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# Approximation orders of FSI spaces in $L_2(\mathbb{R}^d)$

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**Abstract:** A second look at the authors' ([BDR1], [BDR2]) characterization of the approximation order of a Finitely generated Shift-Invariant subspace $S(\Phi)$ of $L_2(\mathbb{R}^d)$ results in a more explicit formulation entirely in terms of the (Fourier transform of the) generators $\varphi \in \Phi$ of the subspace. Further, when the generators satisfy a certain technical condition, then, under the mild assumption that the set of 1-periodizations of the generators is linearly independent, such a space is shown to provide approximation order $k$ if and only if $\operatorname{span}\{\varphi(\cdot - j) : |j| < k, \varphi \in \Phi\}$ contains a $\psi$ (necessarily unique) satisfying $D^j\widehat{\psi}(\alpha) = \delta_j \delta_\alpha$ for $|j| < k$, $\alpha \in 2\pi\mathbb{Z}^d$. The technical condition is satisfied, e.g., when the generators are $O(|\cdot|^{-\rho})$ at infinity for some $\rho > k + d$. In the case of compactly supported generators, this recovers an earlier result of Jia ([J1], [J2]).

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### 1. Introduction

Given a subset $\Phi \subset L_2(\mathbb{R}^d)$, the **shift-invariant**, or **SI**, **space** $S(\Phi)$ **generated by** $\Phi$ is the smallest closed subspace of $L_2(\mathbb{R}^d)$ that contains the set

$$E(\Phi) := \{\phi(\cdot - \alpha) : \ \phi \in \Phi, \ \alpha \in \mathbb{Z}^d\}$$

of all **shifts** of $\Phi$; i.e., $S(\Phi)$ is the $L_2$-closure of the finite span of $E(\Phi)$. We use the abbreviations **FSI** (for 'finitely generated') and **PSI** (for 'principal') in case $\Phi$ is a finite set, respectively a singleton, $\{\phi\}$. In the latter case, we write $S(\phi)$ rather than $S(\{\phi\})$.

FSI spaces play a role in several areas of Analysis. The most relevant to the present article are *Multivariate Approximation Theory* (in particular, Box Splines), and *Wavelets* (in one or more dimensions), particularly, multiwavelets.

In the above-mentioned and other areas, the FSI space $S(\Phi)$ serves as a possible source of approximants for certain subspaces of $L_2(\mathbb{R}^d)$ (e.g., Sobolev spaces). The basic criterion then for assessing the approximation properties of $S(\Phi)$ is the asymptotic decay of the error when approximating from dilates of this space. Precisely, let

$$S_h := \{f(\cdot/h) : \ f \in S(\Phi)\}$$

be the $h$-dilate of $S(\Phi)$. Given $k > 0$, we say that $\Phi$ (or, more correctly, $S(\Phi)$) **provides approximation order** $k$ if

$$\text{dist}(f, S_h) = O(h^k), \quad \text{all } f \in W_2^k. \tag{1.1}$$

Here, dist is the $L_2$-distance between a function and a subset, and $W_2^k := W_2^k(\mathbb{R}^d)$ is the usual potential space. The problem of determining the highest possible approximation order provided by $S(\Phi)$ was first suggested by Strang and Fix in their seminal paper [SF]. We forgo reviewing here to any extent the rich literature concerning the approximation orders of shift-invariant spaces, and refer instead the interested reader to the introduction and bibliography of [BR2] and [BDR1]. Specific discussions of the literature that are pertinent to the present paper can be found in the sequel.

Following a suggestion of Babuška, [SF] studies the possibility that the approximation orders of the FSI space $S$ are already realized by some PSI subspace $S(\psi)$ of it. Such a function $\psi$ (which need not be unique) is sometimes referred to as a 'superfunction' for $S$, and we thus refer to this direction of study as 'superfunction theory of FSI spaces'. The approach is particularly successful if the superfunction is computable and has favorable properties similar to the generating set $\Phi$ (e.g., is compactly supported if the elements of $\Phi$ are). The superfunction approach exploits the fact that the study of approximation orders of PSI spaces, as well as the construction of useful approximation schemes from such spaces, is simpler than for their FSI counterparts. Specifically, the problem of characterizing the $L_2$-approximation order of PSI spaces was completely solved in [BDR1: Theorem 1.6]; in other norms, a complete characterization of the approximation orders of PSI spaces is yet to be found; however, the recent results of Johnson [Jo1], [Jo2] come very close to that target.

The present paper is exclusively devoted to the study of *approximation orders of FSI spaces*. Two of our previous papers, [BDR1] and [BDR2], treat this problem as well (though not exclusively). In what follows, we first briefly discuss the results obtained in these other articles, and then describe the contribution of the present paper to the topic.

Our studies in [BDR1] and [BDR2] of the approximation orders of FSI spaces were focused on the superfunction approach. The basic result on the matter is Theorem 1.9 of [BDR1], a special case of which, of much use in the present paper, is as follows.

**Result 1.2.** *Let $S$ be a closed shift-invariant subspace of $L_2(\mathbb{R}^d)$. Let $\psi$ be the orthogonal projection onto $S$ of the* sinc*-function*

$$g : \omega \mapsto (2\pi)^{d/2} \prod_{j=1}^{d} \frac{\sin(\pi\omega_j)}{\pi\omega_j}.$$

*Then, the approximation order provided by $S$ is the same as the approximation order provided by its PSI subspace $S(\psi)$.*

Improvements of Result 1.2 may be sought for two different reasons. First, even in case $S$ is generated by a 'nice' set $\Phi$, it is not true that the superfunction $\psi$ in the result must inherit any of these favorable properties. Second, the result does not provide any recipe for the construction of $\psi$. Section 4 of [BDR2] deals with these two problems. We state here only its result for the case when $S$ is a **local** space, i.e., $S$ is generated by *finitely many compactly supported* functions. Here and below, reference to $S(\Phi)$ being local means that the $\Phi$ mentioned is a finite set of compactly supported functions.

**Result 1.3.** *Let $S(\Phi)$ be a local space. Then there exists a finite linear combination $\psi$ of the elements of $\Phi$ such that the PSI space $S(\psi)$ provides the same approximation order as that provided by $S$.*

Note that the superfunction of this latter result is certainly compactly supported. Moreover, an explicit construction of (one of the many) possible $\psi$ is given in [BDR2]. Still, one should keep in mind the following: while $\psi$ in Result 1.3 is compactly supported, its mean value may be 0. From an Approximation Theory point-of-view, this is a major drawback. For, it forces any useful approximation scheme that uses the shifts of $\psi$ to be unstable: the coefficients used to approximate, say, a bounded function, can grow at $\infty$.

When writing [BDR2], our primary example of a FSI space was one generated by several box splines. In this case, the fact that the compactly supported superfunction has zero mean value seems to be unavoidable. However, recent examples of FSI spaces (such as the ones considered in [HSS] and [CDP]) are of a different character. In fact, these articles treat a generating set $\Phi$ whose shifts are *linearly independent*. Under a linear independence assumption, and in fact under a much weaker assumption, the superfunction results can be greatly improved. In particular, we will prove (in Section 4; see Theorem 4.2) the following theorem which says that, under conditions that we presently consider to be 'mild' (though we were unwilling to think of them so in the past), spaces whose generators decay suitably at infinity contain a superfunction with non-zero mean-value which is a finite linear combination of shifts of those generators. In the statement of the theorem and later, we make use of the abbreviation

$$\mathcal{Z}_k := \{j \in \mathbb{Z}_+^d : |j| := \sum_{i=1}^{d} j_i < k\}. \tag{1.4}$$

**Theorem 1.5.** *Let $\Phi$ be a finite subset of $L_2(\mathbb{R}^d)$ whose elements are $O(|\cdot|^{-\rho})$ at infinity for some $\rho > k+d$ and assume that $S(\Phi)$ provides approximation order $k$. For each $\phi \in \Phi$, let*

$$\phi^\circ := \sum_{j \in \mathbb{Z}^d} \phi(\cdot - j)$$

*be the periodization of $\phi$. If $\Phi^\circ := \{\phi^\circ : \phi \in \Phi\}$ is linearly independent, then there exists a unique function $\psi$ that has all the following properties:*

*(a) $\psi$ is spanned by the $\mathcal{Z}_k$-shifts of $\Phi$;*

*(b) The zero-moment of $\psi$ equals 1; its $j$-moments, $j \in \mathcal{Z}_k \backslash \{0\}$, all are zero;*

*(c) $S(\psi)$ provides approximation order $k$.*

Under the stronger assumption that the generators are compactly supported, this theorem is essentially proved, by rather different means, in [J2] (and announced in [J1]). Furthermore, under the assumption that the shifts of $\Phi$ are stable, the above result can be found in [LJC: Theorem 5.3] (and is proved there for any $p \in [1\,..\,\infty]$).

**Remark.** We recall from [BDR1: Theorem 1.14] the following. If $\widehat{\psi}$ is bounded on some neighborhood of the origin (a condition obviously satisfied by the $\psi$ in this theorem), then (c) of Theorem 1.5 implies that $\psi$ satisfies the **Strang-Fix conditions of order** $k$, i.e., $\widehat{\psi}$ has a zero of order $k$ at each $\alpha \in 2\pi\mathbb{Z}^d \backslash \{0\}$ (in the sense that $\widehat{\psi}/|\cdot - \alpha|^k$ is bounded in some neighborhood of $\alpha$).

