)} \\
&\leq const(m,\Omega) h^{(d+1)/2} \sum_{k=1}^{\infty} k^{3d/2-1} e^{-ck} \|f - T_\Xi f\|_{\ell_2(\Xi' \cap F_k)}, \text{ by Proposition 4.9}, \\
&\leq const(m,\Omega) h^{m+1} \sum_{k=1}^{\infty} k^{m+2d-1} e^{-ck} \|f\|_{W^{m+1}}, \text{ by Proposition 4.10}, \\
&= const(m,\Omega) h^{m+1} \|f\|_{W^{m+1}}.
\end{aligned}$$

□

In view of (4.1), Lemma 4.2, and Lemma 4.11, we have proved the following.

Theorem 4.12. *There exists $h_0 > 0$ (depending only on m and Ω) such that if $h < h_0$, then*

$$\|f - T_\Xi f\|_{L_1(\Omega)} \leq const(m,\Omega) h^{m+1} \|f\|_{W^{m+1}}.$$

Let us now revoke Assumptions 1.3. Interpolating (see [4, p. 300–302,311] and [19, p. 44,45]) between Theorem 4.12 and Theorem 2.10 ($p = 2$) yields the following.

Theorem 4.13. *Let $\Omega \subset \mathbb{R}^d$ be open, bounded and have the uniform C^{2m}-regularity property. There exists $h_0 > 0$ such that if Ξ is a finite subset of $\overline{\Omega}$ with $h := h(\Xi,\Omega) < h_0$, then*

$$\|f - T_\Xi f\|_{L_p(\Omega)} \leq const(m,\Omega,p) h^{m+1/p} \|f\|_{B_{2,p}^{m+1/p}}$$

for all $f \in B_{2,p}^{m+1/p}$, $1 < p < 2$.

5. Proof of Proposition 4.10

In this section we do not assume Assumptions 1.3 unless stated otherwise.

Definition. For a measurable function $g : \mathbb{R}^d \to \mathbb{C}$ and $\varepsilon > 0$, we define the maximal function $M_\varepsilon g$ by

$$M_\varepsilon g(x) := \|g\|_{L_\infty(x+\varepsilon B)} .$$

The following is essentially proved in [14, p. 416–418].

Theorem 5.1. *Let m be an integer greater than $d/2$, and let $\kappa > 0$ with the case $d = m = \kappa = 1$ excluded. Let $G \subset \mathbb{R}^d$ have the cone property. There exists $\delta_0 > 0$ (depending only on $d, m, \kappa, \varepsilon_G, r_G$) such that if $\Xi \subset \overline{G}$ satisfies $\delta := h(\Xi, G) \le \delta_0$, then*

$$\begin{aligned} &(i) && \|M_\delta g\|_{L_p(G)} \le const(d, m, r_G, \varepsilon_G)\delta^{m+d/p-d/2} \|g\|_{W^m} \quad and \\ &(ii) && \|M_\delta g\|_{L_p(G)} \le const(d, m, \kappa, r_G, \varepsilon_G)\delta^{m+\kappa+d/p-d/2} \|g\|_{B^{m+\kappa}_{2,\infty}} \end{aligned}$$

for all $g \in W^m$ satisfying $g_{|\Xi} = 0$ and for all $p \in [2, \infty]$.

We mention that (ii) is to be understood in the sense that the right side is only finite when $g \in B^{m+\kappa}_{2,\infty}$.

Proof. We employ 'Case 1' of the proof of [14, Theorem 4.1], with $s = 0$ and $\gamma = 0$, to obtain

$$\|g\|_{L_p(G)} \le \|M_\delta g\|_{L_p(G)} \le const(d, m, r_G, \varepsilon_G)\delta^{m+d/p-d/2} \|g\|_{W^m}$$

which proves (i). Using instead $\gamma = \kappa$, we obtain (ii) in exactly the same way provided $\kappa \in (0, m)$. But the general case $\kappa > 0$ follows from this except in the excluded case $d = m = \kappa = 1$. $\square$