**Remark.** More general results than Theorem 1.5 (see Theorem 4.2) are stated and proved in the present paper, though they require a modification of the assumption that $\Phi^\circ$ be linearly independent. Such possible modification is the content of the next remark. $\square$

**Remark.** The linear independence of $\Phi^\circ$ is equivalent to the linear independence of the set

$$\{\widehat{\phi}_{|_{2\pi\mathbb{Z}^d}} : \phi \in \Phi\}$$

of restrictions to $2\pi\mathbb{Z}^d$ of the $\widehat{\phi}$'s. Indeed, for $\phi \in \Phi$, the discrete Fourier transform $\widehat{\phi^\circ}$ of $\phi^\circ$ satisfies $\widehat{\phi^\circ}(j) = \widehat{\phi}(2\pi j)$ by the Poisson summation formula, and the linear independence of $\Phi^\circ$ is equivalent to the linear independence of $\widehat{\Phi^\circ}$. This linear independence requirement is *significantly weaker* than $L_2$-stability (known also as the Riesz basis property) of shifts of $\Phi$ (cf. [BDR2]): the latter property is characterized by the linear independence of

$$\{\widehat{\phi}_{|_{\theta+2\pi\mathbb{Z}^d}} : \phi \in \Phi\},$$

for *every real* $\theta$. The linear independence assumption on the shifts of $\Phi$, used in [HSS] and [CDP], is even stronger than the Riesz basis property. Firstly, it assumes $\Phi$ to be compactly supported. Secondly, it is characterized ([JM]) by the linear independence of the sequences

$$\{\widehat{\phi}_{|_{\theta+2\pi\mathbb{Z}^d}} : \phi \in \Phi\},$$

for every *complex* $\theta$ (note: the Fourier transform of a compactly supported $\phi$ is entire). $\square$

As an application of our "superfunction" results of Section 4, we provide (in Section 5) a characterization of the approximation order of multivariate refinable FSI spaces in terms of their refinement mask. Recall that $\Phi \subset L_2$ is **dyadically refinable** if there exists a square matrix $M$, indexed by $\Phi$ and with $2\pi$-periodic entries, such that

$$\widehat{\Phi} = M(\cdot/2)\widehat{\Phi}(\cdot/2).$$

One result in Section 5 states that (under conditions that are somewhat stronger than those of Theorem 1.5, but still weaker than the Riesz basis assumption) the refinable $\Phi$ provides approximation order $k$ iff there exist trigonometric polynomials $\tau = (\tau_\phi : \phi \in \Phi)$ such that (a): $\tau(2\cdot)M$ has a zero of order $k$ at each $\xi \in \{0,\pi\}^d\backslash 0$; (b): $\tau - \tau(2\cdot)M$ has a zero of order $k$ at 0, and (c) $\tau(0) \neq 0$. The theorem explains the "sum-rules" phenomenon currently highlighted in the literature (cf., e.g., [HSS]): with $y_j := D^j\tau(0)$, $|j| < k$, and $\tau$ as above, the above characterization can be converted, by an application of Leibniz' formula to the equalities $D^j(\tau(2\cdot)M)(\xi) = 0$, $\xi \in \{0,\pi\}^d\backslash 0$, and $D^j(\tau - \tau(2\cdot)M)(0) = 0$, to the equivalent formulation "there exist $(y_j : |j| < k)$ such that $y_0 \neq 0$, and, further,

$$\sum_{j' \le j} \binom{j}{j'} 2^{|j'|} y_{j'} D^{j-j'} M(\xi) = 0, \quad \xi \in \{0,\pi\}^d\backslash 0, \quad |j| < k,$$

and

$$y_j - \sum_{j' \le j} \binom{j}{j'} 2^{|j'|} y_{j'} D^{j-j'} M(0) = 0, \quad |j| < k. \quad "$$

In fact, while we have chosen to outline the above for dyadically refinable functions, the actual result in Section 5 applies to functions that are refinable with respect to general dilation matrices.

**Remark.** It should be emphasized that, in this paper, 'approximation order' always refers to that provided by the corresponding stationary ladder, $S_h := S(\Phi)(\cdot/h)$, as defined in [BDR1]. In the case of refinable functions with dilation matrix $s$, one may also be interested in the approximation order of the nested sequence $V_j := S(\Phi)(s^j\cdot)$. This latter notion of approximation order is investigated by Jia in [J3], [J4] for a compactly supported singleton $\Phi = \{\phi\}$ and for general dilation matrices. In general, the two notions of approximation order differ. However, if we assume the dilation to be isotropic (in particular, if the dilation is dyadic), then there is a simple rigid connection between the two notions, and results in terms of one notion can be equivalently formulated in terms of the other. In view of that, it is correct to attribute the PSI compact support case of Theorem 1.5 to [J4]. □

The superfunction $\psi$ of Theorem 1.5 is said there to be 'unique'. Of course, that uniqueness is in terms of the particular properties asserted in that theorem. For specific applications, other superfunctions with slightly different properties may be desired. For example, if $\Phi$ is refinable with a mask $M$ whose entries are trigonometric polynomials, then, for certain applications, it is desirable to know that there is a generator $\phi \in \Phi$ and a corresponding superfunction $\psi$ so that the vector $\psi \cup (\Phi\backslash\phi)$ is still refinable (and generating), with the entries of its mask still polynomials. Such an assertion can, offhand, not be made for the superfunction of Theorem 1.5 (regardless of the choice of $\phi$), but is proved in Section 5 (Corollary 5.5) for the superfunction obtained in another superfunction result (Theorem 4.12) in Section 4.

While putting together the arguments for the new superfunction observations outlined above, we realized that there is a handy way to characterize approximation orders of FSI spaces directly in terms of the generating set $\Phi$. In fact, that observation extends to the more general case of non-stationary FSI ladders. These characterizations, which are valid without any restriction on the finite set $\Phi$ (other than the obvious restriction, that $\Phi \subset L_2(\mathbb{R}^d)$), are presented and proved in the next section, and their efficacy is illustrated in Section 3 by using them to compute, once again, the exact approximation order of the space of $C^1$-cubics on the 3-direction mesh.

## 2. A characterization of the approximation order of FSI spaces

In this section, we characterize the approximation order of the FSI space directly and explicitly in terms of any particular generating set for it. The characterization extends to the non-stationary case, whose definition is given in the sequel, in a way that is analogous to the non-stationary PSI extensions. The argument we use in the proof of the main result invokes our two main observations from [BDR1], viz., the characterization of the approximation orders of PSI spaces, and the superfunction results such as Result 1.2.

We recall the definition of the **bracket product** of $f, g \in L_2(\mathbb{R}^d)$:

$$[f, g] := \sum_{\alpha \in 2\pi\mathbb{Z}^d} f(\cdot + \alpha)\overline{g(\cdot + \alpha)},$$

i.e., the $2\pi$-periodization of $f\overline{g}$. The sum converges in $L_1$ on compact sets. Given a finite $\Phi \subset L_2(\mathbb{R}^d)$, the **Gramian** $G := G_\Phi$ of $\Phi$ is the square matrix indexed by $\Phi$ whose $(\phi, \varphi)$-entry $((\phi, \varphi) \in \Phi \times \Phi)$ is the corresponding bracket product:

$$G(\phi, \varphi) := [\widehat{\varphi}, \widehat{\phi}];$$

notice the inverted order, here and in [RS], as compared to the definition of $G$ in [BDR2] (forced upon us because we have, following the customary treatment of inner products, made the bracket product skew-linear in its second argument).

In our use of the Gramian, we adopt the convention of treating $\Phi$ as a *sequence* to which we may apply, on the left or the right, matrices of compatible sizes to produce other sequences. This convention permits us to write $\sum_{\phi\in\Phi} c_\phi \widehat{\phi}$ as $c\widehat{\Phi}$ or $\widehat{\Phi}c$. Further, if $A = (a_{\phi,\varphi})$ is a matrix with rows and columns indexed by $\Phi$, then $\widehat{\Phi}A\widehat{\Phi} = \sum_{\phi,\varphi\in\Phi} a_{\phi,\varphi}\widehat{\phi}\widehat{\varphi}$. With this, we recall from [BDR2: Theorem 3.9] that the Fourier transform $\widehat{\psi}$ of the orthogonal projection $\psi$ of $f \in L_2(\mathbb{R}^d)$ onto $S(\Phi)$ can be written

$$\widehat{\psi} \;=\; \tau\widehat{\Phi} = \widehat{\Phi}\tau \qquad (2.1a)$$

with the $2\pi$-periodic function $\tau$ satisfying

$$\tau \;=\; G^{-1}\,[\widehat{f}, \widehat{\Phi}] \qquad (2.1b)$$

at every point at which $G$ is invertible, and with

$$[\widehat{f}, \widehat{\Phi}] := ([\widehat{f}, \widehat{\phi}] : \phi \in \Phi).$$

Here is the main result of this section.

**Theorem 2.2.** *Assume that the Gramian $G = G_\Phi$ for some generating set $\Phi$ for the FSI space $S$ is invertible a.e. in some neighborhood of the origin. Then $S$ provides approximation order $k$ if and only if the function*

$$\Lambda_\Phi : \omega \mapsto \sqrt{1 - (\widehat{\Phi} G^{-1} \overline{\widehat{\Phi}})(\omega)}$$

*is such that $|\cdot|^{-k}\Lambda_\Phi \in L_\infty(B)$ for some neighborhood $B$ of the origin.*

**Example.** If $\Phi$ is the singleton $\{\phi\}$, then the function $\Lambda_\Phi$ reduces to

$$\omega \mapsto (1 - \frac{|\widehat{\phi}|^2}{[\widehat{\phi}, \widehat{\phi}]}(\omega))^{1/2},$$

where here and below, as in [BDR1], we interpret $0/0$ to be 0. The above theorem thus covers, as a special case, the characterization of the approximation order provided by stationary PSI ladders that we obtained in [BDR1: Theorem 1.6].

**Proof of Theorem 2.2.** From Result 1.2, we know that, with $\psi$ the orthogonal projection of the sinc-function $g$ onto $S$, $S(\psi)$ provides the same approximation order as $S$. We will show that, near the origin, the map

$$\Lambda_\psi : \omega \mapsto (1 - \frac{|\widehat{\psi}|^2}{[\widehat{\psi}, \widehat{\psi}]}(\omega))^{1/2}$$

coincides with $\Lambda_\Phi$. Our present FSI theorem will then follow from its special PSI case, i.e., Theorem 1.6 of [BDR1].

To compute $\Lambda_\psi$, note that the error, $g - \psi$, in the orthogonal projection $\psi$ of $g$ to $S$ is necessarily perpendicular to $S(\psi)$, hence, e.g. by [BDR1: Lemma 2.8], $[\widehat{g} - \widehat{\psi}, \widehat{\psi}] = 0$, i.e., $[\widehat{g}, \widehat{\psi}] = [\widehat{\psi}, \widehat{\psi}]$. Since $\widehat{g}$ is the characteristic function of the cube $[-\pi \,..\, \pi]^d$, this shows that, for $\omega$ near the origin, $\overline{\widehat{\psi}}(\omega) = [\widehat{\psi}, \widehat{\psi}](\omega)$, hence

$$|\widehat{\psi}(\omega)|^2 / [\widehat{\psi}, \widehat{\psi}](\omega) = \widehat{\psi}(\omega) \tag{2.4}$$

there. Thus it only remains to show that, for $\omega$ near the origin,

$$\widehat{\psi}(\omega) = (\widehat{\Phi} G^{-1} \overline{\widehat{\Phi}})(\omega) = \sum_{\phi, \varphi \in \Phi} \widehat{\phi}(\omega)\; G(\omega)^{-1}(\phi, \varphi)\; \overline{\widehat{\varphi}}(\omega).$$

But this is evident since, by assumption, $G(\omega)$ is invertible for a.e. $\omega$ in some neighborhood of the origin, hence we have with (2.1), for any such $\omega$,

$$\widehat{\psi}(\omega) = \widehat{\Phi}(\omega) G(\omega)^{-1} [\widehat{g}, \widehat{\Phi}](\omega), \tag{2.5}$$

while $\widehat{g} = 1$ near the origin, hence $[\widehat{g}, \widehat{\Phi}](\omega) = \overline{\widehat{\Phi}}(\omega)$ there. $\square$

**Remark.** The assumption of invertibility of the Gramian a.e. in some neighborhood of the origin is simply a convenience. *The theorem remains true without this assumption provided the symbol $G(\omega)^{-1}$ is interpreted to mean any right inverse of $G(\omega)$ as a map to* $\operatorname{ran}[\widehat{\Phi}^*](\omega)$, *with* $[\widehat{\Phi}^*](\omega) : f \mapsto ([f, \widehat{\phi}](\omega) : \phi \in \Phi)$. Indeed, recall from [BDR2: Result 3.7] that, with the notation

$$\widehat{\varphi}_{\|\omega} := (\widehat{\varphi}(\omega + 2\pi\alpha) : \alpha \in \mathbb{Z}^d),$$

the Fourier transform $\widehat{\psi}$ of the orthogonal projection $\psi$ of $g \in L_2(\mathbb{R}^d)$ to $S(\Phi)$ has the form $\widehat{\Phi}\tau$, with

$$(\widehat{\Phi}\tau)_{\|\omega} = \widehat{\Phi}_{\|\omega}\tau(\omega) := \sum_{\varphi\in\Phi} \widehat{\varphi}_{\|\omega}\tau_\varphi(\omega)$$

the $\ell_2$-projection of $\widehat{g}_{\|\omega}$ onto $\operatorname{span}\widehat{\Phi}_{\|\omega}$, all $\omega \in [-\pi \,..\, \pi]^d$. In other words, for any $\omega \in [-\pi \,..\, \pi]^d$, $\widehat{\psi}(\omega) = \widehat{\Phi}_{\|\omega}\tau(\omega)$ is the orthogonal projection of $\widehat{g}_{\|\omega} \in \ell_2$ onto the range of the linear map

$$V : \mathbb{C}^\Phi \to \ell_2(\mathbb{Z}^d) : c \mapsto \widehat{\Phi}_{\|\omega}c = \sum_{\varphi\in\Phi} \widehat{\varphi}_{\|\omega}c_\varphi.$$

Let

$$V^* : \ell_2(\mathbb{Z}^d) \to \mathbb{C}^\Phi : x \mapsto (\langle x, \widehat{\phi}_{\|\omega}\rangle : \varphi \in \Phi).$$

Then $G(\omega) = V^*V$ maps $\mathbb{C}^\Phi$ *onto* $\operatorname{ran} V^* = \operatorname{ran}[\widehat{\Phi}^*](\omega)$ (since $V^*$ is 1-1 on $\operatorname{ran} V$ and $\dim\operatorname{ran} V^* = \dim\operatorname{ran} V$), hence, as a map to $\operatorname{ran} V^*$, it has right inverses. Let $C$ be any such right inverse. Then $Q := VCV^*$ is the orthogonal projector of $\ell_2(\mathbb{Z}^d)$ to $\operatorname{ran} V = \operatorname{span}\widehat{\Phi}_{\|\omega}$ (since $\operatorname{ran} Q \subset \operatorname{ran} V$ while $V^*(\mathrm{id} - Q) = 0$). Consequently,

$$\widehat{\psi}_{\|\omega} = Q\,\widehat{g}_{\|\omega} = \widehat{\Phi}_{\|\omega}\,C\,([\widehat{g}, \widehat{\varphi}](\omega) : \varphi \in \Phi),$$

since $V^*\widehat{g}_{\|\omega} = (\langle \widehat{g}_{\|\omega}, \widehat{\varphi}_{\|\omega}\rangle : \varphi \in \Phi)$ and $\langle \widehat{g}_{\|\omega}, \widehat{\varphi}_{\|\omega}\rangle = [\widehat{g}, \widehat{\varphi}](\omega)$. This gives (2.5) with $G(\omega)^{-1}$ replaced by $C$. $\square$

The following corollary was established in the course of the proof of the theorem (see (2.4)):

**Corollary 2.6.** *Let $S$ be an FSI space. Let $\psi$ be the orthogonal projection of the* sinc-*function onto $S$. Then $S$ provides approximation order $k$ if and only if the function*

$$\omega \mapsto |\omega|^{-k}(1 - \widehat{\psi})^{1/2}(\omega)$$

*is essentially bounded around the origin. Also, in any case, $0 \le \widehat{\psi} \le 1$ near the origin.*

As we mentioned before, the characterization of approximation order that was obtained here extends to the non-stationary case. In the non-stationary case, each space in the ladder $(S_h)_h$ is still the $h$-dilate of some FSI space, but that FSI space may depend on $h$, i.e.,

$$S_h = S(\Phi_h)(\cdot/h)$$

for some $h$-dependent $\Phi_h \subset L_2(\mathbb{R}^d)$. The notion of 'approximation orders' is defined here exactly as in the stationary case (cf. (1.1)), and the approximation orders are attributed to the ladder $\mathcal{S} = (S_h)_h$. The extension of the above result to the non-stationary case is done exactly in the same manner non-stationary extensions were dealt with in [BDR1]. We state these results without further comment.

**Theorem 2.7.** *Let $\mathcal{S}$ be a non-stationary FSI ladder, i.e., the space $S_h$ is the $h$-dilate of the FSI space $S(\Phi_h)$, with $\Phi_h$ an $h$-dependent finite subset of $L_2(\mathbb{R}^d)$. Let $G_h$ be the Gramian of $\Phi_h$, assumed to be invertible a.e. on some fixed neighborhood of the origin. Then, the ladder $\mathcal{S}$ provides approximation order $k$ if and only if, for some $h_0 > 0$, the functions*

$$(h + |\cdot|)^{-k} \sqrt{1 - \widehat{\Phi}_h G_h^{-1} \overline{\widehat{\Phi}_h}}\ , \quad h < h_0,$$

*are bounded in $L_\infty(B)$, for some neighborhood $B$ of the origin.*

## 3. An application: bivariate $C^1$ cubics on 3-direction mesh

In this section only, let $S$ denote the space of bivariate $C^1$-cubics on the 3-direction mesh. This space was shown in [BH1] to provide approximation order 3 only, even though $\Pi_3$ is contained locally in it. This result made clear that the approximation order of a FSI space might be harder to ascertain than originally thought. It is therefore worthwhile to show how Theorem 2.2 provides the exact approximation order for this space.