In the following, we use $D^m g$ to denote the vector $(D^\alpha g)_{|\alpha|=m}$, and adopt the notation

$$\|D^m g\|_X := \sum_{|\alpha|=m} \|D^\alpha g\|_X .$$

Corollary 5.2. *Let m be an integer greater than $d/2$, and let $\kappa > 0$ with the case $d = m = \kappa = 1$ excluded. Let $\rho > 0$ and let $G \subset \rho B$ have the cone property. There exists $\delta_1 > 0$ (depending only on $d, m, \kappa, \rho, \varepsilon_G, r_G$) such that if $\mathcal{A} \subset \overline{G}$ satisfies $\delta := h(\mathcal{A}, G) \le \delta_1$, then*

$$\begin{aligned} &(i) && \|M_\delta g\|_{L_p(G)} \le const(d, m, \rho, r_G, \varepsilon_G)\delta^{m+d/p-d/2} \|D^m g\|_{L_2} \quad and \\ &(ii) && \|M_\delta g\|_{L_p(G)} \le const(d, m, \kappa, \rho, r_G, \varepsilon_G)\delta^{m+\kappa+d/p-d/2} \|D^m g\|_{B^{\kappa}_{2,\infty}} \end{aligned}$$

for all $g \in H^m$ satisfying $g_{|\mathcal{A}} = 0$ and for all $p \in [2, \infty]$.

Again, (ii) is to be understood in the sense that the right side is only finite when $D^\alpha g \in B^\kappa_{2,\infty}$ for all $|\alpha| = m$.

Proof. We first prove (ii). For the sake of brevity, let us employ the abbreviation $c = const(d, m, \kappa, \rho, r_G, \varepsilon_G)$. Duchon has shown (see [9, p. 328–330]) that for each integer $n > d/2$, there exists $\widetilde{\delta}_n > 0$ (depending only on d, n, ε_G, r_G) such that $\delta \le \widetilde{\delta}_n$ implies that there exists a subset $\mathcal{N}_n \subset \mathcal{A}$ which is correct for interpolation on Π_{n-1} and for which the interpolation operator $\mathcal{I}_{\mathcal{N}_n} : C(\mathbb{R}^d) \to \Pi_{n-1}$ (defined by $(\mathcal{I}_{\mathcal{N}_n} f)_{|\mathcal{N}_n} = f_{|\mathcal{N}_n}$) satisfies

$$\begin{aligned}
&\|\mathcal{I}_{\mathcal{N}_n} f\|_{L_\infty(\rho B)} \le const(d, n, \rho, \widetilde{\delta}_n)\, \|f\|_{L_\infty(\rho B)} \text{ for all } f \in C(\mathbb{R}^d), \text{ and} \\
&\|f - \mathcal{I}_{\mathcal{N}_n} f\|_{W_2^n(2\rho B)} \le const(d, n, \rho, \widetilde{\delta}_n)\, |f|_{H^n} \text{ for all } f \in H^n.
\end{aligned}$$

Let δ_0 be as in Theorem 5.1, and put $\overline{m} := \lceil m + \kappa + 1 \rceil$ and $\delta_1 := \min\{\delta_0, \widetilde{\delta}_m, \widetilde{\delta}_{\overline{m}}, \rho\}$. Let $\mathcal{A} \subset \overline{G}$ be such that $\delta \le \delta_1$ and let $\mathcal{N}_m, \mathcal{N}_{\overline{m}} \subset \mathcal{A}$ be as described above. Assume $g \in H^m$ satisfies $g_{|\mathcal{A}} = 0$ and $\|D^m g\|_{B^\kappa_{2,\infty}} < \infty$. In view of Theorem 5.1, it suffices to show that there exists $\widetilde{g} \in B^{m+\kappa}_{2,\infty}$ such that

$$\widetilde{g}_{|G_\delta} = g_{|G_\delta} \text{ and } \|\widetilde{g}\|_{B^{m+\kappa}_{2,\infty}} \le c\, \|D^m g\|_{B^\kappa_{2,\infty}}, \tag{5.3}$$

where $G_\delta := G + \delta B$. Let $\eta \in C_c^\infty(2B)$ satisfy $\eta = 1$ on B, and define $g_0 \in H^{\overline{m}}$ by $\widehat{g_0} := \eta \widehat{g}$. Put $g_1 := g - g_0 \in B^{m+\kappa}_{2,\infty}$, and note that $\|g_1\|_{B^{m+\kappa}_{2,\infty}} \le const(d, m)\, \|D^m g\|_{B^\kappa_{2,\infty}}$. Put $q_0 := \mathcal{I}_{\mathcal{N}_{\overline{m}}} g_0 \in \Pi_{\overline{m}-1}$. Then we have the estimates