In the interest of brevity, we refer the reader to [BH1], [BH2] as a source for any missing details and for prior literature concerning this particular $S$, which consists of all piecewise cubic functions in $C^1(\mathbb{R}^2)$ for the 3-direction mesh, i.e., with breaklines

$$\mathbb{R}\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\} + \mathbb{Z}^2$$

involving the three 'directions'

$$\mathbf{i}_1 := (1,0), \quad \mathbf{i}_2 := (0,1), \quad \mathbf{i}_3 := (1,1).$$

This space is obviously shift-invariant. It is shown in [BH2] that its approximation power equals that of

$$S_{loc} := S(\Phi),$$

with $\Phi$ comprising the three $C^1$-cubic box splines for the 3-direction mesh. These are obtained by convolving the **hat function** (or, 'Courant' element, or, linear 3-direction box spline) $M_{111}$ with the characteristic function of the parallelepiped spanned by two of the three directions. It follows that $S_{loc}$ is generated by the **Frederickson elements**

$$\varphi_i := \chi_{T_i} * M_{111}, \quad i = 1, 2,$$

where the star indicates convolution, and $T_1$, $T_2$ are the two triangles obtained by cutting the unit square $[0\,..\,1]^2$ by the 'north-east' diagonal.

In order to apply Theorem 2.2, it is enough to determine the order to which the function

$$\sqrt{1 - \widehat{\Phi} G^{-1} \overline{\widehat{\Phi}}}$$

vanishes at 0, with $\Phi = (\varphi_1, \varphi_2)$ and, correspondingly,

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

where

$$g_{ij} := [\widehat{\varphi}_j, \widehat{\varphi}_i].$$

Since $S$ consists of piecewise cubics, its approximation order cannot be bigger than 4. Hence it is sufficient to compute the Taylor coefficients of

$$f := \widehat{\Phi} G^{-1} \overline{\widehat{\Phi}} = \frac{\widehat{\Phi} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix} \overline{\widehat{\Phi}}}{\det G} \tag{3.1}$$

to terms of degree 7 (inclusive), but these coefficients must be computed exactly. It turns out that $\det G$ has a zero of order 4 at the origin. Since $f$ is a rational function of the $\widehat{\varphi}_i$ and the $g_{ij}$, this means that we need (nothing more than) the exact Taylor coefficients to degree 11 (inclusive) of the $\widehat{\varphi}_i$ and the $g_{ij}$.

This is a simple task for the $\widehat{\varphi}_i$ since they can be given in closed form, as follows (with $T_1$ the triangle with vertices 0, $\mathbf{i}_2$, $\mathbf{i}_3$, and with $w := u + v$):

$$\widehat{\varphi}_1(u, v) = \mathrm{i} \frac{(v(1 - e^{-\mathrm{i}w}) - w(1 - e^{-\mathrm{i}v}))(1 - e^{-\mathrm{i}u})(1 - e^{-\mathrm{i}v})(1 - e^{-\mathrm{i}w})}{(uvw)^2}$$

while

$$\widehat{\varphi}_2(u, v) = \widehat{\varphi}_1(v, u).$$

For the $g_{ij} = [\widehat{\varphi}_j, \widehat{\varphi}_i]$, we go the following route. Since

$$[\widehat{\varphi}, \widehat{\phi}] = \sum_{\alpha \in \mathbb{Z}^2} e^{\mathrm{i}\alpha \cdot} a(\alpha)$$

with

$$a(\alpha) := \int_{\mathbb{R}^2} \varphi(x-\alpha)\overline{\phi}(x)\mathrm{d}x,$$

and the $\varphi_i$ are compactly supported, $g_{ij}$ is a trigonometric polynomial. Further, its coefficients

$$a_{ij}(\alpha) = \int_{\mathbb{R}^2} \varphi_j(x-\alpha)\varphi_i(x)\mathrm{d}x$$

are the values at $\alpha \in \mathbb{Z}^2$ of the function

$$N_{ij} := \varphi_j(-\cdot) * \varphi_i = M_{222}(\cdot + 3\mathbf{i}_3) * \begin{cases} \chi_{T_1} * \chi_{T_2}, & i=j; \\ \chi_{T_1} * \chi_{T_1}, & i=1,\, j=2; \\ \chi_{T_2} * \chi_{T_2}, & i=2,\, j=1, \end{cases}$$

hence $N_{21}(s,t) = N_{12}(t,s)$, with $M_{222} = M_{111} * M_{111}$ the bivariate box spline with directions $(\mathbf{i}_1, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_2, \mathbf{i}_3, \mathbf{i}_3)$. (The shift by $-3\mathbf{i}_3$ ensures that 0 is the center of the support of $N_{ij}$, as it should be.) In particular, the $N_{ij}$ are obtainable from the continuous piecewise quadratic functions $F_{ij} := \chi_{T_i} * \chi_{T_j}$ by two-fold convolution with each of the three directions, $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, followed by a shift. Such convolutions can be expressed as simple, linear, local operations on the BB-net of a piecewise polynomial on the 3-direction mesh, involving only simple rational numbers, hence can be easily carried out exactly even in floating-point arithmetic. Here, the BB- (or, Bernstein-Bézier-)net (see, e.g., [BH1]) for a continuous piecewise polynomial of degree $k$ on the 3-direction mesh is the function defined on $k^{-1}\mathbb{Z}^2$ whose restriction to any triangle of the mesh provides the Bernstein-Bézier coefficients, with respect to that triangle, of the polynomial piece associated with that triangle. For example, with $b_{ij}$ the BB-net $b_{ij}$ for $\chi_{T_i} * \chi_{T_j}$,

$$b_{11} : \mathbb{Z}^2/2 \to \mathbb{C} : \alpha \mapsto \begin{cases} \frac{1}{2}, & \alpha = (\frac{1}{2}, 1), (\frac{1}{2}, \frac{3}{2}), (1, \frac{3}{2}); \\ 0, & \text{otherwise}, \end{cases}$$

$$b_{12} = b_{21} : \mathbb{Z}^2/2 \to \mathbb{C} : \alpha \mapsto \begin{cases} \frac{1}{2}, & \alpha = (1,1); \\ 0, & \text{otherwise}, \end{cases}$$

and $b_{22}(i,j) = b_{11}(j,i)$.

These calculations eventually lead to

$$(N_{21}(i-3, j-3) : i,j = 1,\ldots,5) \;=\; \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 28 & 160 & 28 & 0 & 0 \\ 28 & 622 & 622 & 28 & 0 \\ 1 & 160 & 622 & 160 & 1 \\ 0 & 1 & 28 & 28 & 1 \end{pmatrix} /(2 \cdot 7!)$$

and

$$(N_{11}(i-3,j-3) : i,j=1,\ldots,5) \ = \ \begin{pmatrix} 3 & 18 & 3 & 0 & 0 \\ 18 & 494 & 494 & 18 & 0 \\ 3 & 494 & 1950 & 494 & 3 \\ 0 & 18 & 494 & 494 & 18 \\ 0 & 0 & 3 & 18 & 3 \end{pmatrix} /(4\cdot 7!)$$

(with $N_{ij}(\alpha)=0$ otherwise), and, in view of $N_{12}(i,j)=N_{21}(j,i)$ and $N_{22}=N_{11}$, that is all we need.

From this information, we computed that $f$ of (3.1) has the form

$$f(u,v) = 1 - \frac{(uvw)^2}{12\cdot 7!} + \mathbf{h.o.t.},$$

thus verifying that the approximation order of bivariate $C^1$-cubics on the 3-direction mesh is exactly 3.

## 4. Superfunctions in FSI spaces

We now turn our attention to the second purpose of this paper: establishing improved 'superfunction results', under 'mild' conditions on the generating set $\Phi$. The precise assumption on $\Phi$ we shall make is formalized in the following definition.

**Definition 4.1.** *Let $\Phi$ be a finite subset of $L_2(\mathbb{R}^d)$, and let $k$ be a positive number. We say that $\Phi$ has the* **Strong Property H**$(k)$ *if the following two conditions are met:*

*(a) For some neighborhood $B$ of the origin, each $\widehat{\phi}$, $\phi\in\Phi$, as well as each entry of the Gramian $G$ of $\Phi$, is $k$ times continuously differentiable on $B+2\pi\mathbb{Z}^d$.*

*(b) $G(0)$ is invertible.*

**Discussion.** The Gramian $G$ is $2\pi$-periodic, hence the differentiability assumption with respect to it is actually demanded only around the origin. In any event, condition (a) here is 'mild', and is satisfied if (but not only if) each function $\phi\in\Phi$ decays at $\infty$ at a rate $O(|\cdot|^{-\rho})$, for some $\rho > k+d$. For the diagonal entries of $G$, this smoothness is shown in the proof of Proposition 4.1 in [R3] by showing that, for any suitably small neighborhood $B$ of the origin, $\sum_{j\in 2\pi\mathbb{Z}^d} \||\widehat{\phi}|^2\|_{C^k(j+B)} < \infty$ in case $\phi\in L_2(\mathbb{R}^d)$ vanishes to some order $\rho > k+d$ at $\infty$. By Schwarz' Inequality, this implies that also $\sum_{j\in 2\pi\mathbb{Z}^d} \|\widehat{\phi}\overline{\widehat{\varphi}}\|_{C^k(j+B)} < \infty$. Therefore (proceeding as in [R3]), $[\widehat{\phi},\widehat{\varphi}] = \sum_{j\in 2\pi\mathbb{Z}^d} \widehat{\phi}(\cdot + j)\overline{\widehat{\varphi}}(\cdot + j)$ is $k$ times continuously differentiable on $B$.

The more discriminating condition is (b): since $G(0)$ is the Gramian matrix of the sequence $\widehat{\Phi}_{|_{2\pi\mathbb{Z}^d}}$, that condition is equivalent to the linear independence of $\widehat{\Phi}_{|_{2\pi\mathbb{Z}^d}}$, which, as mentioned in the introduction, is equivalent to the linear independence of the finite function set $\Phi^\circ$ (of Theorem 1.5), if that set is well-defined. Thus, in view of the remark at the end of the introduction, requirement (b) here holds if (but not only if) the shifts of $\Phi$ are $L_2$-stable, *a fortiori* whenever these shifts are linearly independent. $\square$

**Remark.** Property H($k$) appears previously in the article [R3], and is defined there differently. Roughly speaking, the definition in [R3] is as follows: "$S(\Phi)$ satisfies the H($k$) property if it provides approximation order $k$ to the entire Sobolev space, the moment it provides approximation order $k$ to some non-zero smooth function"; see [R3] for the precise definition. While our definition here and the definition there may seem to be entirely unrelated, the proof of Proposition 4.2 of [R3] shows that our definition here is stronger (i.e., implies) the definition of [R3], whence the terminology 'Strong H($k$)'. $\square$

The following theorem is the main result of this section, and contains Theorem 1.5 as a special case. Recall the definition $\mathcal{Z}_k := \{j \in \mathbb{Z}_+^d : |j| < k\}$ from (1.4).

**Theorem 4.2.** *Let $k$ be a positive integer, and let $\Phi$ be a finite subset of $L_2(\mathbb{R}^d)$ that satisfies the Strong Property H(k). If $S(\Phi)$ provides approximation order $k$, then there exists a unique function $\psi$ that satisfies the following three properties:*
*(a) It is a linear combination of the $\mathcal{Z}_k$-shifts of $\Phi$.*
*(b) Its mean value is 1, and, further, $D^j\widehat{\psi}(0) = 0$, $j \in \mathcal{Z}_k\backslash\{0\}$.*
*(c) Its corresponding PSI space $S(\psi)$ provides approximation order $k$.*

**Proof.** Let $\psi_0$ be the orthogonal projection of the sinc-function $g$ onto $S(\Phi)$. Since we assume that $S(\Phi)$ provides approximation order $k$, Result 1.2 implies that $S(\psi_0)$ provides that same approximation order. Furthermore, from (2.1), $\widehat{\psi}_0 = \tau\widehat{\Phi}$, with $\tau = (\tau_\phi : \phi \in \Phi)$ a $2\pi$-periodic function which equals $G^{-1}[\widehat{g}, \widehat{\Phi}]$ around the origin. Here, $[\widehat{g}, \widehat{\Phi}] = ([\widehat{g}, \widehat{\phi}] : \phi \in \Phi)$. Since $G$ is $k$ times continuously differentiable at the origin, and since $G(0)$ is non-singular, $G^{-1}$ is well-defined and $k$ times continuously differentiable on some neighborhood $B$ of the origin. Also, near the origin, $[\widehat{g}, \widehat{\Phi}] = \overline{\widehat{\Phi}}$, and $\widehat{\Phi}$ is $k$ times differentiable there, and therefore $\tau$, hence $\widehat{\psi}_0$, is $k$ times continuously differentiable around the origin. Invoking Corollary 2.6, we conclude that $D^j\widehat{\psi}_0(0) = \delta_j$, $j \in \mathcal{Z}_k$. In particular, $\tau(0) \neq 0$.

We have just argued that the $2\pi$-periodic $\tau_\phi$ in the representation $\widehat{\psi}_0 = \tau\widehat{\Phi}$ are $k$ times continuously differentiable around the origin. Therefore, we can find *trigonometric polynomials* $\widetilde{\tau} = (\widetilde{\tau}_\phi : \phi \in \Phi)$ such that
(a) the spectrum of each $\widetilde{\tau}_\phi$ lies in $\mathcal{Z}_k$;
(b) $\tau - \widetilde{\tau}$ vanishes to order $k$ at the origin.

We now define a function $\psi$ by

$$\widehat{\psi} := \widetilde{\tau}\widehat{\Phi}. \tag{4.3}$$

It trivially follows that $\widehat{\psi} - \widehat{\psi}_0$ vanishes to order $k$ at the origin (as a matter of fact, at each $\alpha \in 2\pi\mathbb{Z}^d$), and hence $\psi$ satisfies conditions (a) and (b) of the theorem. It remains to show that $\psi$ provides approximation order $k$, and that it is unique.

Approximation order: Given any function $\eta$ whose Fourier transform $\widehat{\eta}$ is $k$ times differentiable around the origin, and is non-vanishing there, Theorem 1.6 of [BDR1] shows that the PSI space $S(\eta)$ provides approximation order $k$ if and only if

$$[\widehat{\eta}, \widehat{\eta}] - |\widehat{\eta}|^2 = \sum_{\alpha \in 2\pi\mathbb{Z}^d \backslash 0} \widehat{\eta}(\cdot + \alpha)\overline{\widehat{\eta}(\cdot + \alpha)} =: \|\widehat{\eta}\|_0^2$$

has a zero of order $2k$ at the origin. In particular, this condition holds for $\widehat{\eta} = \widehat{\psi}_0 = \tau\widehat{\Phi}$ since $S(\psi_0)$ provides approximation order $k$. Further, since $\widehat{\eta} \mapsto \|\widehat{\eta}\|_0(t)$ is a seminorm, we have *pointwise*

$$(\|\widetilde{\tau}\widehat{\Phi}\|_0 - \|\tau\widehat{\Phi}\|_0)^2 \le \|(\widetilde{\tau} - \tau)\widehat{\Phi}\|_0^2 \le \|G_0\|_2\|\widetilde{\tau} - \tau\|_2^2,$$

with $G_0 := (\sum_{\alpha \in 2\pi\mathbb{Z}^d \backslash 0} \widehat{\phi}(\cdot + \alpha)\overline{\widehat{\varphi}(\cdot + \alpha)} : \varphi, \phi \in \Phi)$, with $\|\cdot\|_2$ the norm in $\ell_2(\Phi)$ and $\|G_0\|_2$ the associated matrix norm. Since, by construction of $\widetilde{\tau}$, $\|\widetilde{\tau} - \tau\|_2$ has a zero of order $k$ at the origin, as does $\|\tau\widehat{\Phi}\|_0$, it follows that so does $\|\widetilde{\tau}\widehat{\Phi}\|_0$, which is what we needed to show.

Uniqueness: Let $\psi$ be any function satisfying (a-c). By (c) (see the Remark following Theorem 1.5), $\psi$ satisfies the Strang-Fix conditions of order $k$, hence, for $j \in \mathcal{Z}_k$, $D^j\widehat{\psi} = 0$ on $2\pi\mathbb{Z}^d \backslash \{0\}$, while, by (b), $D^j\widehat{\psi}(0) = \delta_j$ for $j \in \mathcal{Z}_k$. In short,

$$(D^j\widehat{\psi})(\alpha) \;=\; \delta_\alpha\delta_j, \qquad \alpha \in 2\pi\mathbb{Z}^d,\ j \in \mathcal{Z}_k. \tag{4.4}$$

By (a), $\widehat{\psi} = \tau\widehat{\Phi}$ for some smooth $2\pi$-periodic $\tau$. Therefore, from (4.4) and Leibniz' formula,

$$\sum_{0 \le j' \le j} (D^{j'}\tau)(0)\binom{j}{j'}D^{j-j'}\widehat{\Phi}(\alpha) \;=\; \delta_\alpha\delta_j, \qquad \alpha \in 2\pi\mathbb{Z}^d,\ j \in \mathcal{Z}_k. \tag{4.5}$$

On multiplying through, for each $\phi \in \Phi$, by $\overline{\widehat{\phi}}(\alpha)$ and then summing over $\alpha$, we obtain the system of equations

$$G(0)v_j \;+\; \sum_{j' < j} Q_{j,j'}(0)v_{j'} \;=\; \delta_j\overline{\widehat{\Phi}}(0), \qquad j \in \mathcal{Z}_k, \tag{4.6}$$

for the quantities

$$v_j \;:=\; D^j\tau(0), \quad j \in \mathcal{Z}_k, \tag{4.7}$$

with

$$Q_{j,j'}(\phi,\varphi) := \binom{j}{j'}[D^{j-j'}\widehat{\varphi},\widehat{\phi}], \quad \phi,\varphi\in\Phi. \tag{4.8}$$

Since $G(0)$ is invertible by assumption (i.e., by (b) of the Strong Property $H(k)$), (4.6) is a block-triangular linear system, with invertible diagonal blocks, hence uniquely solvable. Invoking now (a) in full detail, $\tau$ is a trigonometric polynomial with frequencies from $\mathcal{Z}_k$, hence is uniquely determined by the information (4.7), hence so is $\widehat{\psi}$. Therefore, finally, conditions (a-c) determine $\psi$ uniquely. □

The argument just given provides the following.