$$\begin{aligned}
\|q_0\|_{L_\infty(\rho B)} &\le c\, \|g_0\|_{L_\infty(\rho B)} \le c\, \|g_1\|_{L_\infty(\rho B)} + c\, \|g\|_{L_\infty(\rho B)} \quad (\text{note } \mathcal{I}_{\mathcal{N}_m} g = 0) \\
&\le c\, \|g_1\|_{B^{m+\kappa}_{2,\infty}} + c\, \|g - \mathcal{I}_{\mathcal{N}_m} g\|_{W_2^m(2\rho B)} \le c\, \|D^m g\|_{B^\kappa_{2,\infty}} + c\, |g|_{H^m} \le c\, \|D^m g\|_{B^\kappa_{2,\infty}},
\end{aligned}$$

and $\|g_0 - q_0\|_{W_2^{\overline{m}}(2\rho B)} \le c\, |g_0|_{H^{\overline{m}}} \le c\, \|D^m g\|_{B^\kappa_{2,\infty}}$.
Using a strong $\overline{m}$-extension operator E for $2\rho B$ (see [1, p.83–86]), we obtain $E(g_0 - q_0) \in W^{\overline{m}}$ satisfying $E(g_0 - q_0)_{|2\rho B} = (g_0 - q_0)_{|2\rho B}$ and

$$\|E(g_0 - q_0)\|_{W^{\overline{m}}} \le const(d, \overline{m}, \rho)\, \|g_0 - q_0\|_{W_2^{\overline{m}}(2\rho B)}.$$

Let $\sigma \in C_c^\infty(3\rho B)$ satisfy $\sigma = 1$ on $2\rho B$, and define $\widetilde{g} := E(g_0 - q_0) + g_1 + \sigma q_0$. Then $\widetilde{g} = g$ on $2\rho B$ (which contains G_δ) and

$$\|\widetilde{g}\|_{B^{m+\kappa}_{2,\infty}} \le c\, \|E(g_0 - q_0)\|_{W^{\overline{m}}} + \|g_1\|_{B^{m+\kappa}_{2,\infty}} + c\, \|q_0\|_{L_\infty(\rho B)} \le c\, \|D^m g\|_{B^\kappa_{2,\infty}}$$

which, in view of (5.3), proves (ii). By replacing $\overline{m}$ with m, $B^{m+\kappa}_{2,\infty}$ with W^m, and $B^\kappa_{2,\infty}$ with L_2, the above proof can be easily adapted to prove (i). $\square$

Lemma 5.4. *For every $f \in H^m$ there exists $q \in \Pi_{m-1}$ such that*

$$|(f - q)(x)| \le const(d, m)(1 + |x|)^{m - d/(2m)}\, |f|_{H^m} \text{ for all } x \in \mathbb{R}^d.$$

Proof. Let $f \in H^m$ and let ν be as defined in the proof of Lemma 3.5. Recall that $f = \nu + q$, where $q \in \Pi_{m-1}$. We proceed now to estimate $|(f-q)(x)| = |\nu(x)|$ for $x \in \mathbb{R}^d$. Since $\widehat{\nu}$ is the sum of a compactly supported distribution and an integrable function, it follows that

$$(2\pi)^d |\nu(x)| = |\langle e_x, \widehat{\nu}\rangle| = \left| \int_{\mathbb{R}^d \setminus 0} k_x(w) \widehat{f}(w)\, dw \right| \le \left\| |\cdot|^{-m} k_x \right\|_{L_2} |f|_{H^m},$$

where $k_x(w) := (e_x - \sigma P_{m-1} e_x)(w) = e_x(w) - \sum_{|\alpha|<m} \frac{D^\alpha e_x(0)}{\alpha!} \sigma(w) w^\alpha$. Since $|e_x| = 1$ and $\max_{|\alpha|<m} |D^\alpha e_x(0)| \le const(d,m)(1+|x|)^{m-1}$, we have the crude estimate $|k_x(w)| \le const(d,m)(1+|x|)^{m-1}$ for all $w \in \mathbb{R}^d$. Noting that $D^\alpha k_x(0) = 0$ for all $|\alpha| < m$, it follows from Taylor's theorem that for $w \in B$,

$$|k_x(w)| \le const(d,m)\, |w|^m \max_{|\alpha|=m} \|D^\alpha k_x\|_{L_\infty(B)} \le const(d,m)\, |w|^m (1+|x|)^m.$$