**Corollary 4.9.** *Let $\Phi$ be a finite collection of functions in $L_2$ that satisfies the Strong Property H(k). Then, $\Phi$ provides approximation order $k$ if and only if there exist vectors $v := (v_j : j\in\mathcal{Z}_k)$ (each in $\mathbb{C}^\Phi$) that satisfy*

$$\sum_{0\le j'\le j}\binom{j}{j'} v_{j'} D^{j-j'}\widehat{\Phi}(\alpha) \ = \ \delta_\alpha\delta_j, \qquad \alpha\in 2\pi\mathbb{Z}^d,\ j\in\mathcal{Z}_k.$$

*Moreover, these vectors necessarily uniquely solve the linear system (4.6).*

Perhaps the most striking corollary of Theorem 4.2 is the following characterization of FSI ladders that provide approximation order 1. The result should be primarily viewed as 'negative': without satisfying its condition, the FSI ladder (that is known to satisfy the technical Strong Property H(1)) cannot provide any positive approximation order.

**Corollary 4.10.** *Let $\Phi$ be a finite subset of $L_2$ that satisfies the Strong Property H(1). Then $S(\Phi)$ provides approximation order 1 if and only if there exists a function $\psi$ in* span $\Phi$ *whose PSI ladder provides approximation order 1.*

**Proof.** The 'only if' condition follows directly from Theorem 4.2, the 'if' statement is trivial. □

In the case when the functions of $\Phi$ decay at a rate $O(|\cdot|^{-\rho})$ at $\infty$ for some $\rho > k+d$, the differentiability conditions on the Gramian and $\widehat{\Phi}$ that are required in Strong Property H($k$) are automatically satisfied; see the Discussion following Definition 4.1. In that event, the technical conditions imposed on $\Phi$ can be simplified to obtain the following result:

**Corollary 4.11.** *Assume that, for some positive integer $k$ and some $\rho > k+d$, each $\phi$ in the finite $\Phi\subset L_2$ decays at $\infty$ at a rate $O(|\cdot|^{-\rho})$. Assume also that the set $\widehat{\Phi}_{|_{2\pi\mathbb{Z}^d}}$ is linearly independent. Then the characterization of approximation order $k$ that was established in Corollary 4.9 is valid.*

Theorem 4.2 proves, under the assumptions made in it, that the superfunction $\psi$ exists and is *unique*. In particular, the uniqueness depends on the specification that certain derivatives of $\widehat{\psi}$ vanish at the origin, and this may be useful when constructing approximation maps based on the shifts of $\psi$ (cf., e.g., [BR1]), but may be less important for other applications. Specifically, for a certain application concerning a change of generating sets for a refinable FSI space (that we describe in the next section), the following superfunction result, which differs only slightly from Theorem 4.2, is found useful.

**Theorem 4.12.** *Let $k$ be a positive integer, and let $\Phi$ be a finite subset of $L_2(\mathbb{R}^d)$ that satisfies the Strong Property H($k$). If $S(\Phi)$ provides approximation order $k$, then there exists $\phi_0 \in \Phi$ and, for this $\phi_0$, a unique function $\psi$ that satisfies the following three properties:*

*(a) It differs from $\phi_0$ by a linear combination of the $\mathcal{Z}_k$-shifts of $\Phi \backslash \phi_0$.*

*(b) Its mean-value is not zero.*

*(c) Its corresponding PSI space $S(\psi)$ provides approximation order $k$.*

**Proof.** Since the Strong Property H($k$) implies the Strong Property H($\ell$) for every $\ell < k$, Theorem 4.2 supplies, for each $\ell = 1, \ldots, k$, the unique element $\psi_\ell$ in the span $S_\ell$ of the $\mathcal{Z}_\ell$-shifts of $\Phi$ with the property that $\widehat{\psi_\ell}$ has a zero of order $\ell$ at every $\alpha \in 2\pi\mathbb{Z}^d \backslash 0$ while $D^j \widehat{\psi_\ell}(0) = \delta_j$, $j \in \mathcal{Z}_\ell$. Moreover, by the Strong H($k$) property, the $\mathcal{Z}_k$-shifts of the elements of $\Phi$ form a linearly independent sequence, hence form a basis for their span $S_k$. Therefore, with $\Delta$ the forward-difference operator, also the sequence

$$(\Delta^j \phi : \phi \in \Phi, j \in \mathcal{Z}_k) \tag{4.13}$$

is a basis for $S_k$. Further, for each $\phi \in \Phi$, the coefficient $c_{\phi,0}$ of $\phi = \Delta^0 \phi$ in the unique representation of $\psi_\ell$ with respect to the basis (4.13) does not depend on $\ell$. To see this, express $\psi_k$ in terms of the basis (4.13), and evaluate the Fourier transform of that representation on $2\pi\mathbb{Z}^d$. Since the Fourier transform of $\Delta^j \phi$ vanishes on $2\pi\mathbb{Z}^d$ for all $j \neq 0$, we obtain that

$$\widehat{\psi_k} = \sum_{\phi \in \Phi} c_{\phi,0} \widehat{\phi}, \quad \text{on } 2\pi\mathbb{Z}^d. \tag{4.14}$$

However, regardless of the value of $k$, $\widehat{\psi_k} = \delta_0$ on $2\pi\mathbb{Z}^d$, and this property already determines uniquely the coefficients in (4.14) since $G(0)$ is invertible by assumption.

Let $\phi_0$ be any $\phi \in \Phi$ for which $c_0 := c_{\phi,0}$ is nonzero and consider the sequence

$$F := (f_j := \Delta^j \psi_\ell : |j| = k - \ell,\ \ell = 1, \ldots, k-1).$$

Then

$$f_j \in c_0 \Delta^j \phi_0 + \operatorname{span}(W_j, V),$$

with

$$W_j := (\Delta^{j'}\phi_0 : j' > j, j' \in \mathcal{Z}_k)$$

and

$$V := (\Delta^j\phi : j \in \mathcal{Z}_k, \phi \in \Phi\backslash\phi_0).$$

Since $c_0 \neq 0$, one concludes that the sequence $(\phi_0, F, V)$, obtained from (4.13) by replacing $W_0$ there by $F$, is also a basis for $S_k$.

In particular, $\psi_k$ has a unique representation

$$\psi_k =: c_0\phi_0 + f + v, \tag{4.15}$$

with $f$ in the span of $F$, $v$ in the span of $V$, and $c_0 \neq 0$. However, for each $f_j \in F$, $\widehat{f_j}$ vanishes $k$-fold at each $\alpha \in 2\pi\mathbb{Z}^d\backslash 0$ ($\widehat{\psi_\ell}$ has a zero of order $\ell$ at each of these points, and the difference operator is of order $k-\ell$, hence its symbol vanishes to that order on $2\pi\mathbb{Z}^d$). Also, since $\ell < k$ here, $f_j$ has mean-value zero, and consequently, for $f$ in (4.15), $\widehat{f}$ (as all functions in $\operatorname{span}\widehat{F}$) vanishes on $2\pi\mathbb{Z}^d$ and has a zero of order $k$ at each $\alpha \in 2\pi\mathbb{Z}^d\backslash 0$. In view of the known properties of $\psi_k$, it follows that the Fourier transform of the function

$$\psi := \phi_0 + v/c_0$$

does not vanish at 0, but vanishes to order $k$ elsewhere on $2\pi\mathbb{Z}^d$. But this implies that $S(\psi)$ has approximation order $k$: indeed, with $\|\cdot\|_0$ the semi-norm introduced in the proof of Theorem 4.2, we need to show that $\|\widehat{\psi}\|_0$ has a $k$-fold zero at the origin. This folllows from the fact that, pointwise a.e., $\|c_0\widehat{\psi}\|_0 \leq \|\widehat{\psi_k}\|_0 + \|\widehat{f}\|_0$: indeed, $\|\widehat{\psi_k}\|_0$ has a $k$-fold zero at the origin since $\psi_k$ provides approximation order $k$. The complementary fact, i.e., that $\|\widehat{f}\|_0$ also has such a zero, is proved by showing that the transform of each of the functions $f_j = \Delta^j\psi_\ell \in F$ has such a zero; for that, note that, pointwise, $\|\widehat{f_j}\|_0 = |t_j|\|\widehat{\psi_\ell}\|_0$, with $t_j$ the symbol of $\Delta^j$, hence vanishing to order $|j| = k-\ell$ at 0. At the same time, since $\psi_\ell$ provides approximation order $\ell$, $\|\widehat{\psi_\ell}\|_0$ has a zero of order $\ell$ at the origin. Altogether, this proves that $\|\widehat{f_j}\|_0$, hence $\|\widehat{f}\|_0$, hence $\|c_0\widehat{\psi}\|_0$ have the required zero at the origin, thereby that $\psi$ provides the corresponding approximation order.