Put $\rho_x := (1+|x|)^{-1/m}$. Employing these two estimates on $\mathbb{R}^d \setminus \rho_x B$ and $\rho_x B$, respectively, yields

$$\begin{aligned}
\left\| |\cdot|^{-m} k_x \right\|_{L_2}^2 &= \left\| |\cdot|^{-m} k_x \right\|_{L_2(\mathbb{R}^d \setminus \rho_x B)}^2 + \left\| |\cdot|^{-m} k_x \right\|_{L_2(\rho_x B)}^2 \\
&\le const(d,m)(1+|x|)^{2m-2} \int_{\mathbb{R}^d \setminus \rho_x B} |w|^{-2m}\, dw + const(d,m)(1+|x|)^{2m} \int_{\rho_x B} 1\, dw \\
&= const(d,m)(1+|x|)^{2m-2} \rho_x^{d-2m} + const(d,m)(1+|x|)^{2m} \rho_x^d \\
&= const(d,m)(1+|x|)^{2m-d/m}
\end{aligned}$$

which completes the proof. $\square$

In the following result we again employ the notation $A_r := A + rB$ for $r \ge 0$.

Proposition 5.5. *Let A be an open (possibly unbounded) subset of $\mathbb{R}^d$ having the cone property with parameters $r_A, \varepsilon_A \in (0,1]$. Then*

$$\|f\|_{L_\infty(A_r)} \le const(d,m) \left(\frac{1+r}{r_A} \right)^{m-d/(2m)} \left(\|f\|_{L_\infty(A)} + |f|_{H^m} \right)$$

for all $f \in H^m$ and $r \ge 0$.

Proof. Let $x \in A_r$ with $r \ge 0$. Then there exists $y \in A$ such that $|x-y| \le 1+r$ and $y + r_A B \subset A$. We may assume, without loss of generality, that $y = 0$ since otherwise we can replace A with $A - y$ and f with $f(\cdot + y)$. Let $q \in \Pi_{m-1}$ as described in Lemma 5.4. Then

$$|f(x)| \le |(f-q)(x)| + |q(x)| \le const(d,m)(1+r)^{m-d/(2m)} |f|_{H^m} + \|q\|_{L_\infty((1+r)B)}.$$

It is easy to see (using the fact that all norms on a finite dimensional space are equivalent) that $\|p\|_{L_\infty(tB)} \le const(d,m)t^{m-1}\|p\|_{L_\infty(B)}$ for all $p \in \Pi_{m-1}$ and $t \ge 1$. Replacing p with $p(\cdot/t)$ yields $\|p\|_{L_\infty(B)} \le const(d,m)t^{m-1}\|p\|_{L_\infty(B/t)}$ for all $p \in \Pi_{m-1}$ and $t \ge 1$. Hence,

$$\begin{aligned}\|q\|_{L_\infty((1+r)B)} &\le const(d,m)\left(\frac{1+r}{r_A}\right)^{m-1}\|q\|_{L_\infty(r_AB)}\\ &\le const(d,m)\left(\frac{1+r}{r_A}\right)^{m-1}\left(\|f-q\|_{L_\infty(r_AB)}+\|f\|_{L_\infty(r_AB)}\right)\\ &\le const(d,m)\left(\frac{1+r}{r_A}\right)^{m-1}\left(|f|_{H^m}+\|f\|_{L_\infty(A)}\right).\end{aligned}$$

Thus $|f(x)| \le const(d,m)\left(\frac{1+r}{r_A}\right)^{m-d/(2m)}\left(\|f\|_{L_\infty(A)}+|f|_{H^m}\right)$, since $m-1 < m-d/(2m)$, which completes the proof. $\square$

Lemma 5.6. *Let $G \subset \mathbb{R}^d$ have the cone property. If $g \in C(\mathbb{R}^d)$ and $1 \le p \le \infty$, then*

$$\|g\|_{\ell_p(h\mathbb{Z}^d\cap G)} \le const(d,r_G,\varepsilon_G)(1+h^{-d/p})\|M_hg\|_{L_p(G)} \quad \textit{for all } h>0.$$