It remains to prove that $\psi$ is uniquely determined. For this, let $\psi'$ be of the form

$$\psi' = \phi_0 + v', \quad v' \in \operatorname{span} V, \tag{4.16}$$

with nonzero mean-value and with $\widehat{\psi'}$ vanishing $k$-fold at every $\alpha \in 2\pi\mathbb{Z}^d\backslash 0$. Note that for $j' \leq j$, $D^{j'}\widehat{f_j}(0) \neq 0$ if and only if $j' = j$. This implies that there is some $f'$ in $\operatorname{span} F$ for which $D^j(\widehat{\psi'} - \widehat{f'})(0) = 0$, for every $j \in \mathcal{Z}_k\backslash 0$. Remembering that $\widehat{f'}$, as any function in $\operatorname{span}\widehat{F}$, vanishes to order $k$ everywhere on $2\pi\mathbb{Z}^d\backslash 0$, and vanishes at 0, we see that

$$\psi'' := (\psi' - f')/\widehat{\psi'}(0)$$

satisfies all the conditions imposed on the $\psi$ in Theorem 4.2, hence, necessarily, $\psi'' = \psi_k$. In particular, from (4.15) and from (4.16),

$$(\phi_0 + v' - f')/\widehat{\psi}'(0) = \psi_k = c_0\phi_0 + f + v,$$

with $c_0$, $f \in \operatorname{span} F$, and $v \in \operatorname{span} V$ uniquely determined, and $f' \in \operatorname{span} F$, $v' \in \operatorname{span} V$. Since $(\phi_0, F, V)$ is linearly independent, this implies that, necessarily, $v' = v/c_0$. □

## 5. An application: refinable FSI spaces

The FSI spaces that appear in the context of wavelets and uniform subdivisions are *not* given explicitly in terms of a generating set $\Phi$. Rather, a matrix $M$, known as the **mask**, whose rows and columns are indexed by the unknown set $\Phi$ and whose entries are $2\pi$-periodic functions, is given, with the basic assumption that $\Phi$ is **refinable** with respect to the mask $M$, i.e., that, almost everywhere,

$$\widehat{\Phi}(s^*\omega) = M(\omega)\widehat{\Phi}(\omega).$$

Here, $s$ is a $d \times d$ integer matrix which is assumed to be **expansive**, i.e., its spectrum lies outside the closed unit disc.

In this setting, it is desirable to analyse properties of $\Phi$ and $S(\Phi)$ in terms of the readily available information, viz., the mask $M$.

The problem of characterizing the approximation orders of *refinable* FSI spaces via the relevant mask is addressed in [HSS], [P] and [JRZ] (see also the relevant article [CDP]). The analysis in these papers is carried out under one or more of the following assumptions:

(a) $d = 1$, i.e., the functions in $\Phi$ are univariate.

(b) The entries of $M$ are either trigonometric polynomials ([HSS], [JRZ], forcing the functions in $\Phi$ to be compactly supported), or are smooth ([P], forcing $\Phi$ to decay suitably).

(c) The shifts of $\Phi$ are either linearly independent ([HSS]) or $L_2$-stable ([P]) (the reference [JRZ] requires, in the absence of the linear independence of the shifts, some knowledge on the dependence relations among the shifts of $\Phi$).

(d) The dilation is dyadic, i.e., $s = 2I$.

Thus, an attempt to compute approximation orders, with the aid of the mask only via the above-mentioned results, may require a mask-characterization of linear independence and $L_2$-stability. Such results exist for a *univariate singleton* $\Phi$ (cf. [JW], [R2]). Less is known at present in more than one variable (cf. [H1] and [LLS]), and/or when $\Phi$ contains more than one function (cf. [H2], [S], [W]). It seems from these references that a mask-characterization of linear independence for FSI spaces (univariate or not) is non-trivial: it is even hard to determine, by inspecting the mask only, whether the cardinality of $\Phi$ is 'right', i.e., whether $S(\Phi)$ cannot be generated by fewer functions. (It should be emphasized, though, that the problem of characterizing the stronger property of *orthonormality* of the shifts of $\Phi$, or, more generally, the bi-orthogonality of the shifts of $\Phi$ and the shifts of another refinable (dual) system, is more accessible, cf. §3 of [S]).

In what follows, we invoke the results of the previous section for the study of the above problem. Our analysis is carried out in any number of dimensions $d$, and under a Strong H($k$) property. As mentioned in the Discussion following the Definition 4.1 of that property, the first condition of that property is satisfied if each $\phi \in \Phi$ decays at $\infty$ at a rate $O(|\cdot|^{-\rho})$, $\rho > k+d$, with $k$ the desired approximation order. This condition is also implied by smoothness assumptions on the entries of $M$ around the origin, hence can be verified by inspecting the mask entries. The other condition in that property, viz. the invertibility of $G(0)$, was already explained to be significantly weaker than the condition of linear independence, or even of stability. However, we still do not provide here a viable way for checking this relatively mild condition via the mask $M$ only. We hope that Hogan's work will serve in this direction. In any event, one should keep in mind that the invertibility of $G(0)$ is not only *weaker* than linear independence and stability, but also, once $\Phi$ is given, is *easier to check* than any of the properties of approximation order, linear independence and/or stability.

The conditions we assume in the first part of this section on $\Phi$ are somewhat stronger than the ones assumed in the Strong H($k$) Property:

**(5.1) Conditions.** We assume that

($\alpha$) Each $\phi \in \Phi$ decays at $\infty$ at a rate $O(|\cdot|^{-\rho})$, for some $\rho > k+d$, with $k$ the approximation order studied.

($\beta$) One of the two following conditions:

($\beta$1) The Gramian $G = G_\Phi$ is invertible at the origin, i.e., $\widehat{\Phi}_{|_{2\pi\mathbb{Z}^d}}$ is linearly independent.

($\beta$2) The Gramian $G$ is invertible at each $\xi \in 2\pi s^{*-1}\mathbb{Z}^d$, i.e., $\widehat{\Phi}_{|_{\xi+2\pi\mathbb{Z}^d}}$ is linearly independent for each such $\xi$.

**Discussion.** The decay assumption ($\alpha$) above implies (a) of Strong Property H($k$), while ($\beta$1) here is identical with (b) of that property, hence the combined assumption ($\alpha$+ $\beta$1) is a bit stronger than the H($k$) property. The alternative assumption ($\beta$2) is, of course, more demanding than ($\beta$1). However, it is still significantly weaker than a stability or linear independence assumption, and may also be more easily verified than these other two. □

In the statement of the next result, we use the symbol

$$\Gamma$$

to denote any set of representers of the quotient group $2\pi(\mathbb{Z}^d/s^*\mathbb{Z}^d)$. For example, in case the dilation is dyadic, we can choose $\Gamma = \{0, 2\pi\}^d$.

**Theorem 5.2.** *Let $\Phi \subset L_2$ be a finite refinable set with mask $M$ and dilation matrix $s$. Assume further that ($\alpha$) and ($\beta2$) of (5.1) hold. Then the following two conditions are equivalent:*
*(a) $\Phi$ provides approximation order $k$.*
*(b) There exists a sequence $\tau = (\tau_\phi : \phi \in \Phi)$ of $2\pi$-periodic trigonometric polynomials, each with spectrum in $\mathcal{Z}_k$, such that, with $t := \tau M(s^{*-1}\cdot)$, the following is true:*
*(b1) $t(0) \neq 0$.*
*(b2) $t$ has a zero of order $k$ at each of the points $\Gamma\backslash\{0\}$.*
*(b3) $\tau - t(s^*\cdot)$ has a zero of order $k$ at the origin.*
*Moreover, the implication (b)$\Longrightarrow$(a) holds under the weaker assumption that merely ($\beta1$) (rather than ($\beta2$)) of (5.1) holds.*

**Proof.** We first assume ($\alpha$+ $\beta2$) of (5.1), and prove the implication (a) $\Longrightarrow$ (b). Assuming that $\Phi$ provides approximation order $k$, we let $\psi$ be the superfunction of Theorem 4.2, and $\tau$ be the trigonometric polynomials of the representation $\widehat{\psi} = \tau\widehat{\Phi}$. Note that $\psi$ decays at $\infty$ at a $\rho$-rate, since it is a finite combination of the shifts of $\Phi$. In particular, $\widehat{\psi}$ is $k$ times differentiable everywhere, hence, since it provides approximation order $k$, it must satisfy the Strang-Fix conditions of that order.

Since $t = \tau M(s^{*-1}\cdot)$, we have

$$\widehat{\psi}(\omega) = t(\omega)\widehat{\Phi}(s^{*-1}\omega),$$

a fact essential to our proof that these $\tau, t$ satisfy condition (b) of the present theorem. One of these three required conditions, viz. (b1), is obvious: since we know that $\widehat{\psi}(0) = 1$, we obtain that $t(0) \neq 0$. We now prove that the other two conditions in (b) here follow, too.