Proof. There exists $\nu \in (0,1/2]$ (depending only on r_G and ε_G) such that for all $x \in G$ and $t \in (0,r_G]$, there exists $y \in G$ such that

$$y+\nu tB \subset G\cap(x+tB/2).$$

Let $h>0$ and put $\mathcal{A}_h := \mathbb{Z}^d \cap h^{-1}G$ and $\underline{h} := \min\{h,r_G\}$. It follows from the above observation that for each $j \in \mathcal{A}_h$, there exists $y_j \in G$ such that $y_j+\nu\underline{h}B \subset G\cap(hj+\underline{h}B/2)$. Noting that $M_hg(x) \ge |g(hj)|$ whenever $j \in \mathcal{A}_h$ and $x \in y_j+\nu\underline{h}B$, we see that

$$\|M_hg\|_{L_p(y_j+\nu\underline{h}B)} \ge |g(hj)|\operatorname{vol}(\nu\underline{h}B)^{1/p} \ge const(d,r_G,\varepsilon_G)\,|g(hj)|\,\underline{h}^{d/p}.$$

Hence,

$$\begin{aligned}\|g\|_{\ell_p(h\mathbb{Z}^d\cap G)} &\le const(d,r_G,\varepsilon_G)\underline{h}^{-d/p}\left\|j\mapsto\|M_hg\|_{L_p(y_j+\nu\underline{h}B)}\right\|_{\ell_p(\mathcal{A}_h)}\\ &\le const(d,r_G,\varepsilon_G)(1+h^{-d/p})\|M_hg\|_{L_p(G)},\end{aligned}$$

since the balls $\{y_j+\nu hB : j \in \mathcal{A}_h\}$ are pairwise disjoint subsets of G. $\square$

Proof of Proposition 4.10. Let Assumptions 1.3 be in force and assume without loss of generality that $\varepsilon_\Omega, r_\Omega \in (0,1]$. For the sake of brevity we employ the abbreviation $c = const(m,\Omega)$. Let h_0 be the smaller of h_0 from Theorem 2.6 and h_1 from Theorem 2.10. Let δ_1 be as in Corollary 5.2 with $\kappa := 1/2$. We consider first the case $h(k+1) \ge \delta_1$. Note that

$$\begin{aligned}\|f-T_\Xi f\|_{\ell_2(\Xi'\cap F_k)} &\le \sqrt{\#(\Xi'\cap F_k)}\,\|f-T_\Xi f\|_{L_\infty(\Omega_{h(k+1)})}\\ &\le ch^{(1-d)/2}k^{d/2}\|f-T_\Xi f\|_{L_\infty(\Omega_{h(k+1)})}\end{aligned}$$

by Lemma 4.7. It follows from Proposition 5.5 and Theorem 2.6 that

$$\begin{aligned}\|f - T_\Xi f\|_{L_\infty(\Omega_{h(k+1)})} &\le c(1+h(k+1))^m(\|f - T_\Xi f\|_{L_\infty(\Omega)} + |f - T_\Xi f|_{H^m}) \\ &\le c(1+h(k+1))^m\, |f|_{H^m} \le c(hk)^m \|f\|_{W^{m+1}}.\end{aligned}$$

Combining these two estimates yields $\|f - T_\Xi f\|_{\ell_2(\Xi' \cap F_k)} \le c h^{m+(1-d)/2} k^{m+d/2} \|f\|_{W^{m+1}}$. We turn now to the case $h(k+1) < \delta_1$. Put $G := \Omega_{h(k+1)}$. Then by Proposition 4.4, G has the cone property with $\varepsilon_G = \varepsilon_\Omega$ and $r_G = r_\Omega$. Note that by Lemma 5.6

$$\|f - T_\Xi f\|_{\ell_2(\Xi' \cap F_k)} \le \|f - T_\Xi f\|_{\ell_2(h\mathbb{Z}^d \cap G)} \le c h^{-d/2} \|M_h(f - T_\Xi f)\|_{L_2(G)}.$$

Writing $M_h(f - T_\Xi f) \le M_h(f - T_\Omega f) + M_h(T_\Omega f - T_\Xi f)$, we analyze the two terms separately. Since W^{m+1} is continuously embedded in $B^{m+1/2}_{2,1}$, we have $\|f\|^{m+1/2}_{B_{2,1}} \le const(d,m)\|f\|_{W^{m+1}}$. By Theorem 2.9, we have $\|D^m T_\Omega f\|_{B^{1/2}_{2,\infty}} \le c\|f\|_{B^{m+1/2}_{2,1}}$. Since $f - T_\Omega f = 0$ on Ω and $\delta(\Omega, G) \le h(k+1) < \delta_1$, we have by Corollary 5.2 that

$$\|M_h(f - T_\Omega f)\|_{L_2(G)} \le c(h(k+1))^{m+1/2} \|D^m(f - T_\Omega f)\|_{B^{1/2}_{2,\infty}} \le c(hk)^{m+1/2}\|f\|_{W^{m+1}}.$$

By Theorem 2.10, $|T_\Omega f - T_\Xi f|_{H^m} \le c h^{1/2} \|f\|_{B^{m+1/2}_{2,1}}$. Hence, by Corollary 5.2 we have

$$\begin{aligned}\|M_h(T_\Omega f - T_\Xi f)\|_{L_2(G)} &\le c(h(k+1))^m \|D^m(T_\Omega f - T_\Xi f)\|_{L_2} \\ &\le c(hk)^m\, |T_\Omega f - T_\Xi f|_{H^m} \le c(hk)^{m+1/2}\|f\|_{W^{m+1}}.\end{aligned}$$

Combining the above estimates completes the proof. □

6. The width of the affected boundary layer

In this section we assume Assumptions 1.3 with the following modifications: Regarding f, we assume that $f \in W^{2m}_\infty(\mathbb{R}^d) \cap H^m$, and regarding Ω, we assume only that Ω is open, bounded and has the uniform C^1-regularity property.

Definition. For $r > 0$, let Ω_{-r} denote the sub-domain

$$\Omega_{-r} := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > r\}.$$

Matveev [17] has shown that for a fixed $r > 0$

$$\|f - T_\Xi f\|_{L_\infty(\Omega_{-r})} = O(h^{2m}) \text{ as } h \to 0 \tag{6.1}$$

(see also [3]). Our purpose here is to show that (6.1) is still valid when r equals a sufficiently large constant multiple of $h\,|\log h|$. We will thus show that as far as the order of convergence is concerned, the boundary effects (which degrade the rate of convergence) are confined to a boundary layer no wider than a constant multiple of $h\,|\log h|$. Before coming to this result, we prove the following lemma which employs the sets $F_k := \Omega_{h(k+1)} \backslash \Omega_{hk}$.

Lemma 6.2. *If g is a function defined at least on Ξ' and $r > 0$, then*

$$\left\| \sum_{\xi \in \Xi' \cap F_k} g(\xi) L_\xi \right\|_{L_\infty(\Omega_{-r})} \le const(d,m)(k + r/h)^{d-1} e^{-c(k+r/h)} \|g\|_{\ell_\infty(\Xi' \cap F_k)},$$

where c is as in Corollary 2.3.

Proof. Let $r > 0$ and let $x \in \Omega_{-r}$. For $\ell \in \{k, k+1, \dots\}$, let G_ℓ be the annulus defined by

$$G_\ell := \{y \in \mathbb{R}^d : (h\ell + r) \le |y - x| \le (h(\ell+1) + r)\},$$

and note that $F_k \subset \cup_{\ell=k}^\infty G_\ell$. Using the same argument as was used to prove (4.8), we can show that

$$\begin{aligned} &\#((h\mathbb{Z}^d) \cap G_\ell) \le const(d) h^{-d} m_d(G_\ell + hB/2) \\ &= const(d) h^{-d} \left((h(\ell + 3/2) + r)^d - (h(\ell - 1/2) + r)^d \right) \\ &\le const(d) h^{-d} (2h)(h(\ell+3/2) + r)^{d-1} \le const(d)(\ell + r/h)^{d-1}. \end{aligned}$$