Fix $\xi \in s^{*-1}\Gamma\backslash 0$. Then, for any $\alpha \in 2\pi\mathbb{Z}^d$, $s^*(\xi + \alpha)$ is in $2\pi\mathbb{Z}^d\backslash 0$, hence $\widehat{\psi}$ has a zero of order $k$ at $s^*(\xi + \alpha)$, and we know that, for every $j \in \mathcal{Z}_k$,

$$D^j(t\widehat{\Phi}(s^{*-1}\cdot))(s^*(\xi + \alpha)) = 0.$$

Using Leibniz' formula, together with the $2\pi$-periodicity of $t(s^*\cdot)$, we get

$$\sum_{j' \leq j} D^{j'}t(s^*\xi)\binom{j}{j'}(s^{-1}D)^{j-j'}\widehat{\Phi}(\xi + \alpha) = 0, \quad \text{all } \alpha \in 2\pi\mathbb{Z}^d.$$

We now proceed as in the proof of uniqueness in Theorem 4.2. Multiplying this, for each $\phi \in \Phi$, by $\overline{\widehat{\phi}}(\xi + \alpha)$ and summing over $\alpha$ gives the system of equations

$$G(\xi)v_j \ + \sum_{j' < j} Q_{j,j'}(\xi)v_{j'} \ = \ 0, \qquad j \in \mathcal{Z}_k, \tag{5.3}$$

for the quantities

$$v_j \;:=\; D^j t(s^*\xi), \quad j \in \mathcal{Z}_k,$$

with

$$Q_{j,j'}(\phi,\varphi) := \binom{j}{j'}[(s^{-1}D)^{j-j'}\widehat{\varphi},\widehat{\phi}], \quad \phi,\varphi \in \Phi,$$

and $G(\xi)$ invertible by ($\beta$2). Hence the coefficient matrix in (5.3) is block-triangular with invertible diagonal blocks, therefore invertible, hence $D^j t(s^*\xi) = 0$ for all $j \in \mathcal{Z}_k$. This proves (b2).

Condition (b3) is implied by a similar argument: the properties of the 'superfunction' $\psi$ of Theorem 4.2 imply that $F := \widehat{\psi} - \widehat{\psi}(s^*\cdot)$ has a zero of order $k$ at all points of $2\pi\mathbb{Z}^d$. On the other hand, $F = (\tau - t(s^*\cdot))\widehat{\Phi}$. Repeating the argument in the preceding paragraph, with $t$ replaced by the $2\pi$-periodic $\tau - t(s^*\cdot)$, and with $\xi = 0$, implies that $\tau - t(s^*\cdot)$ has the required zero at the origin. This completes the proof of the implication (a)$\Longrightarrow$(b).

We now assume ($\alpha$+ $\beta$1) from (5.1), and prove that (b)$\Longrightarrow$(a). Consider the function $\psi$ defined by $\widehat{\psi} := \tau\widehat{\Phi} = t\widehat{\Phi}(s^{*-1}\cdot)$. First, by assumption (b2), $t$ has a zero of order $k$ at each point in $\Gamma\backslash 0$, and, since $t$ is $2\pi s^*\mathbb{Z}^d$-periodic, it follows that $\widehat{\psi}$ has a zero of order $k$ at each point in $2\pi\mathbb{Z}^d\backslash 2\pi s^*\mathbb{Z}^d$. Second, from assumption (b3) we conclude that $\widehat{\psi}(s^*\cdot) - \widehat{\psi}$ vanishes to order $k$ at each point of $2\pi\mathbb{Z}^d$. Using that fact for $\alpha \in 2\pi(\mathbb{Z}^d\backslash s^*\mathbb{Z}^d)$, we see that both $\widehat{\psi}(s^*\cdot) - \widehat{\psi}$ and $\widehat{\psi}$ have zeros of order $k$ at such $\alpha$, and hence $\widehat{\psi}(s^*\cdot)$ also has such a zero there; i.e., $\widehat{\psi}$ has zeros of order $k$ at each point of $2\pi s^*(\mathbb{Z}^d\backslash 2s^*\mathbb{Z}^d)$. Proceeding by induction, we conclude (from the expansiveness of $s$) that $\psi$ satisfies the Strang-Fix conditions of order $k$. Therefore, since $\psi$ decays at a $\rho$-rate, it remains to show (cf., e.g., [BDR1: Corollary 5.15]) that $\widehat{\psi}(0) \neq 0$, in order to conclude that $\psi$ provides approximation order $k$, which trivially implies that $\Phi$ provides that order $k$. Note that, so far, we have not used ($\beta$1) of (5.1).

By (b1) and (b3), $\tau(0) \neq 0$, hence, with ($\beta$1) of (5.1), $\widehat{\psi}_{|_{2\pi\mathbb{Z}^d}} = \tau(0)\widehat{\Phi}_{|_{2\pi\mathbb{Z}^d}}$ is not zero. However, we already proved that $\widehat{\psi} = 0$ on $2\pi\mathbb{Z}^d\backslash 0$. Hence, $\widehat{\psi}(0) \neq 0$. □

We now turn our attention to a second application to refinable functions of the superfunction results of Section 4. This application deals with changing a generating set for a refinable function space $S := S(\Phi)$. In various applications, one would like to replace the original refinable $\Phi$ by another generating set $\Phi_0$ for the space $S$, so that the new vector $\Phi_0$ contains a superfunction $\psi$ as one of its elements. If the relation between $\Phi$ and $\Phi_0$ is given as

$$\widehat{\Phi}_0 = U\widehat{\Phi}$$

for some transition matrix $U$ (whose entries are $2\pi$-periodic), and if $\Phi$ is refinable with mask $M$ and dilation matrix $s$, then one easily computes that $\Phi_0$ is refinable with mask

$$M_0 := U(s^*\cdot)MU^{-1}. \tag{5.4}$$

**Corollary 5.5.** *Let $\Phi$ be refinable with mask $M$. Assume that, for some positive $k$, $\Phi$ satisfies the Strong H(k) property. Then there exists a function $\psi$ in the finite span of $E(\Phi)$ such that:*

*(a) $\psi$ is a superfunction, i.e., $S(\psi)$ provides approximation order $k' \le k$ if $S(\Phi)$ does so.*

*(b) $\widehat{\psi}(0) \neq 0$.*

*(c) There exists $\phi_0 \in \Phi$ such that the vector $\Phi_0 := \psi \cup (\Phi \backslash \phi_0)$ is refinable with mask $M_0$ of the form*

$$M_0 = U_1 M U_2,$$

*where the entries of $U_j$, $j = 1, 2$, are trigonometric polynomials.*

The highlight in this corollary is the trigonometric polynomiality of the entries of the transition matrices: this guarantees that the entries of $M_0$ are trigonometric polynomials in case the entries of $M$ are. The superfunction in the corollary is chosen below to be that from Theorem 4.12. In contrast, the superfunction of Theorem 4.2 does not seem to yield a result as clean as Corollary 5.5: if one replaces any one $\phi_0 \in \Phi$ by $\psi$ from that result, the resulting transition matrix $U_2$ cannot be guaranteed to have polynomial entries (regardless of the choice of $\phi_0$; compare with (5.4)).

The corollary above is used in [PR] in the context of *factorization of univariate masks*, and estimating the *smoothness* of refinable functions via their masks. In that latter context, another highlight of the above corollary is that the adjunction of the superfunction $\psi$ to the generating set results in the removal of some generator $\phi_0$, but in no other change to any of the remaining generators: this proves to be significant when one attempts to study one by one the smoothness of each of the original generators. Theorem 4.2 could still be used inductively to yield the original factorization technique of [P]. However, that technique shuffles completely the entire generating set, leaving no hope for a separate study of the smoothness of each individual entry.

**Proof of Corollary 5.5:** We take $\psi$ and $\phi_0$ to be as in Theorem 4.12. Then conditions (a-b) of the current corollary follow directly from assertions in Theorem 4.12. We then note that

$$\widehat{\psi} = \widehat{\phi}_0 + \sum_{\phi \in (\Phi \backslash \phi_0)} \tau_\phi \widehat{\phi},$$

where, for every $\phi \in (\Phi \backslash \phi_0)$, $\tau_\phi$ is some trigonometric polynomial with spectrum in $\mathcal{Z}_k$. This means that, upon ordering $\Phi$ in any way that puts $\phi_0$ first, and ordering $\Phi_0$ correspondingly (i.e., $\psi$ first, no change in the rest), we obtain that

$$\widehat{\Phi}_0 = U \widehat{\Phi},$$

with the matrix $U$ differing from the identity matrix only in its first row, whose diagonal entry is 1 and the other entries being $\tau_\phi$, $\phi \in \Phi \backslash \phi_0$. Now, $\Phi_0$ is indeed refinable with mask

$M_0 = U_1 M U_2$, with $U_1$ obtained by dilating the above $U$, and $U_2$ by inverting the above $U$. The polynomiality of $U_1$, thus, is not in question. The polynomiality of $U_2$ follows from the special structure of $U$: indeed, $U_2$ also differs from the identity in the first row only: that first row has again 1 as the diagonal entry, and $-\tau_\phi$, $\phi \in \Phi\backslash\phi_0$, elsewhere. $\square$

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