Therefore

$$\begin{aligned} &\left| \sum_{\xi \in \Xi' \cap F_k} g(\xi) L_\xi(x) \right| \le \sum_{\ell=k}^\infty \sum_{\xi \in \Xi' \cap F_k \cap G_\ell} |g(\xi) L_\xi(x)| \\ &\le const(d,m) \sum_{\ell=k}^\infty (\ell + r/h)^{d-1} \|g\|_{\ell_\infty(\Xi' \cap F_k)} \exp(-c(\ell + r/h)), \qquad \text{by Cor. 2.3,} \\ &\le const(d,m)(k + r/h)^{d-1} \|g\|_{\ell_\infty(\Xi' \cap F_k)} \exp(-c(k + r/h)). \end{aligned}$$

□

Theorem 6.3. *Let c be as in Lemma 2.3 and define $\kappa := (2m - 1 + d)/c$. There exists $h_0 > 0$ (depending only on m, Ω) such that if $h := h(\Xi, \Omega) \le h_0$ and $r := \kappa h |\log h|$, then*

$$\|f - T_\Xi f\|_{L_\infty(\Omega_{-r})} \le const(m, \Omega) h^{2m} \|f\|_{W_\infty^{2m}} \quad \forall f \in W_\infty^{2m}(\mathbb{R}^d) \cap H^m.$$

Proof. Let h_0 be as in Theorem 2.6 and assume without loss of generality that $h_0 < 1$. Assume $h \le h_0$. Let $f \in W_\infty^{2m}(\mathbb{R}^d) \cap H^m$ and note, as before, that

$$f - T_\Xi f = f - T_{\Xi''} f + T_{\Xi''}(f - T_\Xi f).$$

It follows from Theorem 2.4 that

$$\|f - T_{\Xi''} f\|_{L_\infty(\mathbb{R}^d)} \le const(d,m) h^{2m} \|f\|_{W_\infty^{2m}(\mathbb{R}^d)}.$$

By Proposition 3.1, we can write the latter term as $T_{\Xi''}(f - T_{\Xi}f) = \sum_{\xi \in \Xi'} g(\xi) L_\xi$, where $g := f - T_{\Xi} f$. We then employ Corollary 2.3 to obtain

$$\begin{aligned} \|T_{\Xi''}(f - T_{\Xi'} f)\|_{L_\infty(\Omega_{-r})} &\le \sum_{\xi \in \Xi'} |g(\xi)| \, \|L_\xi\|_{L_\infty(\Omega_{-r})} \\ &\le const(d, m) \sum_{k=1}^{\infty} \#(\Xi' \cap F_k) \, \|g\|_{L_\infty(\Omega_{h(k+1)})} \, e^{-c(k+r/h)}, \end{aligned}$$

where F_k is as defined in section 4. By Theorem 2.6 and Proposition 5.5, we have

$$\|g\|_{L_\infty(\Omega_{h(k+1)})} \le const(m, \Omega)(1 + hk)^m \, |f|_{H^m} \, .$$

Therefore, by the above and Lemma 4.7,

$$\begin{aligned} \|T_{\Xi''}(f - T_{\Xi'} f)\|_{L_\infty(\Omega_{-r})} &\le const(m, \Omega) \sum_{k=1}^{\infty} h^{1-d} k^d (1 + hk)^m e^{-c(k+r/h)} \, |f|_{H^m} \\ &\le const(m, \Omega) h^{1-d} \sum_{k=1}^{\infty} k^{m+d} e^{-c(k+\kappa|\log h|)} \, \|f\|_{W^{2m}} = const(m, \Omega) h^{2m} \, \|f\|_{W^{2m}} \, . \end{aligned}$$

which completes the proof. $\square$

7. A numerical experiment

For $2 < p \le \infty$ it is known that the L_p-approximation order of surface spline interpolation is bounded below by $m + 1/2 + d/p - d/2$ and bounded above by $m + 1/p$. For the purpose of directing future research, it is helpful to have an opinion, based on experimental evidence, as to which bound is closer to the true value. The simplest case for experimentation is $m = d = 2$ which is known as *thin-plate spline interpolation*. Note that the above-mentioned bounds on the L_∞-approximation order of thin-plate spline interpolation are $3/2$ and 2, respectively. The basic idea in a numerical experiment is to choose a domain $\Omega \subset \mathbb{R}^d$ having a very smooth boundary and choose a very smooth data function f, and then compute the $L_\infty(\Omega)$-norm of the error $f - T_\Xi f$ for several choices of the interpolation points Ξ. One then tries to identify a relation of the form

$$\|f - T_\Xi f\|_{L_\infty(\Omega)} \approx a h^b,$$

where $h := h(\Xi, \Omega)$. If the experimental results fit the above relation well, for a particular choice of the constants a and b, then the experiment leads one to expect that the L_∞-approximation order of thin-plate spline interpolation is close to b.

Let us choose $\Omega := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ to be the open unit ball in $\mathbb{R}^2$ and we choose f to be the homogeneous polynomial

$$f(x, y) := x^4 - x^2 y^2, \quad (x, y) \in \mathbb{R}^2.$$

For simplicity, we assume that the interpolation points Ξ satisfy the condition

$$h \leq const \min_{\xi \neq \xi' \in \Xi} |\xi - \xi'| \,. \tag{7.1}$$

It follows from Theorem 6.3 that if $\kappa_1 > 0$ is sufficiently large and $\kappa_2 > 0$, then

$$\|f - T_\Xi f\|_{L_\infty(\Omega \setminus A_t)} = O(h^4) \text{ as } h \to 0,$$

where $t := \kappa h(\kappa_2 + |\log h|)$ and $A_t := \{(x,y) \in \mathbb{R}^2 : 1 - t < |(x,y)| \leq 1\}$. Thus, in our experiment, it suffices to observe the $L_\infty(A_t)$-norm of $f - T_\Xi f$. Moreover, it can be shown, using the techniques of section 6, that if Ξ_{2t} denotes $\Xi \cap A_{2t}$, then

$$\|T_\Xi f - T_{\Xi_{2t}} f\|_{L_\infty(A_t)} = O(h^4) \text{ as } h \to 0.$$

Thus, in our experiment, it suffices to observe the $L_\infty(A_t)$-norm of $f - T_{\Xi_{2t}} f$. This reduction is significant because the cardinality of Ξ_{2t} is likely to be much less than the cardinality of Ξ which means that computing $T_{\Xi_{2t}} f$ is far easier than computing $T_\Xi f$. Given a large integer N, we define Ξ as follows:
For each $r \in (0,1]$, let K_r denote the

$$\#K_r := \begin{cases} N & \text{if } 2^{-2} \leq r \leq 1 \\ \lceil 2^{-j} N \rceil & \text{if } 2^{-j-2} \leq r < 2^{-j-1}, \ j \in \mathbb{N} \end{cases}$$

equispaced points on the circle $x^2 + y^2 = r^2$, one being $(r, 0)$. We then define

$$\Xi := \cup \{K_r : r \in \{1, 1 - 2\pi/N, 1 - 4\pi/N, \ldots\} \cap (0,1]\}.$$

With $\kappa_1 := 3$, $\kappa_2 := 0.347$ and for the values $N \in \{125, 251, 502, 1005, 2010\}$, we have numerically computed $T_{\Xi_{2t}} f$ and measured

$$E_0 = \|f - T_{\Xi_{2t}} f\|_{L_\infty(A_t)} \text{ and } E_1 \approx \|f - T_{\Xi_{2t}} f\|_{L_\infty(A_t \setminus A_{t/2})},$$

where the interpolation equations were solved using a domain decomposition technique proposed by Beatson, Light and Billings [2]. Here are the results:

N	h	$\#\Xi$	$\#\Xi_{2t}$	E_0	E_1
125	.03510	2112	1500	3.325×10^{-3}	1.8×10^{-4}
251	.01759	8397	3765	8.465×10^{-4}	1.0×10^{-5}
502	.008828	33497	9036	2.143×10^{-4}	1.4×10^{-7}
1005	.004414	134047	21105	5.383×10^{-5}	7.4×10^{-9}
2010	.002209	536047	48240	1.350×10^{-5}	2.4×10^{-10}

The fact that E_1 decays very fast with h indicates that our choice of κ_1 and κ_2 are appropriate. The obtained values of E_0 fit the formula $E_0 \approx 2.722\, h^2$ quite well. Precisely, we can write

$$E_0 = 2.722\, h^{b_N}, \text{ with } b_N \in [1.997, 2.003].$$

On the basis of this experiment, I conjecture that the L_p-approximation order of surface spline interpolation is $m + 1/p$ for $2 < p \leq \infty$.

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